Quiz 2

Course: Algebraic Topology II (KSM4E02) **Instructor:** Aritra Bhowmick

Time: 2:00PM – 4:00PM, 23rd April, 2026 **Total marks:** 20

Attempt any question. You can get maximum **15 marks**.

You may use the homology groups / cohomology rings (with $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients) of standard spaces like $S^n, \mathbb{RP}^n, \mathbb{T}^n$, etc without showing the computation. If you are using UCT or Künneth theorem, mention them clearly. If any $\mathrm{Tor}/\mathrm{Ext}$ module vanishes, mention with justification.

Q1. Given an R-module M, suppose $\mathrm{Tor}_1^R(M,N)=0$ for any R-module N. Show that $\mathrm{Tor}_n^R(M,N)=0$ for any R-module N, and for any $n \geq 2$. [4]

Solution : Suppose $\mathrm{Tor}_1^R(M,N)=0$ for any R-module N. Consider a short exact sequence

$$0 \to A \to B \to C - 0.$$

We have the long exact sequence

$$\cdots \to \mathrm{Tor}_1^R(M,C) \to M \otimes A \to M \otimes B \to M \otimes C \to 0.$$

Since $\mathrm{Tor}_1^R(M,C)=0$, we have the short exact sequence

$$0 \to M \otimes A \to M \otimes B \to M \otimes C \to 0.$$

This shows that M is a flat R-module. But then tensoring by M preserves any long exact sequence as well.

Now, let N be an arbitrary R-module. Get a projective resolution

$$\cdots \to P_2 \overset{d_2}{\to} P_1 \overset{d_1}{\to} P_0 \overset{d_0}{\to} N \to 0.$$

Denote $K_i = \mathrm{im}(d_{i+1}) = \ker(d_i)$. Then, we have short exact sequences

$$0 \to K_i \to P_i \to K_{i-1} \to 0.$$

Since M is flat, we have short exact sequence

$$0 \to M \otimes K_i \to M \otimes P_i \to M \otimes K_{i-1} \to 0.$$

Thus,

$$\ker(1 \otimes d_i) = M \otimes K_i = M \otimes \ker(d_i), \quad \mathrm{im}(1 \otimes d_i) = M \otimes K_{i-1} = M \otimes \mathrm{im}(d_i),$$

which implies,

$$\ker(1 \otimes d_i) = M \otimes \ker(d_i) = M \otimes \mathrm{im}(d_{i+1}) = \mathrm{im}(1 \otimes d_{i+1}).$$

Consequently, we have long *exact* sequence

$$\cdots \to M \otimes P_1 \to M \otimes P_0 \to M \otimes N \to 0.$$

Hence, for $n \geq 2$ we have

$$\mathrm{Tor}_n^R(M,N) = H_n(\cdots \to M \otimes P_1 \to M \otimes P_0 \to 0) = 0.$$

Q2. Suppose $n > m$. Show that there are no maps $\mathbb{RP}^n \to \mathbb{RP}^m$ inducing a nontrivial map in the 1st cohomology with $\mathbb{Z}_2$-coefficients. [4]

Solution : Suppose $f : \mathbb{RP}^n \to \mathbb{RP}^m$ is a map which induces a nontrivial map $f^\star : H^1(\mathbb{RP}^m;\mathbb{Z}_2) \to H^1(\mathbb{RP}^n;\mathbb{Z}_2)$. Recall, the cohomology ring of $\mathbb{RP}^k$ is generated by the nonzero element $x \in H^1(\mathbb{RP}^k;\mathbb{Z}_2)$ such that $x^k \neq 0$ but $x^{k+1}=0$.

Say $x \in H^1(\mathbb{RP}^m;\mathbb{Z}_2)$ be the generator. Since $y = f^\star(x) \neq 0$, it must be the genrator of $H^\star(\mathbb{RP}^n;\mathbb{Z}_2)$. We have

$$y^{m+1} = (f^\star(x))^{m+1} = f^\star(x^{m+1}) = f^\star(0) = 0 \Rightarrow m+1 \geq n+1 \Rightarrow m \geq n.$$

Thus, for $n > m$, there does not exist any map $\mathbb{RP}^n \to \mathbb{RP}^m$, which induces a nontrivial map in the first cohomology with $\mathbb{Z}_2$ coefficient.

Q3. Prove or disprove : $\mathbb{RP}^3$ is homotopy equivalent to $\mathbb{RP}^2 \vee S^3$. [4]

Solution : We have cohomology ring

$$H^\star(\mathbb{RP}^3;\mathbb{Z}_2) = \mathbb{Z}_2[X]/\langle X^4\rangle, \quad |X| = 1.$$

As for $X = \mathbb{RP}^2 \vee \mathbb{S}^3$, the cohomology groups with $\mathbb{Z}_2$ coefficients are

$$H^k(X;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & k = 0,1,2,3 \\ 0, & \text{otherwise.} \end{cases}$$

Pick the generator $y \in H^1(\mathbb{RP}^2;\mathbb{Z}_2)$. Then, $(y,0)$ is the generator of $H^1(\mathbb{RP}^2 \vee S^3;\mathbb{Z}_2)$, since $H^1(S^3;\mathbb{Z}_2) = 0$. Now,

$$(y,0)^3 = (y^3,0) = (0,0) = 0.$$

Thus, the third power of any element of $H^1(\mathbb{RP}^2 \vee S^3;\mathbb{Z}_2)$ vanishes. But on the other hand, there is $x \in H^1(\mathbb{RP}^3;\mathbb{Z}_2)$ so that $x^3 \neq 0$ (in fact, it generates $H^3(\mathbb{RP}^3;\mathbb{Z}_2) = \mathbb{Z}_2$).

Thus, the cohomology rings are not isomorphic, and consequently, the two spaces cannot be homotopy equivalent.

Note : both spaces have identical homology groups with $\mathbb{Z}$-coefficients, and hence, via UCT, have the same (co)homology groups with any coefficients. To distinguish them, we needed the cup product.

Q4. Assuming $k, l > 0$, show that there *does not* exist a map $f : S^{k+l} \to S^k \times S^l$ which induces a nontrivial map in the $(k+l)^{\text{th}}$-homology with $\mathbb{Z}$ coefficients. [4]

Solution : Suppose $f : S^{k+l} \to S^k \times S^l$ induces a nontrivial map $f_\star : H_{k+l}(S^{k+l}) \to H_{k+l}(S^k \times S^l)$. By UCT, it follows that $f^\star$ is dual to $f_\star$, and hence, $f^\star : H^{k+l}(S^k \times S^l) \to H^{k+l}(S^{k+l})$ is nontrivial.

Now, the cohomology ring is $H^\star(S^k \times S^l) = H^\star(S^k) \otimes H^\star(S^l)$. Denote $f_1 : S^{k+l} \to S^k$, $f_2 : S^{k+l} \to S^l$ as the components of f. The naturality of the Künneth formula gives the commuting diagram

$$\begin{tikzcd}
H^k(S^k) \otimes H^l(S^l) \arrow[r, "\times", "\cong"'] \arrow[d, "f_1^\star \otimes f_2^\star"'] & H^{k+l}(S^k \times S^l) \arrow[d, "(f_1 \times f_2)^\star"] \\
\underbrace{H^k(S^{k+l})}_{0} \otimes \underbrace{H^l(S^{k+l})}_{0} \arrow[r, "\times"'] & H^{k+l}(S^{k+l} \times S^{k+l})
\end{tikzcd}$$

where the top cross product is an isomorphism. Now, for the generators $a \in H^k(S^k), b \in H^l(S^l)$, we have

$$(f_1 \times f_2)^\star(a \times b) = f_1^\star(a) \times f_2^\star(b) = 0.$$

For the diagonal map $\Delta : S^{k+l} \to S^{k+l} \times S^{k+l}$, we have $(f_1 \times f_2) \circ \Delta = f$. Then, we get

$$f^\star(a \times b) = \Delta^\star(f_1 \times f_2)^\star = \Delta^\star(f_1^\star(a) \times f_2^\star(b)) = \Delta^\star(0) = 0.$$

This contradicts the nontriviality of $f^\star$, since $a \times b$ is the generator of $H^{k+l}(S^k \times S^l)$ by the Künneth theorem.

Q5. A based space X is called a *co-H space* if there is a map $\mu : X \to X \vee X$ such that the diagram commutes up to homotopy

$$\begin{tikzcd}
X \arrow[rr, "\Delta"] \arrow[dr, bend right, "\mu"'] & \circlearrowleft & X \times X \\
& X \vee X \arrow[ur, hook, bend right, "j"'] &
\end{tikzcd}$$

Show that the cup product in the cohomology ring of X (with coefficients in any ring) is trivial in every positive degree. [4]

Solution : Let $a \in H^p(X), b \in H^q(X)$ be cohomology classes, with $p, q \geq 1$, and coefficients in some fixed ring. Consider the two proejction maps $\pi_1, \pi_2 : X \times X \to X$, and denote $p_i = \pi_i \circ j : X \vee X \to X$ to be the projections from the wedge. Recall that in the cohomology ring of the wedge, cup product in positive grading coming from different components always vanishes. We compute

$$a \smile b = \Delta^\star(a \times b) = (j \circ \mu)^\star(a \times b) = \mu^\star j^\star(a \times b) = \mu^\star j^\star(\pi_1^\star a \smile \pi_2^\star b) = \mu^\star(p_1^\star a \smile p_2^\star b) = \mu^\star(0) = 0.$$

Hence, the claim follows.

Note: This proves that a co-H space has LS-category ≤ 2; in fact, the converse is also true : a space of CW type and having LS-category ≤ 2, is a co-H space. Any suspension ΣX is always a co-H space, where the comultiplication map is quotient map $q : \Sigma X \to \Sigma X \vee \Sigma X$ which collapses the subspace $X \times \left\{\frac{1}{2}\right\} \hookrightarrow \Sigma X$. Thus, every S^n is a co-H space, but $\mathbb{RP}^n$ is not for $n \geq 2$ (as it has nontrivial cup product).