Quiz 1

Course: Algebraic Topology II (KSM4E02) **Instructor:** Aritra Bhowmick

Time: 2:00PM – 4:00PM, 22nd February, 2026 **Total marks:** 20

Attempt any question. You can get maximum **15 marks**.

Q1. Suppose $A \subset X$ is a *homotopy retract*, i.e, there is a map $r : X \to A$ such that $r \circ \iota \simeq \mathrm{Id}_A$. Show that $H_n(X) = H_n(A) \oplus H_n(X, A)$ for all n.

Solution : It follows from the functoriality and the homotopy invariance of the homology groups,

$$\mathrm{Id}_{H_n(A)} = H_n(\mathrm{Id}_A) = H_n(r \circ \iota) = H_n(r) \circ H_n(\iota).$$

This means, $H_n(i)$ is injective. Now, consider the long exact sequence of the pair (X, A)

$$\cdots \longrightarrow H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \underset{H_n(r)}{\overset{H_n(\iota)}{\rightleftarrows}} H_n(X) \longrightarrow H_n(X, A) \longrightarrow \cdots$$

From exactness,

$$\mathrm{im}(\partial) = \ker H_n(\iota) = 0.$$

Hence, we have the short exact sequence

$$0 \xrightarrow{\partial} H_n(A) \underset{H_n(r)}{\overset{H_n(\iota)}{\rightleftarrows}} H_n(X) \longrightarrow H_n(X, A) \longrightarrow 0$$

which is split. Hence, we have $H_n(X) = H_n(A) \oplus H_n(X, A)$ holds for all n.

Q2. A pointed space (X, x_0) is called *good* if there is an open neighborhood $x_0 \in U \subset X$ such that U strongly deformation retract onto x_0. Given good pointed spaces $(A, a_0), (B, b_0)$, consider the wedge $X = A \vee B$. Show that $H_n(X) = \tilde{H}_n(A) \oplus \tilde{H}_n(B)$ for all n. Show by example that the claim is not true if we do not consider reduced homology.

Solution : Fix open sets $a_0 \in U \subset A$ and $b_0 \in V \subset V$ such that U (resp. V) deformation retracts onto a_0 (resp. b_0). Consider the subspaces $\tilde{A} = A \vee V$ and $\tilde{B} = U \vee B$ of the wedge X. Since the deformation retract is *strong*, it follows that $\tilde{A}$ (resp. $\tilde{B}$) also deformation retracts onto A (resp. onto B). Moreover, we have $\tilde{A} \cup \tilde{B} = X$, and $\tilde{A} \cap \tilde{B} = U \vee V$ deformation retracts onto the wedge point, say, x_0. Next, observe that

$$\left(\tilde{A}, \tilde{A} \cap \tilde{B}\right) \hookrightarrow (X, \tilde{B}), \quad \left(\tilde{B}, \tilde{A} \cap \tilde{B}\right) \hookrightarrow \left(X, \tilde{A}\right),$$

are excisive since we can excise out the closed set $B \setminus V$ (resp. $A \setminus U$). Now, consider the reduced Mayer-Vietoris sequence, we have

$$\cdots \to \tilde{H}_n\left(\tilde{A} \cap \tilde{B}\right) \to \tilde{H}_n\left(\tilde{A}\right) \oplus \tilde{H}_n\left(\tilde{B}\right) \to \tilde{H}_n(X) \to \tilde{H}_{n-1}\left(\tilde{A} \cap \tilde{B}\right) \to \cdots$$

Since $\tilde{A} \cap \tilde{B}$ is contractible, the reduced homology groups vanish. Hence, we have the isomorphism $\tilde{H}_n(X) = \tilde{H}_n\left(\tilde{A}\right) \oplus \tilde{H}_n\left(\tilde{B}\right) = \tilde{H}_n(A) \oplus \tilde{H}_n(B)$.

Now, consider $A = B = S^0$, the 0-sphere. Then, $A \vee B$ is a discrete space of 3 points. By finite additivity of homology theory, $H_0(A) = H_0(B) = H_0(\star)^2$, and $H_0(A \vee B) = H_0(\star)^3$. Assuming we are working with singular homology, we have $H_0(A) \oplus H_0(B) = \mathbb{Z}^4$, and $H_0(X) = \mathbb{Z}^3$, which are not isomorphic.

Q3. Suppose $X = [0, 1]$ and $A = \{0\} \cup \left\{\frac{1}{n} \mid n \geq 1\right\} \subset X$. Show that $H_1(X, A) \not\cong \tilde{H}_1(X/A)$ for the singular homology.

Hint: You may use the fact that there exists a surjection $\pi_1(X/A) \twoheadrightarrow \prod_{i=0}^{\infty} \mathbb{Z}$.

Solution : Note that the path components of A are singletons. Hence, $H_0(A) = \oplus_{i=0}^{\infty} \mathbb{Z}$ and $H_1(A) = 0$. Also, $X \simeq \star \Rightarrow \tilde{H}_n(X) = 0$. Then, from the reduced long exact sequence of the pair (X, A), we have

$$\underbrace{\tilde{H}_1(A)}_{0} \to H_1(X, A) \to \tilde{H}_0(A) \to \underbrace{\tilde{H}_0(X)}_{0}$$

Thus, $H_1(X, A) = \tilde{H}_0(A) = \oplus_{i=1}^{\infty} \mathbb{Z}$. Note that $H_1(X, A)$ is countable.

Now, it is given that $\pi_1(X/A)$ surjects onto $G := \prod_{i=0}^{\infty} \mathbb{Z}$, which is an uncountable Abelian group. By the universal property of the Abelianization, it follows that the surjection $\varphi : \pi_1(X/A) \twoheadrightarrow G$ factors through $\tilde{\varphi} : \pi_1(X/A)^{\text{ab}} \twoheadrightarrow G$, which is also a surjection. As G is uncountable, it follows that $\pi_1(X/A)^{\text{ab}}$ is uncountable. But then by the Hurewicz theorem, we have $H_1(X/A) = \pi_1(X/A)^{\text{ab}}$ must be uncountable. Hence, $H_1(X, A) \ncong \tilde{H}_1(X/A) = H_1(X/A)$

Note: It is easy to see that X/A is in fact homeomorphic to the Hawaiian earring.

Q4. Compute the singular homology groups of the following spaces.

- X is the space obtained from S^2 by pinching two antipodal points.
- Y is the space obtained from S^2 by attaching an equatorial disc.
- Z is the space obtained from the torus by attaching a S^2 along the equator in the middle hole.
- U is the space obtained from $\mathbb{R}^3$ by removing the unit circle in the xy-plane with center at the origin.
- V is the space obtained from $S^3 = \mathbb{R}^3 \cup \{\infty\}$ by removing the unit circle in the xy-plane with center at the origin.

Solution : The space X is homotopy equivalent to the wedge $S^2 \vee S^1$. Hence, it follows that

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

The space Y is homotopy equivalent to the wedge $S^2 \vee S^2$. Hence, it follows that

$$H_2(Y) = \begin{cases} \mathbb{Z}^2, & n = 2 \\ \mathbb{Z}, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

The space Z is homotopy equivalent to the wedge $S^2 \vee S^2 \vee S^1$. Hence, it follows that

$$H_2(Z) = \begin{cases} \mathbb{Z}^2, & n = 2 \\ \mathbb{Z}, & n = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$

The space U is homotopy equivalent to $S^2 \vee S^1$. Hence, it follows that

$$H_n(U) = \begin{cases} \mathbb{Z}, & n = 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

The space V is homotopy equivalent to S^1. To visualize this, think of S^3 as the union of two solid tori $S^1 \times D^2$ with their boundary attached. Both of them deformation ratract onto the center circle $S^1 \times \{0\}$. Now, imagine one of the circle as the unit circle in the xy-plane. Then, removing it means your are removing one of the solid tori. So up to homotopy equivalence, you are left with a circle. Hence,

$$H_n(V) = \begin{cases} \mathbb{Z}, & n = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$