End-semester Examination

Course: Algebraic Topology II (KSM4E02) **Instructor:** Aritra Bhowmick

Time: 2:00PM – 5:00PM, 7th May, 2026 **Total marks:** 45

Attempt any question. You can get maximum **40 marks**.

Fact: If M is a compact n-fold, then M is homotopy equivalent to a finite CW complex of dimension n.

Fact: If G is a finitely generated Abelian group, then G can be uniquely written as $G = \mathbb{Z}^k \oplus T$ for some $k \geq 0$, where T is the torsion subgroup defined as $T = \{g \in G \mid \operatorname{ord}(g) < \infty\}$. Moreover, T can be written as a finite direct sum of cyclic groups.

Fact: Tensor naturally distributes over direct sum : $A \otimes (\oplus_{i \in I} B_i) = \oplus_{i \in I} A \otimes B_i$

Fact: Given any Abelian group A, we have $\mathbb{Z}/n\mathbb{Z} \otimes A = A/nA$, where $nA = \{na \mid a \in A\}$, and $\operatorname{Tor}(\mathbb{Z}/n\mathbb{Z}, A) = {}_nA = \{a \in A \mid na = 0\}$.

You may also use the (co)homology groups of standard spaces like $S^n, \mathbb{RP}^n, \mathbb{CP}^n$ etc without computation. If you are using any UCT like theorem to compute something, mention why the $\operatorname{Tor}/\operatorname{Ext}$ term vanishes (if it does vanish). You may use any part of any question, whether attempted or not, in solving another question.

Q1. Fix R-modules A, B, M. Show that $\operatorname{Tor}_n^R(M, A \oplus B) = \operatorname{Tor}_n^R(M, A) \oplus \operatorname{Tor}_n^R(M, B)$ for all n.

Solution : Get a projective resolutions $P_\bullet \to A \to 0, Q_\bullet \to B \to 0$. Note that $P_i \oplus Q_i$ is again projective, being direct sum of projectives. Thus, we have a projective resolution

$$\cdots \to P_1 \oplus Q_1 \to P_0 \oplus Q_0 \to A \oplus B \to 0.$$

It is easy to see that kernel and images are computed component-wise, and hence it is indeed a resolution. Moreover, $M \otimes (P_i \oplus Q_i) = (M \otimes P_i) \oplus (M \otimes Q_i)$ naturally. Now, we have

$$\begin{aligned}
\operatorname{Tor}_n^R(M, A \oplus B) &= H_n(\cdots \to M \otimes (P_1 \oplus Q_1) \to M \otimes (P_0 \oplus Q_0) \to 0) \\
&= H_n(\cdots \to (M \otimes P_1) \oplus (M \otimes Q_1) \to (M \otimes P_0) \oplus (M \otimes Q_0) \to 0) \\
&= H_n(\cdots \to M \otimes P_1 \to M \otimes P_0 \to 0) \oplus H_n(\cdots \to M \otimes Q_1 \to M \otimes Q_0 \to 0) \\
&= \operatorname{Tor}_n^R(M, A) \oplus \operatorname{Tor}_n^R(M, B).
\end{aligned}$$

Q2. Compute the cohomology rings (with $\mathbb{Z}$-coefficients) of the following spaces.

a. $\Sigma\mathbb{CP}^2$.

b. $S^3 \vee S^5$.

Are these two spaces homotopy equivalent?!

Solution : By cellular homology, we have

$$H_k(\mathbb{CP}^2) = \begin{cases} \mathbb{Z}, & k = 0, 2, 4 \\ 0, & \text{otherwise.} \end{cases}$$

By the suspenesion isomorphism $\tilde{H}_k(\Sigma X) \cong \tilde{H}_{k-1}(X)$, we get

$$H_k(\Sigma\mathbb{CP}^2) = \begin{cases} \mathbb{Z}, & k = 0, 3, 5 \\ 0, & \text{otherwise.} \end{cases}$$

Since all the groups are free, applying UCT, it follows that cohomology is dual to homology. In particular,

$$H^k(\Sigma\mathbb{CP}^2) = \begin{cases} \mathbb{Z}, & k = 0, 3, 5 \\ 0, & \text{otherwise.} \end{cases}$$

Since this is a suspension space, (or alternatively, because of degree reasons) there is no nontrival cup product.

Since reduced homology splits as direct sum for wedge of spaces, we have

$$H_k(S^3 \vee S^5) = \begin{cases} \mathbb{Z}, & k = 0, 2, 4 \\ 0, & \text{otherwise.} \end{cases}$$

This is same as $\Sigma\mathbb{CP}^2$. Hence, again by UCT, we have cohomology is dual to homology and thus,

$$H^k(S^3 \vee S^5) = \begin{cases} \mathbb{Z}, & k = 0, 2, 4 \\ 0, & \text{otherwise.} \end{cases}$$

Again, for degree reason (or because $S^3 \vee S^5 = \Sigma(S^2 \vee S^4)$ is a suspension), we have there is no nontrivial cup product.

Thus, in both the cases, the cohomology ring can is isomorphic to

$$\frac{\mathbb{Z}[X, Y]}{\langle X^2, Y^2 \rangle}, \quad |X| = 3, |Y| = 5.$$

Note: Since we are working with graded commutative rings, the relation $XY = 0$ is implicit in the above presentation.

Any homotopy equivalence will induce an isomorphism between the cohomology rings. Since the cohomology rings $H^\star(\mathbb{CP}^2)$ and $H^\star(S^3 \vee S^5)$ are abstractly isomorphic, it follows that the cup product structure is not enough to distinguish between the spaces.

Note: The spaces are in fact *not* homotopy equivalent to each other. To justify this, one requires *cohomology operations*, namely, Steenrod square maps. These are natural transformation Sq^i : $H^k(X, \mathbb{Z}_2) \to H^{k+i}(X, \mathbb{Z}_2)$, which in particular satisfies that $\mathrm{Sq}^n(a) = a \smile a$ whenever $|a| = n$. Moreover, Sq^0 is the Bockstein map. We have,

$$H^\star(\Sigma\mathbb{CP}^2; \mathbb{Z}_2) = H^\star(S^3 \vee S^5, \mathbb{Z}_2) = \frac{\mathbb{Z}_2[X, Y]}{\langle X^2, Y^2 \rangle}, \quad |X| = 3, |Y| = 5.$$

One can show the following.

- $\mathrm{Sq}^2 : H^3(\Sigma\mathbb{CP}^2, \mathbb{Z}_2) \to H^5(\Sigma\mathbb{CP}^2, \mathbb{Z}_2)$ is an isomorphism. This is because the Steenrod square maps commute with the suspension isomorphism. Now, from consider the generator $a \in H^2(\mathbb{CP}^2, \mathbb{Z}_2)$, so that Σa generates $H^3(\Sigma\mathbb{CP}^2, \mathbb{Z}_2)$. Then,

$$\mathrm{Sq}^2(\Sigma a) = \Sigma(\mathrm{Sq}^2(a)) = \Sigma(a \smile a),$$

which generates $H^5(\Sigma\mathbb{CP}^2, \mathbb{Z}_2)$. Note, $\mathrm{Sq}^2(a) = a \smile a$ as $|a| = 2$.

- On the other hand, $\mathrm{Sq}^2 : H^3(S^3 \vee S^5; \mathbb{Z}_2) \to H^5(S^3 \vee S^5; \mathbb{Z}_2)$ is the 0 map. This is because the generator $a \in H^3(S^3; \mathbb{Z}_2)$ gives rise to the generator $\iota^\star(a) \in H^3(S^3 \vee S^5; \mathbb{Z}_2)$, where $\iota : S^3 \hookrightarrow S^3 \vee S^5$ is the inclusion. Now, naturality of the Steenrod square gives

$$\mathrm{Sq}^2(\iota^\star a) = \iota^\star(\mathrm{Sq}^2(a)) = \iota^\star(0) = 0,$$

as $\mathrm{Sq}^2(a) \in H^5(S^3; \mathbb{Z}_2) = 0$.

Since Steenrod squares are natural maps, homotopy equivalent spaces must have the same structure. Thus, the spaces $\Sigma\mathbb{CP}^2$ and $S^3 \vee S^5$ cannot be homotopy equivalent.

Further Note: The point of this exercise is the following. Recall the Hopf map $h : S^3 \to S^2$. One can show that the mapping cone of h is $\mathbb{CP}^2$. Now, taking suspension, we have $\Sigma h : S^4 \to S^3$. Then, (from the cofibration sequence) one gets that $\Sigma\mathbb{CP}^2$ is the mapping cone of Σh. We would like to say that $\Sigma h \not\simeq 0$. If $\Sigma h \simeq 0$, then we have the mapping cone is $S^3 \vee S^5$, which is a contradiction by the above discussion (recall : homotopic maps have homotopy equivalent mapping cones). Thus, Σh is a nontrivial element in $\pi_4(S^3)$.

Further further note: One can moreover show that Σh has order 2, and $\pi_4(S^3) = \mathbb{Z}_2$ is generated by this map. The fact that $\pi_4(S^3) = \mathbb{Z}_2$ is classical (e.g, via EHP sequence). In 2016, in his PhD thesis, Gillaume Brunerie reproved the same via homotopy type theory formalism; 2 is sometimes called the Brunerie number!

Q3. Suppose M is a compact, connected, oriented, 4-fold, without boundary. Show that the nontrivial (co)homology groups of M with $\mathbb{Z}$ coefficients are

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$H_k(M)$	$\mathbb{Z}$	$F_1 \oplus T$	$F_2 \oplus T$	F_1	$\mathbb{Z}$
$H^k(M)$	$\mathbb{Z}$	F_1	$F_2 \oplus T$	$F_1 \oplus T$	$\mathbb{Z}$

Here F_1, F_2 are some free Abelian groups, and T is an Abelian torsion group (i.e, every element of T has finite order).

Solution : Let us observe the following.

- Since M is compact and oriented, by the Poincare duality, we have $H_i(M) \cong H^{4-i}(M)$. Moreover, all the groups are finitely generated.
- As M is connected, we have $H_0(M) = \mathbb{Z}$, and hence, $H^4(M) = \mathbb{Z}$ as well.
- By UCT, it follows $H^0(M) = \mathbb{Z}$, and hence, $H_4(M) = \mathbb{Z}$. Alternatively, M being oriented implies $H_4(M) = \mathbb{Z}$ as well.
- Let us write $H_1(M) = F_1 \oplus T$ for some free group F_1 and a torsion group T. This gives $H^3(M) = F_1 \oplus T$ as well.
- From UCT, we have the split exact sequence

$$0 \to \mathrm{Ext}(H_0(M), \mathbb{Z}) \to H^1(M) \to \hom(H_1(M), \mathbb{Z}) \to 0.$$

Since $\text{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$, and $\hom(F_1 \oplus T, \mathbb{Z}) = \hom(F_1, \mathbb{Z}) = F_1$, we have $H^1(M) = F_1$. This gives $H_3(M) = F_1$ as well.

- Again, from UCT, we have the split exact sequence

$$0 \to \text{Ext}(H_1(M), \mathbb{Z}) \to H^2(M) \to \hom(H_2(M), \mathbb{Z}) \to 0.$$

Now, $\text{Ext}(H_1(M), \mathbb{Z}) = \text{Ext}(F_1 \oplus T, \mathbb{Z}) = \text{Ext}(T, \mathbb{Z}) = T$. Let us write $H_2(M) = F_2 \oplus T'$ for a free group F_2 and torsion group T'. Then, we have a split exact sequence

$$0 \to T \to H^2(M) \to F_2 \to 0.$$

We get, $H_2(M) = H^2(M) = F_2 \oplus T$. By the structure theorem of finitely generated Abelian groups, we must have $T' = T$.

Thus, we have justified the table.

Q4. Suppose M is a compact, connected, oriented n-fold.

a. If the fundamental group of M is torsion (i.e, every element is of finite order), then show that $H_{n-1}(M) = 0$.

b. If $n = 2k$ and $H_{k-1}(M)$ is torsionfree (i.e, every nonzero element is of infinite order), then show that $H_k(M)$ is also torsionfree.

Solution :

a. Since M is connected (and hence path connected), by Hurewicz theorem, we have $H_1(M) = \pi_1(M)^{\text{ab}}$ is a torsion group. In particular, $\hom(H_1(M), \mathbb{Z}) = 0$. Now, by UCT, we have

$$H^1(M) = \text{Ext}(H_0(M), \mathbb{Z}) \oplus \hom(H_1(M), \mathbb{Z}) = \text{Ext}(\mathbb{Z}, \mathbb{Z}) \oplus 0 = 0.$$

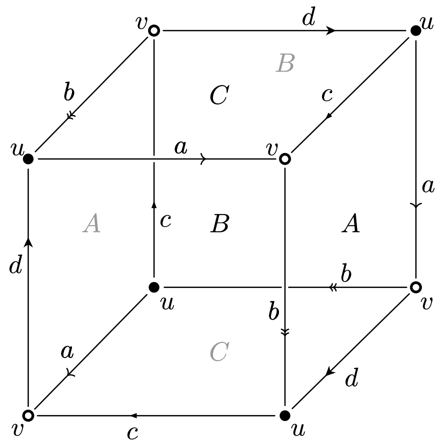
Then, by the Poincare duality, we have $H_{n-1}(M) = H^1(M) = 0$.

b. Note that being free and torsionfree is equivalent for a finitely generated Abelian group. Thus, $H_{k-1}(M)$ is free. By the UCT, we have

$$0 \to \text{Ext}(H_{k-1}(M), \mathbb{Z}) \to H^k(M) \to \hom(H_k(M), \mathbb{Z}) \to 0.$$

The Ext term vanishes as $H_{k-1}(M)$ is free. Also, $\hom(H_k(M), \mathbb{Z})$ is free, since hom-ing into $\mathbb{Z}$ kills any torsion. Thus, $H^k(M) = \hom(H_k(M), \mathbb{Z})$ is free.

Q5. Consider the space X obtained from the cube in the following way: identify each face of the cube with the one opposite after twisting it by $90°$ in the right-handed corkscrew motion. We have the following cell structure on X.



Using cellular homology (or otherwise) show that

$$H_k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k=0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & k=1 \\ 0, & k=2 \\ \mathbb{Z}, & k=3 \\ 0, & \text{otherwise.} \end{cases}$$

Compute the cohomology groups $H^k(X;\mathbb{Z}_2)$.

Solution : The cellular chain complex of X with $\mathbb{Z}$ coefficient is given as

$$0 \to \mathbb{Z}\langle \Delta \rangle \to \mathbb{Z}\langle A, B, C\rangle \to \mathbb{Z}\langle a, b, c, d\rangle \to \mathbb{Z}\langle u, v\rangle \to 0.$$

Note $d_1(a) = v - u = -\partial_1(b) = \partial_1(c) = -\partial_1(d)$, and hence,

$$H_0(X) = \mathbb{Z}\langle [u] = [v]\rangle.$$

Next, we have

$$\begin{aligned} \ker d_1 &= \{t_1 a + t_2 b + t_3 c + t_4 d \mid t_1 - t_2 + t_3 - t_4 = 0\} \\ &= \{t_1(a+d) + t_2(b-d) + t_3(c+d) \mid t_1, t_2, t_3 \in \mathbb{Z}\} \\ &= \mathbb{Z}\langle a+d, b-d, c+d\rangle. \end{aligned}$$

On the other hand,

$$d_2(A) = a - b - c + d, \quad d_2(B) = a + b + c + d, \quad d_2(C) = -a - b + c + d.$$

This gives,

$$d_2(A+B) = 2(a+d), \quad d_2(A+C) = -2(b-d), \quad d_2(B+C) = 2(c+d),$$

and

$$d_2(A+B+C) = (a+d) - (b-d) + (c+d).$$

Let us consider the quotient map

$$
\begin{aligned}
q : \mathbb{Z}\langle a+d, b-d, c+d\rangle &\to \mathbb{Z}_2\langle [a+d],[c+d]\rangle \\
(a+d) &\mapsto [a+d] \\
(b-d) &\mapsto [a+d]+[c+d] \\
(c+d) &\mapsto [c+d].
\end{aligned}
$$

Then, $\operatorname{im} d_2 \subset \ker q$. On the other hand, an element in $\ker q$ is

$$t_1(a+d)+t_2(b-d)+t_3(c+d), \qquad t_1+t_2, \quad t_2+t_3 \in 2\mathbb{Z}.$$

This means, t_1, t_2, t_3 must have same parity. If they are all even, clearly we can write them in terms of elements of $\operatorname{im} d_2$. If they are all odd, we have

$$
\begin{aligned}
&2(k_1+1)(a+d)+(2k_2+1)(b-d)+(2k_3+1)(c+d) \\
&= k_1 d_2(A+B) - k_2 d_2(A+C) + k_3 d_2(B+C) + (b-d) + (a+d) + (c+d) \\
&= k_1 d_2(A+B) - k_2 d_2(A+C) + k_3 d_2(B+C) \\
&\qquad +(b-d) + d_2(A+B+C) + (b-d) \\
&= k_1 d_2(A+B) - k_2 d_2(A+C) + k_3 d_2(B+C) + d_2(A+B+C) + 2(b-d) \\
&= k_1 d_2(A+B) - (k_2+1) d_2(A+C) + k_3 d_2(B+C) + d_2(A+B+C) \in \operatorname{im} d_2.
\end{aligned}
$$

Thus,

$$H_1(X) = \mathbb{Z}_2\langle [a+d],[c+d]\rangle = \mathbb{Z}_2^2.$$

Next, we compute

$$
\begin{aligned}
0 &= d_2(q_1A+q_2B+q_3C) \\
&= q_1(a-b-c+d) + q_2(a+b+c+d) + q_3(-a-b+c+d) \\
&= (q_1+q_2-q_3)a + (-q_1+q_2-q_3)b + (-q_1+q_2+q_3)c + (q_1+q_2+q_3)d.
\end{aligned}
$$

This gives,

$$q_1 = q_2+q_3, \quad q_2 = q_3+q_1, \quad q_3 = q_1+q_2, \quad q_1+q_2+q_3 = 0.$$

The last condition is superfluous since the first three gives $q_1+q_2+q_3 = 2(q_1+q_2+q_3) \Rightarrow q_1+q_2+q_3 = 0$. Also, $q_1-q_2 = q_2-q_1 \Rightarrow q_1 = q_2$, and similarly, $q_2 = q_3$. But then, $q_1 = q_2 = q_3 = 0$. Hence, we have $\ker d_2 = 0$, which implies $H_2(X) = 0$. Finally, $d_3\Delta = 0$, and hence $H_3(X) = 0$. Thus, we have

$$
H_k(X;\mathbb{Z}) = \begin{cases}
\mathbb{Z}, & k=0 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2, & k=1 \\
0, & k=2 \\
\mathbb{Z}, & k=3 \\
0, & \text{otherwise.}
\end{cases}
$$

Using UCT, we can compute $H_\star(X;\mathbb{Z}_2)$. Since $\mathbb{Z}_2$ is a field, cohomology is then dual to homology. We have,

$$H^k(X;\mathbb{Z}_2) = H_k(X;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & k = 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & k = 1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & k = 2 \\ \mathbb{Z}_2, & k = 3 \\ 0, & \text{otherwise.} \end{cases}$$

Q6. Given a finitely generated Abelian group G, define the *rank* $\mathrm{rk}(G)$ as the rank of the free part of G, i.e, if $G = \mathbb{Z}^k \oplus \text{torsion}$, then $\mathrm{rk}(G) = k$. Given a space X, such that $\oplus_{i \geq 0} H_i(X)$ is finitely generated, define the *Euler characteristic* as

$$e(X) := \sum_{i=0}^{\infty} (-1)^i \ \mathrm{rk} \ H_i(X).$$

Assume that X (resp. G) is such that $e(X)$ (resp. $\mathrm{rk}(G)$) is defined.

a. If $\mathbb{F}$ is a field of characteristic 0, show that $\dim_{\mathbb{F}} G \otimes \mathbb{F} = \mathrm{rk}\ G$.

b. Suppose $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated Abelian groups. Show that $\mathrm{rk}\ B = \mathrm{rk}\ A + \mathrm{rk}\ C$. **Hint:** $\mathrm{Tor}(G, \mathbb{Q}) = 0$ for any G.

c. If $\mathbb{F}$ is a field of characteristic p, show that $\dim_{\mathbb{F}} G \otimes \mathbb{F} = \mathrm{rk}(G) +$ the number of $\mathbb{Z}/n\mathbb{Z}$ summands of G with $p \mid n$.

d. Show that $e(X) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{F}} H_i(X;\mathbb{F})$, where $\mathbb{F}$ is any field.

e. Suppose M is a compact manifold of odd dimension. Show that $e(M) = 0$.

Solution :

a. Suppose $\mathbb{F}$ is a field of characteristic 0. Then, we have $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{F} = \mathbb{F}/n\mathbb{F} = \mathbb{F}/\mathbb{F} = 0$, as any n is invertible in $\mathbb{F}$. Then, writing $G = \mathbb{Z}^k \oplus \text{torsion}$, we have $G \otimes \mathbb{F} = \mathbb{Z}^k \otimes \mathbb{F} = (\mathbb{Z} \otimes \mathbb{F})^k = \mathbb{F}^k$. Thus, $\dim_{\mathbb{F}} G \otimes \mathbb{F} = \mathrm{rk} G$.

b. Since $\mathbb{Q}$ is a torsion-free field of characteristic 0, we have a short exact sequence

$$0 \to A \otimes \mathbb{Q} \to B \otimes \mathbb{Q} \to C \otimes \mathbb{Q} \to 0.$$

By the rank-nullity theorem, we have $\dim_{\mathbb{Q}} B \otimes \mathbb{Q} - \dim_{\mathbb{Q}} A \otimes \mathbb{Q} = \dim_{\mathbb{Q}} C \otimes \mathbb{Q}$. But then by the previous problem, $\mathrm{rk}\ B = \mathrm{rk}\ A + \mathrm{rk}\ C$.

c. Suppose $\mathbb{F}$ is a field of characterisic p. Recall that for any Abelian group A, we have $\mathbb{Z}/n\mathbb{Z} \otimes A = A/nA$. Then, for each summand $\mathbb{Z}/n\mathbb{Z}$, where $p \mid n$, we have $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{F} = \mathbb{F}/n\mathbb{F} = \mathbb{F}$, as $n = 0$ in $\mathbb{F}$. The claim then follows easily.

d. If $\mathbb{F}$ is a field of charateristic 0, then the Tor term in the UCT vanishes, and we get $H_i(X;\mathbb{F}) = H_i(X) \otimes \mathbb{F}$. The claim is then immediate since

$$e(X) = \sum_i (-1)^i \ \mathrm{rk} \ H_i(X) = \sum_i (-1)^i \dim_{\mathbb{F}} H_i(X) \otimes \mathbb{F} = \sum_i (-1)^i \dim_{\mathbb{F}} H_i(X;\mathbb{F}).$$

Let us assume $\mathbb{F}$ is a field of characteristic p. Denote, $a_i = \mathrm{rk}(H_i(X))$, and $b_i =$ the number of $\mathbb{Z}/n\mathbb{Z}$ summand of $H_i(X)$ such that $p \mid n$. Recall tha we have $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) = \{x \in A \mid nx = 0\}$ for any Abelian group A. In particular, if $p \mid n$, then $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{F}) = \mathbb{F}$ (as $n = 0$ in $\mathbb{F}$), and if $p \nmid n$, then $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{F}) = 0$ (as n is invertible in $\mathbb{F}$). Then, we have $\mathrm{Tor}(H_i(X), \mathbb{F}) = \mathbb{F}^{b_i}$. Now, the UCT gives

$$0 \to H_i(X) \otimes \mathbb{F} \to H_i(X; \mathbb{F}) \to \mathrm{Tor}(H_{i-1}(X), \mathbb{F}) \to 0.$$

Using the discussion above, we have the split exact sequences

$$0 \to \mathbb{F}^{a_i + b_i} \to H_i(X; \mathbb{F}) \to \mathbb{F}^{b_{i-1}} \to 0,$$

where we take $b_{-1} = 0$ for notational convenience. Now, we have

$$\dim_{\mathbb{F}} H_i(X; \mathbb{F}) = a_i + b_i + b_{i-1}.$$

Taking alternatig sum, we see that the b_i terms vanish. Thus, we have

$$\sum_i (-1)^i \dim_{\mathbb{F}} H_i(X; \mathbb{F}) = \sum_i (-1)^i a_i = \sum_i (-1)^i \text{ rk } H_i(X) = e(X).$$

e. Let us work with $\mathbb{Z}_2$-coefficients, so that M is $\mathbb{Z}_2$-orientable. Then, Poincare duality is applicable. In particular, $H_i(M; \mathbb{Z}_2) = H^{n-i}(M; \mathbb{Z}_2)$ for each $0 \leq i \leq n = \dim M$. Also, over a Field coefficient, cohomology is dual to homology. Thus, $H_i(M; \mathbb{Z}_2) = \mathrm{hom}(H_{n-i}(M; \mathbb{Z}_2), \mathbb{Z}_2) = H_{n-i}(M; \mathbb{Z}_2)$. Since $n = \dim M$ is odd, it follows that the terms in $e(M) = \sum_{i=0}^{n} (-1)^i \dim H_i(M; \mathbb{Z}_2)$ cancels pairwise. Hence, $e(M) = 0$.