End-semester Examination

Course: Algebraic Topology II (KSM4E02) **Instructor:** Aritra Bhowmick

Time: 2:00PM – 5:00PM, 7th May, 2026 **Total marks:** 45

Attempt any question. You can get maximum **40 marks**.

Fact: If M is a compact n-fold, then M is homotopy equivalent to a finite CW complex of dimension n.

Fact: If G is a finitely generated Abelian group, then G can be uniquely written as $G = \mathbb{Z}^k \oplus T$ for some $k \geq 0$, where T is the torsion subgroup defined as $T = \{g \in G \mid \operatorname{ord}(g) < \infty\}$. Moreover, T can be written as a finite direct sum of cyclic groups.

Fact: Tensor naturally distributes over direct sum : $A \otimes (\oplus_{i \in I} B_i) = \oplus_{i \in I} A \otimes B_i$

Fact: Given any Abelian group A, we have $\mathbb{Z}/n\mathbb{Z} \otimes A = A/nA$, where $nA = \{na \mid a \in A\}$, and $\operatorname{Tor}(\mathbb{Z}/n\mathbb{Z}, A) = {}_nA = \{a \in A \mid na = 0\}$.

You may also use the (co)homology groups of standard spaces like $S^n, \mathbb{RP}^n, \mathbb{CP}^n$ etc without computation. If you are using any UCT like theorem to compute something, mention why the $\operatorname{Tor}/\operatorname{Ext}$ term vanishes (if it does vanish). You may use any part of any question, whether attempted or not, in solving another question.

Q1. Fix R-modules A, B, M. Show that $\operatorname{Tor}_n^R(M, A \oplus B) = \operatorname{Tor}_n^R(M, A) \oplus \operatorname{Tor}_n^R(M, B)$ for all n. [5]

Q2. Compute the cohomology rings (with $\mathbb{Z}$-coefficients) of the following spaces. $[2\frac{1}{2} + 2\frac{1}{2} + 0 = 5]$

a. $\Sigma\mathbb{CP}^2$.

b. $S^3 \vee S^5$.

Are these two spaces homotopy equivalent?!

Q3. Suppose M is a compact, connected, oriented, 4-fold, without boundary. Show that the nontrivial (co)homology groups of M with $\mathbb{Z}$ coefficients are

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$H_k(M)$	$\mathbb{Z}$	$F_1 \oplus T$	$F_2 \oplus T$	F_1	$\mathbb{Z}$
$H^k(M)$	$\mathbb{Z}$	F_1	$F_2 \oplus T$	$F_1 \oplus T$	$\mathbb{Z}$

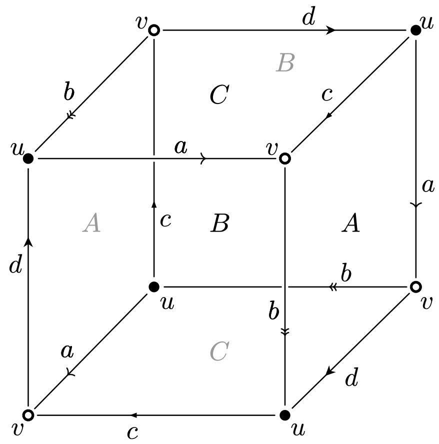
Here F_1, F_2 are some free Abelian groups, and T is an Abelian torsion group (i.e, every element of T has finite order). [5]

Q4. Suppose M is a compact, connected, oriented n-fold. $[2\frac{1}{2} \times 2 = 5]$

a. If the fundamental group of M is torsion (i.e, every element is of finite order), then show that $H_{n-1}(M) = 0$.

b. If $n = 2k$ and $H_{k-1}(M)$ is torsionfree (i.e, every nonzero element is of infinite order), then show that $H_k(M)$ is also torsionfree.

Q5. Consider the space X obtained from the cube in the following way: identify each face of the cube with the one opposite after twisting it by $90°$ in the right-handed corkscrew motion. We have the following cell structure on X.



Using cellular homology (or otherwise) show that

$$H_k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k=0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & k=1 \\ 0, & k=2 \\ \mathbb{Z}, & k=3 \\ 0, & \text{otherwise.} \end{cases}$$

Compute the cohomology groups $H^k(X;\mathbb{Z}_2)$. $[7+3=10]$

Q6. Given a finitely generated Abelian group G, define the *rank* $\mathrm{rk}(G)$ as the rank of the free part of G, i.e, if $G = \mathbb{Z}^k \oplus \text{torsion}$, then $\mathrm{rk}(G) = k$. Given a space X, such that $\oplus_{i\geq 0} H_i(X)$ is finitely generated, define the *Euler characteristic* as

$$e(X) := \sum_{i=0}^{\infty} (-1)^i \ \mathrm{rk}\ H_i(X).$$

Assume that X (resp. G) is such that $e(X)$ (resp. $\mathrm{rk}(G)$) is defined. $[3+3+2+4+3=15]$

a. If $\mathbb{F}$ is a field of characteristic 0, show that $\dim_{\mathbb{F}} G \otimes \mathbb{F} = \mathrm{rk}\ G$.

b. Suppose $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated Abelian groups. Show that $\mathrm{rk}\ B = \mathrm{rk}\ A + \mathrm{rk}\ C$. **Hint:** $\mathrm{Tor}(G, \mathbb{Q}) = 0$ for any G.

c. If $\mathbb{F}$ is a field of characteristic p, show that $\dim_{\mathbb{F}} G \otimes \mathbb{F} = \mathrm{rk}(G) +$ the number of $\mathbb{Z}/n\mathbb{Z}$ summands of G with $p \mid n$.

d. Show that $e(X) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{F}} H_i(X;\mathbb{F})$, where $\mathbb{F}$ is any field.

e. Suppose M is a compact manifold of odd dimension. Show that $e(M) = 0$.