Course notes for

Algebraic Topology II (KSM4E02)

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Day 1 : 13th January, 2026

`basic category theory — functors — chain complexes`

1.1 Categories and Functors

Category theory is the language of mathematics, it lets you to identify pattens among disparate topics. Although we shall see some definitions, they are not set in stone, and depending on the context you may need to assume extra structure.

Definition 1.1: (*Category*)

A *category* $\mathcal{C}$ consists of the following data.

1. A collection of objects, denoted $\mathrm{Ob}(\mathcal{C})$.
2. For any two objects $A, B \in \mathcal{C}$ a set $\hom_{\mathcal{C}}(A, B)$.
3. For any three objects $A, B, C \in \mathcal{C}$, a binary operation

$$\begin{aligned} \circ : \hom_{\mathcal{C}}(A, B) \times \hom_{\mathcal{C}}(B, C) &\to \hom_{\mathcal{C}}(A, C) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

which satisfies the following.

   a. **(Associativity)** For morphisms $f : A \to B, g : B \to C, h : C \to D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

   b. **(Identity)** For each object A, there is a morhphism $\mathrm{Id}_A : A \to A$, such that for $g : A \to B$ and any $h : C \to A$ we have

$$g \circ \mathrm{Id}_A = g, \quad \mathrm{Id}_A \circ h = h.$$

Remark 1.2:

Definition of a category is highly dependent on the context! One can add many adjectives to specify different tpyes of categories. The above should be called a *locally small* category, as we insists that $\hom_{\mathcal{C}}(A, B)$ is a **set**. When the class of object $\mathrm{Ob}(\mathcal{C})$ is also a set, we say the category is *small*. Such set theoretic issues crop up all over category theory. Recall Russell's paradox : can you define a *set* of all sets?!

Example 1.3: (*Some categories*)

Categories crop up all over mathematics (and other fields as well).

- $\mathbf{Sets}$: Category of sets and set functions.
- $\mathbf{Top}$: Category of topological spaces and continuous functions.
- $\mathbf{Top}_*$: Category of topological spaces X with a fixed point $*_X \in X$ (called the *basepoint*), and continuous maps $f : (X, *_X) \to (Y, *_Y)$, which are basepoint preserving, i.e., $f(*_X) = *_Y$.
- $\mathbf{Grp}$: Category of groups and group homomorphisms.
- $\mathbf{Ab}$: Category of Abelian groups and group homomorphisms.
- $\boldsymbol{R} - \mathbf{Mod}$: Given a ring R, category of (left) R-modules, and R-module maps.
- $\mathbf{Cat}$: Category of all locally small categories. Note that this category itself is not locally small, as the collection of all functors between two categories need not be a set. Thus, Cat maybe called a *large* category.
- $\mathbf{Kit}$: (not a standard notaion!) Category of all *small* categories (i.e, both objects and morphisms form a set) indeed gives rise to a locally small category.
- $\boldsymbol{\Delta}$: For each $n \geq 0$, denote $[n] := \{0, 1, \cdots, n\}$. A function $f : [m] \to [n]$ is called *non-decreasing* (or *order-preserving*) if $i < j \Rightarrow f(i) \leq f(j)$. The *simplicial category* Δ consists of $[n]$ for each $n \geq 0$ as objects, and $\hom_\Delta([m], [n])$ is the set of non-decreasing functions $[m] \to [n]$.

Exercise 1.4: (*Groups as Categories*)

Let G be group. Check that we have a natural (small) category with a single object say $*$, and G as the hom set $\hom(*, *)$.

Example 1.5: (*Discrete Category*)

Given any set X, we can consider a small category whose object set is X, and given any $x, y \in X$, there is no morphism if $x \neq y$. For $x = y$, the definition forces us to consider the identity morphism $1_x : x \to x$. This is called a *discrete category*. Any small discrete category is always obtained in this way.

Definition 1.6: (*Functor*)

Given two categories $\mathcal{C}, \mathcal{D}$, a *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ is the following data.

1. For each object $c \in \mathcal{C}$, there is an object $F(c) \in \mathcal{D}$.
2. For each morphism $f : c_1 \to c_2$ in $\mathcal{C}$, there is a morphism $F(f) : F(c_1) \to F(c_2)$ in $\mathcal{D}$, that satisfies the following.
   a. F preseves the identity, i.e., $F(\mathrm{Id}_c) = \mathrm{Id}_{\{F(c)\}}$ for any object $c \in \mathcal{C}$.
   b. F preseves the composition, i.e., given morphisms $f : A \to B, g : B \to C$ in $\mathcal{C}$ we have
   $$F(g \circ f) = F(g) \circ F(f).$$

We say $F : \mathcal{C} \to \mathcal{D}$ is a *contravariant functor* if it *reverses the morphisms and compositions*, i.e., given morphism $f : c_1 \to c_2$ in $\mathcal{C}$, we have the morphism $F(f) : F(c_2) \to F(c_1)$, and the composition is preseved as $F(g \circ f) = F(f) \circ F(g)$.

Example 1.7: (*Some functors*)

- Given any category, we can always consider the identity functor.
- Given two sets X, Y and a function $f : X \to Y$, we have a functor between the two discrete categories, and conversely.
- For any mathematical object defined as a set with some additional structure, lets us define a *forgetful functor* by forgetting the extra structure. Thus, we have functors $\mathbf{Top} \to \mathbf{Sets}$, $\mathbf{Grp} \to \mathbf{Sets}$ etc. Similarly, we have forgetful functors $\mathbf{Ab} \to \mathbf{Grp}$ and $\boldsymbol{R}\text{-}\mathbf{Mod} \to \mathbf{Ab}$.
- Recall Δ is the simplicial category, where $\mathrm{Ob}(\Delta) = \{[n] := \{0, 1, \cdots, n\} \mid n \geq 0\}$, and $\hom_\Delta([m], [n])$ is the set of order-preserving functions. A *contravariant* functor $K : \Delta \to \mathrm{Set}$ is known as a *simplicial set*. Simplicial sets are of fundamental importance in homotopy theory.
- A covariant functor $\Delta \to \mathrm{Set}$ is known as a *cosimplicial set*, which might be an unfortunate nomenclature!

Definition 1.8: (*Opposite Category*)

Given a category $\mathcal{C}$, the *opposite category*, denoted as $\mathcal{C}^{\mathrm{op}}$, is the category which has the following data.

- $\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Obj}(\mathrm{cal}(\mathrm{C})$.
- For any $A, B \in \mathcal{C}^{\mathrm{op}}$, we have $\hom_{\mathcal{C}^{\mathrm{op}}}(A, B) := \hom_{\mathcal{C}}(B, A)$.
- For $f \in \hom_{\mathcal{C}^{\mathrm{op}}}(A, B), g \in \hom_{\mathcal{C}^{\mathrm{op}}}(B, C)$, we have

$$g \circ_{\mathcal{C}^{\mathrm{op}}} f := f \circ_{\mathcal{C}} g \in \hom_{\mathcal{C}}(C, A) = \hom_{\mathcal{C}^{\mathrm{op}}}(A, C)$$

In other words, $\mathcal{C}^{\mathrm{op}}$ is obtained from $\mathcal{C}$ by reversing the arrows (i.e., morphisms).

Exercise 1.9: (*Group Homomorphism as Functor*)

Given a group homomorphism $f : G \to H$ interpret it as a functor between the two associated one-object categories as in Exercise 1.4. Is the converse true? That is, is any (covariant) functor between these categories induced by a group homomorphism?

Given any (locally small) category, one of the most important functors are the $\hom$ functors.

Definition 1.10: ($\hom$ *functors*)

Let $\mathcal{C}$ be a category, and fix an object $X \in \mathcal{C}$. Then, the *covariant* $\hom$*-functor* is the functor

$$\begin{aligned} \hom_{\mathcal{C}}(X, \_) : \mathcal{C} &\to \mathrm{Set} \\ Y &\mapsto \hom_{\mathcal{C}}(X, Y), \end{aligned}$$

and the *contravariant* $\hom$*-functor* is the contravariant functor

$$\begin{aligned}\hom_{\mathcal{C}}(\_, X) : \mathcal{C} &\to \mathrm{Set}\\ Y &\mapsto \hom_{\mathcal{C}}(Y, X).\end{aligned}$$

Often times the $\hom$ itself may have some extra structure, in which case the range of the functors can be a different category.

Example 1.11:

Let $\mathcal{C}$ be the category of vector spaces over some field $\Bbbk$, where morphisms are $\Bbbk$-linear maps. Then, $\hom_{\mathcal{C}}(V, W)$ itself is a vector space. Thus, the $\hom$-functors for some fixed vector space V can be realized as $\hom_{\mathcal{C}}(V, \_) : \mathcal{C} \to \mathcal{C}$, and $\hom_{\mathcal{C}}(\_, V) : \mathcal{C} \to \mathcal{C}$.

1.2 Category of Chain Complexes

One of the crucial interest in homological algebra is the category of chain complexes of Abelian groups (or more generally, R-modules).

Definition 1.12: (*Chain Complex*)

A *chain complex* of Abelian groups, is a collection $\{C_n\}_{n\in\mathbb{Z}}$ of Abelian groups, and a collection $\{\partial_n : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ of group homomorphisms, called boundary maps. We denote this as

$$(C_\bullet, \partial) : \cdots \to C_n \underset{\partial_n}{\longrightarrow} C_{n-1} \to \cdots,$$

such that $\partial_n \circ \partial_{n+1} = 0$. We say $(C^\bullet, \partial)$ is a *cochain complex* if we have the maps in the opposite direction, i.e.,

$$(C^\bullet, \partial) : \cdots \leftarrow C^{n+1} \underset{\partial^n}{\longleftarrow} C^n \leftarrow \cdots,$$

Although there is no explicit rules for indexing the boundary maps, it is standard to put the index of the source object as the index for the boundary map for both chain and cochain complexes.

Definition 1.13: (*Chain map*)

A *chain map* $f_\bullet : C_\bullet \to D_\bullet$ between two chain complexes $(C_\bullet, \partial_\bullet^C)$ and $(D_\bullet, \partial_\bullet^D)$ is a collection of homomorphisms $f_n : C_n \to D_n$ such that, we have a commutative diagram as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\partial_n^C"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_n \arrow[r, "\partial_n^D"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Explicitly, we have $f_{n-1} \circ \partial_n^C = \partial_n^D \circ f_n$. Similarly, one can define *cochain maps* of cochain complexes.

Remark 1.14:

If it is clear from the context, we often drop index from the ∂_n, f_n. In particular, the definition of chain map may also be understood as $\partial \circ f = f \circ \partial$. Moreover, we sometimes use the same ∂ to denote the boundary maps of different chain complexes, so that the notation is kept light!

Definition 1.15:

A (co)chain complex $(C_\bullet, \partial)$ is called

a. *bounded below* by some $n_0 \in \mathbb{Z}$ if $C_n = 0$ for all $n < n_0$,

b. *bounded above* by some $m_0 \in \mathbb{Z}$ if $C_n = 0$ for all $n > m_0$, and

c. *bounded* if it is both bounded below and above.

Definition 1.16: (*Category of Chain Complex*)

The *category of chain complexes* $\mathrm{Ch}(\mathrm{Ab})$ is the category whose objects are chain complexes, and morphisms are the chain maps.

Remark 1.17:

Depending on the requirement, we can restrict ourselves to bounded below, bounded above, or just bounded chain complexes. Among these, we are primarily interested in the $C_\bullet$ when $C_n = 0$ for $n < 0$, which we simply write as

$$C_\bullet : \cdots \to C_2 \to C_1 \to C_0 \to 0,$$

and $C^\bullet$ when $C^n = 0$ for $n < 0$, which we simply write as

$$C^\bullet : 0 \to C^0 \to C^1 \to C^2 \to \cdots$$

Definition 1.18: (*Graded Abelian Groups*)

A *graded Abelian group* is a collection $G_\bullet = \{G_n\}_{n\in\mathbb{Z}}$ of Abelian groups. A map $f_\bullet : G_\bullet \to H_\bullet$ between two such graded groups is a collection of group homomorphisms $\{f_n : G_n \to H_n\}_{n\in\mathbb{Z}}$

Exercise 1.19: (*Graded Groups are Chain Complex*)

Interpret graded group as a chain complex. Verify that a morphism of graded groups is precisely a chain map.

Day 2 : 16th January, 2026

`natural transformations — adjunction — Eilenberg-Steenrod axioms`

2.1 Natural Transformations

Definition 2.1: (*Natural transformation*)

Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta : F \Rightarrow G$ is a collection of morphisms $\eta_c : F(c) \to G(c)$ in $\mathcal{D}$, one for each $c \in \mathcal{C}$, such that given any morphism $f : X \to Y$ in $\mathcal{C}$, we have the following commutative diagram

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_Y"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

Exercise 2.2: (*Composition of Natural Transformations*)

Try to figure out two distinct ways compose natural transformations, namely, horizontally and vertically. The following diagram may help.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & \mathcal{D} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & \mathcal{E}
\end{tikzcd}$$

Try to determine the associativity of these compositions.

Definition 2.3: (*Natural Isomorphism*)

A natural transformation $\eta : F \Rightarrow G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is called a *natural isomorphism* if $\eta_c : F(c) \to G(c)$ is an isomorphism for all $c \in \mathcal{C}$.

Remark 2.4: (*Natural Transformation and Homotopy*)

You can imagine functors as maps between two spaces. Then, natural transformation can be interprated as homotopy between them! In fact this analogy can be made rigourous, and one can consider a level 3 morphism between two natural transformations, and so on and on to infinity.

Example 2.5:

Consider $\mathcal{C}$ to be the category of vector spaces over some field $\Bbbk$, with linear maps as morphisms. We have a functor $F : \mathcal{C} \to \mathcal{C}$ given by

$$F(V) = (V^*)^* = \hom_{\Bbbk}(\hom_{\Bbbk}(V, \Bbbk), \Bbbk).$$

Now, for each V, we have linear map given by evaluation

$$\begin{aligned} \eta_V : V &\to F(V) \\ \mathbf{v} &\mapsto (T \mapsto T(\mathbf{v})). \end{aligned}$$

Then, $\eta : \mathrm{Id} \Rightarrow F$ is a natural transformation (Check!).

2.2 Adjoint functors

Definition 2.6: (*Adjoint Functor*)

Let $\mathcal{C}, \mathcal{D}$ be two categories, and we have two fuctors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$. We say L is *left adjoint* to R, and R is *right adjoint* to L if there exists natural isomorphisms (i.e, set bijections)

$$\eta_{c,d} : \hom_{\mathcal{D}}(L(c), d) \to \hom_{\mathcal{C}}(c, R(d)), \quad c \in \mathcal{C}, d \in \mathcal{D}.$$

Here naturality means that given $c \in \mathcal{C}$ fixed, and any morphism $g : d_1 \to d_2$ in $\mathcal{D}$, we have a commutative diagram

$$\begin{tikzcd} \hom_{\mathcal{D}}(L(c), d_1) \arrow[r, "\eta_{c,d_1}"] \arrow[d, "\hom_{\mathcal{D}}(L(c), g)"'] & \hom_{\mathcal{C}}(c, R(d_1)) \arrow[d, "\hom_{\mathcal{C}}(c, R(g))"] \\ \hom_{\mathcal{D}}(L(c), d_2) \arrow[r, "\eta_{c,d_2}"'] & \hom_{\mathcal{C}}(c, R(d_2)) \end{tikzcd}$$

Similarly, given $d \in \mathcal{D}$ fixed, and any morphism $f : c_1 \to c_2$ in $\mathcal{C}$, we have a commutative diagram

$$\begin{tikzcd} \hom_{\mathcal{D}}(L(c_1), d) \arrow[r, "\eta_{c_1,d}"] & \hom_{\mathcal{C}}(c_1, R(d)) \\ \hom_{\mathcal{D}}(L(c_2), d) \arrow[r, "\eta_{c_2,d}"'] \arrow[u, "\hom_{\mathcal{D}}(L(f), d)"] & \hom_{\mathcal{C}}(c_2, R(d)) \arrow[u, "\hom_{\mathcal{C}}(f, R(d))"'] \end{tikzcd}$$

The pair of functors (L, R) is called an *adjunct pair*.

Exercise 2.7: (*Adjunction and Natural Isomorphisms*)

Let $L : \mathcal{C} \to \mathcal{D}, R : \mathcal{D} \to \mathcal{R}$ be a pair of functors. Show that (L, R) is an adjuct pair if and only if for each $c \in \mathcal{C}$ and each $d \in \mathcal{D}$ there are natural isomorphisms of functors

$$\hom_{\mathcal{D}}(L(c), \_) \Rightarrow \hom_{\mathcal{C}}(c, R(\_)), \quad \text{and} \quad \hom_{\mathcal{D}}(L(\_), d) \Rightarrow \hom_{\mathcal{C}}(\_, R(d)).$$

Exercise 2.8: (*Left and Right Adjoint of Forgetful Functor* $\mathrm{Top} \to \mathrm{Set}$)

Let $U : \mathrm{Top} \to \mathrm{Set}$ be the forgetful functor. Given a set X, we can either give it the discrete topology, or the indiscrete topology. This defines two functors $D, I : \mathrm{Set} \to \mathrm{Top}$. Show that D is left adjoint of U, and I is right adjoint of U.

Exercise 2.9: (*hom-tensor Adjunction*)

On the category Ab of Abelian groups, we have two functors for each $X \in \mathrm{Ab}$

$$\hom(X, \_) : \mathrm{Ab} \to \mathrm{Ab}, \quad \_ \otimes X : \mathrm{Ab} \to \mathrm{Ab}$$

Prove that the tensor is left adjoint to hom, i.e, for $X, Y, Z \in \mathrm{Ab}$ show that there exists natural isomorphism

$$\hom(X \otimes Y, Z) \to \hom(X, \hom(Y, Z)).$$

In fact, the isomorphism is an isomorphism of Abelian groups as well. Same statement holds true for R -modules as well.

2.3 Eilenberg-Steenrod Axioms of (co)Homology Theories

Consider the category $\mathrm{TopPair}$ whose objects are pairs of topological space (X, A) where $A \subset X$, and morphisms $f : (X, A) \to (Y, B)$ are continuous maps $f : X \to Y$, such that $f(A) \subset B$. For simplicity, we shall denote $X = (X, \emptyset)$. Given any pair (X, A), we have the following inclusion maps

$$\begin{tikzcd}
 & & (X, \emptyset) \arrow[dr, "j"] & & \\
(\emptyset, \emptyset) \arrow[r] & (A, \emptyset) \arrow[ur, "\iota"] \arrow[dr] & & (X, A) \arrow[r] & (X, X) \\
 & & (A, A) \arrow[ur] & &
\end{tikzcd}$$

For $I = [0, 1]$, we have $(X, A) \times I = (X \times I, A \times I)$. Two maps $f_0, f_1 : (X, A) \to (Y, B)$ are called homotopic if there is a map $h : (X \times A) \times I \to (Y, B)$ such that $h|_{(X,A)\times\{0\}} = f_0$ and $h|_{(X,A)\times\{1\}} = g$ holds. In particular, a homtopy of pairs restricts to a homotopy of $f_0|_A =$. Later on we shall put more restrictions on this category, one such possibility is to assume that (X, A) is a CW-pair, i.e, X is a CW complex, and A is a subcomplex.

Definition 2.10: (*Homology Theory*)

A *homology theory*, according to Eilenberg-Steenrod, is a collection, indexed by $\mathbb{Z}$, of

- functors $H_n : \mathrm{TopPair} \to \mathrm{Ab}$ (or, more generally, to category of R-mod), and

- natural transformations $\partial^n_{(X,A)} : H_n(X,A) \to H_{n-1}(A)$,

satisfying the following axioms. Without loss of generaligy, given $f : (X,A) \to (Y,B)$, we shall denote $f_* = H_n(f) : H_n(X,A) \to H_n(Y,B)$, provided the index n is understood from the context. Similarly, we shall denote $\partial = \partial^n_{(X,A)}$.

I. **Exactness Axiom** : Given any pair (X,A), there exists a long exact sequence of Abelian groups

$$\cdots \to H_{n+1}(X,A) \to H_n(A) \underset{\iota_\star}{\to} H_n(X) \underset{j_\star}{\to} H_n(X,A) \underset{\partial}{\to} H_{n-1}(A) \to \cdots$$

Here $\iota : A \hookrightarrow X$ and $j : X \hookrightarrow (X,A)$ are the inclusions.

II. **Homotopy Axiom** : Homotopic maps induce the same map in homology. That is, given two homotopic maps $f, g : (X,A) \to (Y,B)$, we have $f_\star = g_\star : H_n(X,A) \to H_n(Y,B)$.

III. **Excision Axiom** : Given $(X,A) \in \mathrm{TopPair}$ and $U \subset A$ satisfying $\overline{U} \subset \mathring{A}$, the inclusion map $\iota : (X \setminus U, A \setminus U) \hookrightarrow (X,A)$ induces isomoephism (called *excision isomorphism*) in all homology groups.

IV. **Dimension Axiom** : For the one-point space $P = \{\star\}$, we have $H_n(P) = 0$ for $n \neq 0$.

Remark 2.11: (*Extraodinary Homology Theory*)

A homology theory without the dimension axiom is called *extraodinary homology theory*. Topological K-theory is an example of extraodinary homology theory.

Remark 2.12: (*Excision*)

In their original work, Eilenberg-Steenrod assumed a weaker version of the excision axiom, namely they only considered *open* sets $U \subset X$ such that $\overline{U} \subset \mathring{A}$ holds. One of the most common homology theory (i.e, the singular homology theory) satisfies the stronger excision axiom where *any* subset $U \subset X$ with $\overline{U} \subset \mathring{A}$ can be excised out. We shall see that any (ordinary) homology theory defined on reasonable pairs (e.g. CW-paris) is isomorphic to the singular homology, and thus, satisfies the stronger excision axiom.

Exercise 2.13: (*Two Excision Axioms Can Be Equivalent*)

Consider $\mathcal{C}$ to be the category of pairs (X,A) of *compact* topological spaces, and continuous maps between them. Justify that the two excision axioms must be equivalent for this category.

Exercise 2.14:

Let $f : (X,A) \to (Y,B)$ be map. From the axiom, show that we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X,A) \arrow[r] \arrow[d] & H_n(A) \arrow[r] \arrow[d] & H_n(X) \arrow[r] \arrow[d] & H_n(X,A) \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n+1}(X,A) \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & H_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical arrows are induced by f. That is, the homology long exact sequence is natural.

Day 3 : 20th January, 2026

`additivity in homology theory – homology of homtopy equivalent spaces – homology long exact sequence of triple`

3.1 Additivity in Homology Theory

Proposition 3.1: (*Finite Additivity*)

Suppose $X = \sqcup_{j=1}^{k} X_j$ is a finite disjoint union of spaces. Then, the inclusion maps $\iota_j : X_j \hookrightarrow X$ induces isomorphism of $H_n(X)$ onto a direct summand of $H_n(X_j)$, and $H_n(X)$ is naturally isomorphic to $\oplus_{j=1}^{n} H_n(X_j)$ for each n.

Proof : We only conider the case $X \sqcup Y$, the general case follows by induction. Note that $\varphi_X : X \hookrightarrow (X \sqcup Y, Y)$ and $\varphi_Y : Y \hookrightarrow (X \sqcup Y, X)$ induces excision isomorphism in homology. Thus, we have the commuting diagrams

$$\begin{tikzcd} H_n(X) \arrow[r, hook, "(\iota_X)_\star"] \arrow[rr, bend right, "(\varphi_X)_\star"'] & H_n(X \sqcup Y) \arrow[r, two heads, "(j_X)_\star"] & H_n(X \sqcup Y, Y) \end{tikzcd} \qquad \begin{tikzcd} H_n(Y) \arrow[r, hook, "(\iota_Y)_\star"] \arrow[rr, bend right, "(\varphi_Y)_\star"'] & H_n(X \sqcup Y) \arrow[r, two heads, "(j_Y)_\star"] & H_n(X \sqcup Y, X) \end{tikzcd}$$

Then, from the long exact sequence for the pair $(X \sqcup Y, X)$, we get the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_n(X) \arrow[r] & H_n(X \sqcup Y) \arrow[r] & H_n(X \sqcup Y, X) \arrow[l, bend right, "(\iota_X)_\star \circ (\varphi_X)_\star^{-1}"'] \arrow[r] & 0. \end{tikzcd}$$

Thus, $(\iota_X)_\star$ maps $H_n(X)$ isomorphically onto a direct summand of $H_n(X \sqcup Y)$. Similarly, $(\iota_Y)_\star$ maps $H_n(Y)$ isomorphically onto a direct summand of $H_n(X \sqcup Y)$ as well. Finally, by the exicision isomorphism, we have $H_n(X \sqcup Y) \cong H_n(X) \oplus H_n(X \sqcup Y, X) \cong H_n(X) \oplus H_n(Y)$. $\square$

Exercise 3.2: (*Finite Union of Pairs*)

Let (X_j, A_j) be pairs of spaces for $1 \leq i \leq k$. Denote $(X, A) = (\sqcup X_i, \sqcup A_i)$. Show that $\iota_j : (X_j, A_j) \hookrightarrow (X, A)$ induces an isomorphism of $H_n(X_j, A_j)$ onto a direct summand of $H_n(X, A)$, and moreover, $H_n(X, A)$ is naturally isomorphic to the direct sum $\oplus_{j=1}^{k} H_n(X_j, A_j)$.

Hint : Use Proposition 3.1 and the naturality of the long exact sequence (Exercise 2.14).

The finite additivity of homology theory cannot be improved to arbitrary sum, and there are examples of homology theories which does not split for arbitrary union of spaces. On the other hand, we shall see later that singular homology abides by this. Hence, Milnor added one extra axiom.

Definition 3.3: (*Additive Homology Theory*)

A homology theory is called an *additive homology theory* if given a collection of spaces $\{X_\alpha\}_{\alpha\in I}$, the inclusion maps $\iota_\alpha : X_\alpha \hookrightarrow X := \sqcup X_\alpha$ induces an isomorphism of $H_n(X_\alpha)$ onto a direct summand of $H_n(X)$, and moreover, $H_n(X)$ is naturally isomorphic to $\oplus H_n(X_\alpha)$.

3.2 Homology of Homotopy Equivalent Spaces

Recall, $f : X \to Y$ is a homotopy equivalence if there is a map $g : Y \to X$ such that

$$g \circ f \simeq \mathrm{Id}_X, \quad f \circ g \simeq \mathrm{Id}_Y.$$

One can similarly define homotopy equivalence $f : (X, A) \to (Y, B)$ for pairs, which essentially requires that that the homotopy restricts to a homotopy of the restricted map.

Proposition 3.4: (*Homotopy Equivalence induces Homology Isomorphism*)

Let $f : X \to Y$ be a homotopy equivalence. Then, $f_\star : H_n(X) \to H_n(Y)$ is an isomorphism for all n. Similarly, a homotopy equivalence $(X, A) \to (Y, B)$ also induces isomorphism at the homology groups.

Proof : Let $g : Y \to X$ be a homotopy inverse. Now, from the functoriality and the homotopy invariance, we have

$$\mathrm{Id} = H_n(\mathrm{Id}_X) = H_n(g \circ f) = H_n(g) \circ H_n(f) = g_\star \circ f_\star,$$

and similarly, we have $f_\star \circ g_\star = \mathrm{Id}$. Hence, $f_\star : H_n(X) \to H_n(Y)$ is an isomorphism, with inverse $g_\star$. Similar argument holds for $(X, A) \simeq (Y, B)$ as well. □

Corollary 3.5: (*Homeomorphism induces Homology Isomorphism*)

Let $f : X \to Y$ be a homeomorphism of spaces. Then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism. Similarly, homeomorphism $(X, A) \cong (Y, B)$ induces homology isomorphism as well.

3.3 Homology Long Exact Sequence of Triples

Let us consider a triple (X, A, B) of spaces where $B \subset A \subset X$. Then, we have inclusions

$$\iota : (A, B) \hookrightarrow (X, B), \quad j : (X, B) \hookrightarrow (X, A).$$

A map $f : (X, A, B) \to (Y, C, D)$ of triples is a continuous map $f : X \to Y$ such that $f|_A : A \to C$ and $f|_B : B \to D$ holds. A triple leads to a natural long exact sequence of homology groups. We shall need the following.

Lemma 3.6: (*Relative Homology of Space w.r.t. itself*)

For any space X, we have $H_n(X, X) = 0$.

Proof : By the exicision isomorphism, it follows that $H_n(X, X) \cong H_n(\emptyset, \emptyset) = H_n(\emptyset)$. Now, for the pair $(\emptyset, \emptyset)$, we have long exact sequence $\cdots \to H_n(\emptyset) \underset{\iota_\star}{\to} H_n(\emptyset) \underset{j_\star}{\to} H_n(\emptyset) \to \cdots$. Since $\iota = \mathrm{Id} = j$, we have

$i_\star = \mathrm{Id} = j_\star$. But by exactness, we have $j_\star \circ \iota_\star = 0$. Hence, we have $\iota_* = 0 = j_\star$, and consequently, $H_n(\emptyset) = 0$. Thus, $H_n(X, X) = 0$ for any space X. $\square$

Exercise 3.7:

Give a proof of $H_n(X, X) = 0$ without using excision.

Hint : Use the long exact sequence of the pair (X, X).

Theorem 3.8: (*Homology Long Exact Sequence of Triples*)

Let (X, A, B) be a triple with $B \subset A \subset X$. Then, there exists a natural long exact sequence

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A, B) \xrightarrow{\iota_\star} H_n(X, B) \xrightarrow{j_\star} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B) \cdots,$$

which is natural with respect to maps of triples. The boundary map is given as the composition

$$\partial : H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{j_\star} H_{n-1}(A, B),$$

where the first boundary is from the long exact sequence of the pair (X, A), and the second map is induced by the inclusion $A \hookrightarrow (A, B)$.

Proof : The proof boils down to checking exactness at each point, using the respective long exact sequences associated to the pairs (A, B), (X, B), and (X, A). Let us color code the arrows as follows.

$$\cdots \longrightarrow H_{n+1}(A, B) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(A, B) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_{n+1}(X, B) \longrightarrow H_n(X) \longrightarrow H_n(B) \longrightarrow H_n(X, B) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_{n+1}(X, A) \longrightarrow H_n(X) \longrightarrow H_n(A) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

Now, we have the following commutative diagram.

$$\begin{tikzcd}
H_n(A) \arrow[r] \arrow[d] & H_n(X) \arrow[d] \arrow[dr, bend left] & & \\
H_n(A,B) \arrow[r, "\iota_\star"] \arrow[dr, bend right] & H_n(X,B) \arrow[r, "j_\star"] \arrow[d] & H_n(X,A) \arrow[d] \arrow[dr, bend left, "\partial"] & \\
& H_{n-1}(B) \arrow[r] \arrow[dr, bend right] & H_{n-1}(A) \arrow[r] \arrow[d] & H_{n-1}(A,B) \arrow[d] \\
& & H_{n-1}(X) \arrow[r] & H_{n-1}(X,B)
\end{tikzcd}$$

The horizontal arrows are induced by the inclusions $(A, B) \hookrightarrow (X, B) \hookrightarrow (X, A)$, and the commutativity of each squeare follows from the naturality of the long exact sequence of pairs. Let us now check the exactness at each position.

1. $\ker j_\star \supset \operatorname{im} \iota_\star$: We need to show $j_\star \circ \iota_\star = 0$. We have a commuting diagram of spaces

$$\begin{tikzcd}
(A,B) \arrow[r, "\iota"] \arrow[dr, bend right] & (X,B) \arrow[r, "j"] & (X,A) \\
& (A,A) \arrow[ur, bend right] &
\end{tikzcd}$$

Since $H_n(A,A) = 0$ (Lemma 3.6), by functoriality, we have $j_\star \circ \iota_\star = 0$.

2. $\mathbf{ker}\, \boldsymbol{j_\star} \subset \mathbf{im}\, \boldsymbol{\iota_\star}$: We look at the following diragram

$$\begin{tikzcd}
\overset{w}{H_n(A)} \arrow[r, "\iota_\star"] \arrow[d, "j_\star"'] & \overset{z}{H_n(X)} \arrow[d, "j_\star"'] & \\
\overset{y}{H_n(A,B)} \arrow[r, "\iota_\star"] \arrow[dr, bend right, "\partial"'] & \overset{x}{H_n(X,B)} \arrow[r, "j_\star"] \arrow[d, "\partial"'] & H_n(X,A) \arrow[d, "\partial"] \\
& H_{n-1}(B) \arrow[r, "\iota_\star"'] & H_{n-1}(A)
\end{tikzcd}$$

Suppose $j_\star(x) = 0$ for some $x \in H_n(X,B)$. Then,

$$\iota_\star \partial(x) = \partial j_\star x = 0 \Rightarrow \partial(x) \in \ker(\iota_\star) = \operatorname{im}(\partial).$$

So, there exists some $y \in H_n(A,B)$ such that

$$\partial x = \partial y = \partial \iota_\star(y) \Rightarrow \partial(x - \iota_\star(y)) = 0 \Rightarrow x - \iota_\star(y) \in \ker \partial = \operatorname{im} j_\star.$$

So, there exists some $z \in H_n(X)$ such that

$$j_\star(z) = x - \iota_\star(y) \Rightarrow j_\star(z) = j_\star(x - \iota_\star(y)) = j_\star(x) - j_\star \iota_\star(x) = 0 \Rightarrow z \in \ker j_\star = \operatorname{im} \iota_\star.$$

So, there exists some $w \in H_n(A)$ such that $\iota_\star(w) = z$. Define $y_1 = y + j_\star(w) \in H_n(A,B)$. Then,

$$\iota_\star(y_1) = \iota_\star(y) + j_\star \iota_\star(w) = \iota_\star(y) + j_\star(z) = \iota_\star(y) + x - \iota_\star(y) = x.$$

This proves the claim.

3. $\mathbf{ker}\, \boldsymbol{\partial} \supset \mathbf{im}\, \boldsymbol{j_\star}$: We only need to show $\partial \circ j_\star = 0$. We have the following diagram.

$$\begin{tikzcd}
H_n(X,B) \arrow[r, "j_\star"] \arrow[d] & H_n(X,A) \arrow[d] \arrow[dd, bend left=60, "\partial"] \\
H_{n-1}(B) \arrow[r] \arrow[dr, bend right, "0"'] & H_{n-1}(A) \arrow[d] \\
& H_{n-1}(A,B)
\end{tikzcd}$$

The 0 map is a consequence of the long exact sequence of the pair (A,B). Then, chasing the diagram, it follows that $\partial \circ j_\star = 0$.

4. $\mathbf{ker}\, \boldsymbol{\partial} \subset \mathbf{im}\, \boldsymbol{j_\star}$: We look at the following diagram.

$$\begin{tikzcd}
\overset{w}{H_n(X)} \arrow[d, "j_\star"'] \arrow[dr, bend left, "j_\star"] & & \\
\overset{z}{H_n(X,B)} \arrow[r, "j_\star"] \arrow[d, "\partial"'] & \overset{x}{H_n(X,A)} \arrow[d, "\partial"] \arrow[dr, bend left, "\partial"] & \\
\overset{y}{H_{n-1}(B)} \arrow[r, "\iota_\star"] \arrow[dr, bend right, "\iota_\star"'] & H_{n-1}(A) \arrow[r, "j_\star"] \arrow[d, "\iota_\star"] & H_{n-1}(A,B) \\
& H_{n-1}(X) &
\end{tikzcd}$$

Suppose $\partial(x) = 0$ for some $x \in H_n(X,A)$. Now,

$$j_\star(\partial(x)) = 0 \Rightarrow \partial(x) \in \ker(j_\star) = \operatorname{im}(\iota_\star).$$

So, there exists some $y \in H_{n-1}(B)$ such that $\iota_\star(y) = \partial(x)$. Since $B \hookrightarrow A \hookrightarrow X$, we get

$$\iota_\star(y) = \iota_\star(\iota_\star(y)) = \iota_\star(\partial(x)) = 0 \Rightarrow y \in \ker(\iota_\star) = \operatorname{im}(\partial).$$

So, there exists some $z \in H_n(X,B)$ such that $\partial(z) = y$. Now,

$$\partial(x - j_\star(z)) = \partial(x) - \iota_\star\partial(z) = \partial(x) - \iota_\star(y) = 0 \Rightarrow x - j_\star(z) \in \ker(\partial) = \operatorname{im}(j_\star).$$

So, there exists some $w \in H_n(X)$ such that $j_\star(w) = x - j_\star(z)$. Define, $z_1 = z + j_\star(w) \in H_n(X,B)$. Then, we have

$$j_\star(z_1) = j_\star(z) + j_\star(w) = j_\star(z) + x - j_\star(z) = x.$$

This proves the claim.

5. $\mathbf{ker}\,\boldsymbol{\iota_\star} \supset \mathbf{im}\,\boldsymbol{\partial}$: We only need to show $\iota_\star \circ \partial = 0$. We have the following diagram.

$$\begin{tikzcd}
H_n(X,A) \arrow[d] \arrow[dr, bend left, "0"] \arrow[dd, bend right=60, "\partial"'] & \\
H_{n-1}(A) \arrow[r] \arrow[d] & H_{n-1}(X) \arrow[d] \\
H_{n-1}(A,B) \arrow[r, "\iota_\star"'] & H_{n-1}(X,B)
\end{tikzcd}$$

The 0 map is a consequence of the long exact sequence of the pair (X,A). Then, chasing the diagram, it follows that $\iota_\star \circ \partial = 0$.

6. $\mathbf{ker}\,\boldsymbol{\iota_\star} \subset \mathbf{im}\,\boldsymbol{\partial}$: We look at the following diagram.

$$
\begin{tikzcd}
 & \overset{z}{H_{n-1}(B)} \arrow[d, "\iota_\star"'] \arrow[dr, bend left, "\iota_\star"] & & \\
\overset{w}{H_n(X,A)} \arrow[r, "\partial"] \arrow[dr, bend right, "\partial"'] & \overset{y}{H_{n-1}(A)} \arrow[r, "\iota_\star"] \arrow[d, "j_\star"'] & H_{n-1}(X) \arrow[d, "j_\star"'] \\
 & \overset{x}{H_{n-1}(A,B)} \arrow[r, "\iota_\star"] \arrow[dr, bend right, "\partial"'] & H_{n-1}(X,B) \arrow[d, "\partial"'] \\
 & & H_{n-2}(B)
\end{tikzcd}
$$

Suppose for some $x \in H_{n-1}(A, B)$ we have $\iota_\star(x) = 0$. Then,

$$\partial(x) = \partial(\iota_\star(x)) = 0 \Rightarrow x \in \ker(\partial) = \operatorname{im}(j_\star).$$

So, there exists some $y \in H_{n-1}(A)$ such that $j_\star(y) = x$. Now,

$$j_\star(\iota_\star(y)) = \iota_\star(j_\star(y)) = \iota_\star(x) = 0 \Rightarrow \iota_\star(y) \in \ker(j_\star) = \operatorname{im}(\iota_\star).$$

So, there exists some $z \in H_{n-1}(B)$ such that $\iota_\star(z) = \iota_\star(y)$. As $B \hookrightarrow A \hookrightarrow X$, we have

$$\iota_\star(y - \iota_\star(z)) = \iota_\star(y) - \iota_\star(z) = 0 \Rightarrow y - \iota_\star(z) \in \ker(\iota_\star) = \operatorname{im}(\partial).$$

So, there exists some $w \in H_n(X, A)$ such that $\partial(w) = y - \iota_\star(z)$. We then have,

$$\partial(w) = j_\star(\partial(w)) = j_\star(y - \iota_\star(z)) = j_\star(y) - 0 = x.$$

This proves the claim.

Hence, we have proved that the sequence is exact at all points. Since all the maps involved are natural, one can *easily* show that the sequence is natural with respect to maps of triple as well (Check!). $\square$

Day 4 : 23rd January, 2026

`reduced homology — triads — long exact sequence of proper triads`

4.1 Reduced Homology

Given a space X, we have the unique map $f : X \to \{\star\}$. If X is nonempty, fixing a point $x_0 \in X$, we have a continuous map $r : X \to P := \{x_0\}$. Then, P is a retract of X. Hence, by functoriality, we see $H_n(P)$ is a direct summand of $H_n(X)$ for all n. Let us write $H_n(X) = \tilde{H}_n(X) \oplus H_n(P)$. By the dimension axiom, for all $n \neq 0$, it follows that $\tilde{H}_n(X) = H_n(X)$.

Definition 4.1: (*Reduced Homology*)

Given a space X, its *reduced homology* groups are defined as $\tilde{H}_n(X) := \ker(H_n(f) : H_n(X) \to H_n(\star))$, for $f : X \to \{\star\}$ the unique constant map. If X is nonempty, we have $H_n(X) \cong \tilde{H}_n(X) \oplus H_n(\star)$.

Corollary 4.2: (*Reduced Homology of a Point*)

We have $\tilde{H}_n(X) = H_n(X)$ for all $n \neq 0$, and $\tilde{H}_n(\star) = 0$ for all n.

Lemma 4.3:

Let $f : X \to Y$ be a map. Then, we have an induced map $f_\star : \tilde{H}_0(X) \to \tilde{H}_0(Y)$.

Proof : We have a diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \tilde{H}_0(X) \arrow[r, hook] \arrow[d, dashed] & H_0(X) \arrow[r, two heads] \arrow[d, "H_0(f)"'] & H_0(\star) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \tilde{H}_0(Y) \arrow[r, hook] & H_0(Y) \arrow[r, two heads] & H_0(\star) \arrow[r] & 0
\end{tikzcd}$$

Note that for any $x \in \tilde{H}_0(X)$, we have $H_0(f)(x) \in \ker(H_0(Y) \to H_0(\star)) = \tilde{H}_0(Y)$. Hence, $H_0(f)$ restricts to a map $\tilde{H}_0(X) \to \tilde{H}_0(Y)$. $\square$

Theorem 4.4: (*Long Exact Sequence of Reduced Homology Groups*)

Given a pair (X, A), we have the long exact sequence

$$\cdots \to H_{n+1}(X, A) \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X, A) \to \cdots$$

Proof : We only need to consider the case $n = 0$, i.e, we need to prove the exactness of the part

$$\cdots \to H_1(X, A) \to \tilde{H}_0(A) \to \tilde{H}_0(X) \to H_0(X, A) \to \cdots$$

Let us consider the long exact sequence for the pair $(\star, \star)$. Naturality gives the following commutative diagram of long exact sequences.

$$\begin{tikzcd}
 & & \tilde{H}_0(A) \arrow[r] & \tilde{H}_0(X) \arrow[dr, bend left] & & \\
\cdots \arrow[r] & H_1(X,A) \arrow[r] \arrow[ur, bend left] \arrow[d] & H_0(A) \arrow[r] \arrow[d] & H_0(X) \arrow[r] \arrow[d] & H_0(X,A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \underbrace{H_1(\star,\star)}_{0} \arrow[r] & H_0(\star) \arrow[r] & H_0(\star) \arrow[r] & \underbrace{H_0(\star,\star)}_{0} \arrow[r] & \cdots
\end{tikzcd}$$

Since $\tilde{H}_0(A)$ and $\tilde{H}_0(X)$ is defined as the kernel of the vertical maps, we get the induced blue arrows, and mroreover, the exactness for the reduced homology groups follows from the commutativity of the two rows (Check!). □

Definition 4.5: (*Homologically Trivial Space*)

A space X is called *homologically trivial* if $H_n(X) = 0$ for all $n \neq 0$, and $\tilde{H}_0(X) = 0$ (or in other words, $\tilde{H}_n(X) = 0$ for all n). For $A \neq \emptyset$, the pair (X, A) is called homologically trivial if $H_n(X, A) = 0$ for all n.

Exercise 4.6:

Suppose $A \neq \emptyset$, and (X, A) is homologically trivial. Show that $\iota : A \hookrightarrow X$ induces isomorphism of both unreduced and reduced homology groups. Conversely, if $\iota_\star$ is an isomorphism (for either unreduced or the reduced homology groups), then show that (X, A) is homologically trivial.

4.2 Homology Exact Sequence of Triads

Definition 4.7: (*Proper Triad*)

A *triad* is a triple $(X; X_1, X_2)$, where $X_1, X_2 \subset X$ are subspaces. We say $(X; X_1, X_2)$ is a *proper triad* if the inclusion maps

$$\kappa_1 : (X_2, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_1), \quad \kappa_2 : (X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2)$$

induce isomorphisms of homology groups.

Note that $(X; X_1, X_2)$ and $(X; X_2, X_1)$ are distinct triads, but one of them is proper if and only if the other one is as well. A triad maybe proper with resepect to one homology theory, but fails to be proper with respect to another.

Exercise 4.8:

Suppose $X_1, X_2 \subset X$ are closed subsets, such that $X = X_1 \cup X_2$, and $\overline{X_1 \setminus (X_1 \cap X_2)} \cap \overline{X_2 \setminus (X_1 \cap X_2)} = \emptyset$. Then, the triad $(X; X_1, X_2)$ is a proper triad for any homology theory.

<u>Hint</u>: Check that the inclusion maps κ_1, κ_2 are excision maps.

Exercise 4.9: (*Triad and Triple*)

Suppose (X, A, B) is a triple, i.e, $B \subset A \subset X$. Then, show that $(X; A, B)$ is a proper triad.

> **Hint :** Recall Lemma 3.6.

Lemma 4.10: (*Direct Sum Decomposition*)

Consider the commutative diagram of groups.

$$\begin{tikzcd}
C_1 & & & & C_2 \\
& & B \arrow[llu, "j_1"'] \arrow[rru, "j_2"] & & \\
A_2 \arrow[uu, "k_1"] \arrow[rru, "\iota_2"'] & & & & A_1 \arrow[uu, "k_2"'] \arrow[llu, "\iota_1"]
\end{tikzcd}$$

Suppose, $\ker(j_\alpha) = \operatorname{im}(\iota_\alpha)$ for $\alpha = 1, 2$. Then, the following are equivalent.

1. k_1, k_2 are isomorphisms
2. $0 \to A_i \to B \to C_i \to$ is exact for $i = 1, 2$. Moreover, $B = \operatorname{im}(\iota_1) \oplus \operatorname{im}(\iota_2)$ is a direct sum.

Proof : Suppose k_1, k_2 are isomorphisms. Commutativity implies that j_1, j_2 are epic, and ι_1, ι_2 are monic. Thus, $0 \to A_i \to B \to C_i \to$ is exact for $i = 1, 2$. Note that $\ker(j_1) \cap \ker(j_2) = \ker(j_1) \cap \operatorname{im}(\iota_2) = 0$, since otherwise k_1 will fail to be iso. Let us prove the direct sum decomposition. For any $b \in B$, consider

$$a_2 = (k_1)^{-1} j_1(b) \in A_2, \quad a_1 = (k_2)^{-1} j_2(b) \in A_1.$$

Set $b' = \iota_1(a_1) + \iota_2(a_2)$. Now, $j_1(b') = j_1(b)$ and $j_2(b') = j_2(b)$. Thus, $b - b' \in \ker(j_1) \cap \ker(j_2) = 0$, which means $b = b'$. Thus, every $b \in B$ can be written as a sum from $\operatorname{im}(\iota_1) + \operatorname{im}(\iota_2)$. On the other hand, $\operatorname{im}(\iota_1) \cap \operatorname{im}(\iota_2) = \ker(j_1) \cap \ker(j_2) = 0$. Hence, we have direct sum decomposition. $\square$

Exercise 4.11: (*Finite Additivity*)

Prove Proposition 3.1 for relative homology directly using Lemma 4.10.

> **Hint :** Given (X_i, A_i), consider $(X = X_1 \sqcup X_2, A = A_1 \sqcup A_2)$. We have a diagram of spaces.

$$\begin{tikzcd}
& (X, X_2 \sqcup A) & & (X, X_1 \sqcup A_2) & \\
& & (X, A) \arrow[lu] \arrow[ru] & & \\
(X_1, A_1) \arrow[r] & (X_1 \sqcup A_2, A) \arrow[uu] \arrow[ru] & & (X_2 \sqcup A, A) \arrow[uu] \arrow[lu] & (X_2, A_2) \arrow[l]
\end{tikzcd}$$

Theorem 4.12:

A triad $(X; X_1, X_2)$ is proper if and only if $\iota_\alpha : (X_\alpha, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_1 \cap X_2)$ induces a monomorphism, and gives a direct sum decomposition of $H_\star(X_1 \cup X_2, X_1 \cap X_2)$.

Proof : At the space level, we have the following diagram.

$$\begin{tikzcd}
(X_1 \cup X_2, X_1) & & & & (X_1 \cup X_2, X_2) \\
& & (X_1 \cup X_2, X_1 \cap X_2) \arrow[ull, "j_1"'] \arrow[urr, "j_2"] & & \\
(X_2, X_1 \cap X_2) \arrow[uu, "\kappa_1"] \arrow[urr, "\iota_2"'] & & & & (X_1, X_1 \cap X_2) \arrow[uu, "\kappa_2"'] \arrow[ull, "\iota_1"]
\end{tikzcd}$$

We have the triples $(X_1 \cup X_2, X_1, X_1 \cap X_2)$ and $(X_1 \cup X_2, X_2, X_1 \cap X_2)$. From Theorem 3.8, we we have $\ker (j_\alpha)_\star = \operatorname{im} (\iota_\alpha)_\star$ for $\alpha = 1, 2$. We conclude the proof from Lemma 4.10. □

Definition 4.13: (*Boundary Operator of Triad*)

Given a proper triad $(X; X_1, X_2)$, the *boundary operator* is defined as the composition

$$\partial : H_n(X, X_1 \cup X_2) \xrightarrow{\partial} H_{n-1}(X_1 \cup X_2) \longrightarrow H_{n-1}(X_1 \cup X_2, X_2) \xrightarrow{(\kappa_2)_\star^{-1}} H_n(X_1, X_1 \cap X_2).$$

It should be noted that this boundary map is the boundary map of the triple $(X, X_1 \cup X_2, X_1)$, followed by (the inverse of) an excision isomorphism.

Theorem 4.14: (*Homology Long Exact Sequence of Proper Triad*)

Given a proper triad $(X; X_1, X_2)$ we have a long exact sequence of homology groups

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X, X_1 \cup X_2) \arrow[d, "\partial"'] & & \\
& H_n(X_1, X_1 \cap X_2) \arrow[r] & H_n(X, X_2) \arrow[r] & H_n(X, X_1 \cup X_2) \arrow[d, "\partial"'] \\
& & & H_{n-1}(X_1, X_1 \cap X_2) \arrow[r] & \cdots
\end{tikzcd}$$

Moreove, the sequence is natural with respect to map of triads.

Proof : We have the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X, X_2) \arrow[r] & H_{n+1}(X, X_1 \cup X_2) \arrow[r, "\partial"] \arrow[dr, bend right, "\partial"'] & H_n(X_1 \cup X_2, X_2) \arrow[r] & H_n(X, X_2) \arrow[r] & \cdots \\
& & & H_n(X_1, X_1 \cap X_2) \arrow[u, "(\kappa_2)_\star", "\cong"'] \arrow[ur, bend right] & &
\end{tikzcd}$$

The top row is exact, being the homology long exact sequence of the triple $(X, X_1 \cup X_2, X_2)$. Hence, the blue sequence is also exact by commutativity of the triangles. Naturality can also be proved *easily*, as the top row is natural and so is the boundary map (Check!). $\square$

Day 5 : 27th January, 2026

`Mayer-Vietoris sequence – suspension isomorphism – homology of spheres – relative Mayer-Vietoris sequence – singular chains – singular homology`

5.1 Mayer-Vietoris Sequence of a Proper Triad

Before describing the sequence, let us observe the following hexagonal lemma.

Lemma 5.1: (*Hexagonal Lemma*)

Suppose, we have a diagram of groups, where each triangle commutes.

$$\begin{tikzcd}
 & A_0 \arrow[dl, "l_1"'] \arrow[dr, "l_2"] \arrow[dd, "\iota_0"] & \\
C_1 & & C_2 \\
 & B \arrow[ul, "j_1"] \arrow[ur, "j_2"'] \arrow[dd, "j_0"] & \\
A_2 \arrow[uu, "k_1"] \arrow[ur, "\iota_2"] \arrow[dr, "h_1"'] & & A_1 \arrow[uu, "k_2"'] \arrow[ul, "\iota_1"'] \arrow[dl, "h_2"] \\
 & C_0 &
\end{tikzcd}$$

Assume that $\ker(\iota_\alpha) = \mathrm{im}(j_\alpha)$ for $\alpha = 1, 2$, $j_0 \circ \iota_0 = 0$, and k_1, k_2 are isomorphisms. Then, the left and right sides of the hexagon differs by a side, i.e, $h_1 \circ k_1^{-1} \circ l_2 = -h_2 \circ k_2^{-1} \circ l_2$.

Proof : We can apply Lemma 4.10 to the blue part of the diagram. In particular, for any $a \in A_0$, we can uniquely write

$$\iota_0(a) = \iota_1(k_2^{-1} j_2 \iota_0(a)) + \iota_2(k_1^{-1} j_1 \iota_0(a)).$$

Applying j_0, and using $j_0 \circ \iota_0 = 0$, we have

$$0 = j_0 \iota_0(a) = h_2 k_2^{-1} j_2 \iota_0(a) + h_1 k_1^{-1} j_1 \iota_0(a) = h_2 k_2^{-1} l_2(a) + h_1 k_1^{-1} l_1(a).$$

As $a \in A_0$ is arbitrary, we have the claim. □

Let us consider a proper triad $(X; X_1, X_2)$ with $X = X_1 \cup X_2$, and set $A = X_1 \cap X_2$. At the space level, we have the following commuting diagram.

$$
\begin{tikzcd}
 & X_1 \arrow[dr, "m_1"] \arrow[rrrr, bend left, "n_1"] & & (X, X_2) & & (X_1, A) \arrow[ll, "k_2"'] \arrow[dl, "\iota_1"] \\
A \arrow[ur, "h_1"] \arrow[rr, "\iota"] \arrow[dr, "h_2"'] & & X \arrow[ur, "l_2"] \arrow[rr, "j"] \arrow[dr, "l_1"'] & & (X, A) \arrow[ul, "j_2"'] \arrow[dl, "j_1"] & \\
 & X_2 \arrow[ur, "m_2"'] \arrow[rrrr, bend right, "n_2"'] & & (X, X_1) & & (X_2, A) \arrow[ll, "k_1"] \arrow[ul, "\iota_2"']
\end{tikzcd}
$$

Passing to homology, we have the following commuting diagram.

$$
\begin{tikzcd}
 & H_n(X_1) \arrow[dr, "m_{1\star}"] \arrow[rrrr, bend left, "n_{1\star}"] & & H_n(X, X_2) & & H_n(X_1, A) \arrow[ll, "k_{2\star}", "\cong"'] \arrow[dl, hook, "\iota_{1\star}"] \arrow[dr, "\partial_2"] & \\
H_n(A) \arrow[ur, "h_{1\star}"] \arrow[rr, "\iota_\star"] \arrow[dr, "h_{2\star}"'] & & H_n(X) \arrow[ur, "l_{2\star}"] \arrow[rr, "j_\star"] \arrow[dr, "l_{1\star}"'] & & H_n(X, A) \arrow[ul, two heads, "j_{2\star}"'] \arrow[dl, two heads, "j_{1\star}"] \arrow[rr, "\partial"] & & H_{n-1}(A) \\
 & H_n(X_2) \arrow[ur, "m_{2\star}"'] \arrow[rrrr, bend right, "n_{2\star}"'] & & H_n(X, X_1) & & H_n(X_2, A) \arrow[ll, "\cong", "k_{1\star}"'] \arrow[ul, hook, "\iota_{2\star}"'] \arrow[ur, "\partial_1"'] &
\end{tikzcd}
$$

As the triad is proper, we have $k_{1\star}, k_{2\star}$ are isomorphisms, which gives the diagonal short exact sequences by Lemma 4.10. The commutativity involving the boundary maps follows from the naturality of the homology long exact sequence. The colored arrows are part of long exact sequence of (X_1, A), (X_2, A), and (X, A). We define a sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{\psi} H_n(X_1) \oplus H_n(X_2) \xrightarrow{\varphi} H_n(X) \xrightarrow{\Delta} H_{n-1}(A) \longrightarrow \cdots$$

where the maps are as follows:

$$
\begin{aligned}
\psi(u) &= (h_{1\star}(u), -h_{2\star}(u)), & u &\in H_n(A) \\
\varphi(v_1, v_2) &= m_{1\star}(v_1) + m_{2\star}(v_2), & v_1 &\in H_n(X_1), v_2 \in H_n(X_2) \\
\Delta(w) &= -\partial_1 k_{1\star}^{-1} l_{1\star}(w) = \partial_2 k_{2\star}^{-1} l_{2\star}(w), & w &\in H_n(X).
\end{aligned}
$$

The definition of Δ is a consequence of Lemma 5.1.

Theorem 5.2: (*Mayer-Vietoris Sequence for Proper Triad*)

Given a proper triad $(X; X_1, X_2)$ with $X = X_1 \cup X_2$ and $A = X_1 \cap X_2$, we have a long exact sequence,

$$\cdots \longrightarrow H_n(A) \xrightarrow{\psi} H_n(X_1) \oplus H_n(X_2) \xrightarrow{\varphi} H_n(X) \xrightarrow{\Delta} H_{n-1}(A) \longrightarrow \cdots$$

known as the *Mayer-Vietoris sequence*, which is natural with respect to morphisms of triads.

Proof : As usual, the proof requires explicitly checking the exactness at each point.

1. $\ker \varphi \supset \operatorname{im} \psi$ **:** For any $u \in H_n(A)$, we have

$$\varphi\psi(u) = \varphi(h_{1\star}(u), -h_{2\star}(u)) = m_{1\star}h_{1\star}(u) - m_{2\star}h_{2\star}(u) = \iota_\star(u) - \iota_\star(u) = 0.$$

Thus, $\varphi \circ \psi = 0 \Rightarrow \ker \varphi \supset \operatorname{im} \psi$.

2. $\ker \varphi \subset \operatorname{im} \psi$ **:** Suppose $\varphi(v) = 0$ for some $v = (v_1, v_2) \in H_n(X_1) \oplus H_n(X_2)$. Then,

$$0 = j_\star\varphi(v) = j_\star m_{1\star}(v_1) + j_\star m_{2\star}(v_2) = \iota_{1\star}n_{1\star}(v_1) + \iota_{2\star}n_{2\star}(v_2)$$

Now, by Lemma 4.10, we have $\iota_{1\star}, \iota_{2\star}$ are monic, and $\operatorname{im}(\iota_{1\star}) \cap \operatorname{im}(\iota_{2\star}) = 0$. Hence, $n_{1\star}(v_1) = 0 = n_{2\star}(v_2)$. By exactness of the long exact sequence of (X_1, A) and (X_2, A) respectively, there exists $u_1, u_2 \in H_n(A)$ such that $v_1 = h_{1\star}(u_1), v_2 = h_{2\star}(u_2)$. In other words,

$$0 = \varphi(v) = m_{1\star}h_{1\star}(u_1) + m_{2\star}h_{2\star}(u_2) = \iota_\star(u_1) + \iota_\star(u_2) = \iota_\star(u_1 + u_2).$$

Now, $u_1 + u_2 \in \ker(\iota_\star) = \operatorname{im} \partial$. Thus, there is some $w \in H_{n+1}(X, A)$ such that $\partial(w) = u_1 + u_2$. Again by Lemma 4.10, there are (unique) $w_1 \in H_{n+1}(X_1, A), w_2 \in H_{n+1}(X_2, A)$ such that $w = \iota_{1\star}(w_1) + \iota_{2\star}(w_2)$. Then,

$$u_1 + u_2 = \partial(w) = \partial\iota_{1\star}(w_1) + \partial\iota_{2\star}(w_2) = \partial_2(w_1) + \partial_1(w_2).$$

Set, $u = u_1 - \partial_2(w_1) = -(u_2 - \partial_1(w_2))$. We then have,

$$h_{1\star}(u) = h_{1\star}(u_1) - \cancel{h_{1\star}\partial_2(w_1)} = v_1, h_{2\star}(u) = -(h_{2\star}(u_2) - \cancel{h_{2\star}\partial_1(w_2)}) = -v_2.$$

Hence, $\psi(u) = (h_{1\star}(u), -h_{2\star}(u)) = (v_1, v_2) = v$. This proves the claim.

3. $\ker \Delta \supset \operatorname{im} \varphi$ **:** For $v = (v_1, v_2) \in H_n(X_1) \oplus H_2(X_2)$, we have

$$\Delta\varphi(v) = \Delta(m_{1\star}(v_1) + m_{2\star}(v_2)) = -\partial_1 k_{1\star}^{-1} l_{1\star} m_{1\star}(v_1) + \partial_2 k_{2\star}^{-1} l_{2\star} m_{2\star}(v_2) = 0.$$

Thus, $\Delta \circ \varphi = 0 \Rightarrow \ker \Delta \supset \operatorname{im} \varphi$.

4. $\ker \Delta \subset \operatorname{im} \varphi$ **:** Suppose for some $w \in H_n(X)$, we have $\Delta(w) = 0$. Thus, $\partial_1 k_{1\star}^{-1} l_{1\star}(w) = 0 = \partial_2 k_{2\star}^{-1} l_{2\star}(w)$. By exactness, for $\alpha = 1, 2$, there is $v_\alpha \in H_n(X_\alpha)$ such that $n_{\alpha\star}(v_\alpha) = k_{\alpha\star}^{-1} l_{\alpha\star}(w)$. By Lemma 4.10, we can (uniquely) write

$$\begin{aligned} j_\star(w) &= i_{1\star}k_{2\star}^{-1}j_{2\star}j_\star(w) + i_{2\star}k_{1\star}^{-1}j_{1\star}j_\star(w) \\ &= i_{1\star}k_{2\star}^{-1}l_{2\star}(w) + \iota_{2\star}k_{1\star}^{-1}l_{1\star}(w) \\ &= \iota_{1\star}n_{1\star}(v_1) + \iota_{2\star}n_{2\star}(v_2) \\ &= j_\star m_{1\star}(v_1) + j_\star m_{2\star}(v_2). \end{aligned}$$

Thus, $w - m_{1\star}(v_1) - m_{2\star}(v_2) \in \ker(j_\star) = \operatorname{im}(\iota_\star)$. Hence, there exists some $u \in H_n(A)$ such that $\iota_\star(u) = w - m_{1\star}(v_1) - m_{2\star}(v_2)$. Set, $v_1' = v_1 + h_{1\star}(u), v_2' = v_2$. Then

$$\varphi(v_1', v_2') = m_{1\star}(v_1 + h_{1\star}(u)) + m_{2\star}(v_2)$$

$$= m_{1\star}(v_1) + (w - m_{1\star}(v_1) - m_{2\star}(v_2)) + m_{2\star}(v_2) = w.$$

This proves the claim.

5. $\mathbf{ker}\,\boldsymbol{\psi} \supset \mathbf{im}\,\boldsymbol{\Delta}$ **:** For $w \in H_n(X)$, we have

$$\psi\Delta(w) = (h_{1\star}\partial_2 k_{2\star}^{-1})l_{2\star}(w), h_{2\star}\partial_1 k_{1\star}^{-1}l_{1\star}(w) = (0,0) = 0.$$

Thus, $\psi \circ \Delta = 0 \Rightarrow \ker\psi \supset \operatorname{im}\Delta$.

6. $\mathbf{ker}\,\boldsymbol{\psi} \subset \mathbf{im}\,\boldsymbol{\Delta}$ **:** Suppose for some $u \in H_n(A)$ we have $\psi(u) = 0$. Thus, $h_{1\star}(u) = 0 = h_{2\star}(u)$. By exactness, for $\alpha = 1, 2$ there are $x_\alpha \in H_{n+1}(X_\alpha, A)$ such that $\partial_1(x_2) = u$ and $\partial_2(x_1) = -u$. Then,

$$\partial\iota_{1\star}(x_1) + \partial\iota_{2\star}(x_2) = \partial_2(x_1) + \partial_1(x_2) = -u + u = 0 \Rightarrow \iota_{1\star}(x_1) + \iota_{2\star}(x_2) \in \ker(\partial) = \operatorname{im}(j_\star).$$

Thus, there exists $w \in H_{n+1}(X)$ such that $j_\star(w) = -\iota_{1\star}(x_1) - \iota_{2\star}(x_2)$. Then,

$$\begin{aligned} \Delta(w) &= -\partial_1 k_{1\star}^{-1} l_{1\star}(w) = -\partial_1 k_{1\star}^{-1} j_{1\star} j_\star(w) \\ &= \partial_1 k_{1\star}^{-1} j_{1\star}(\iota_{1\star}(x_1) + \iota_{2\star}(x_2)) \\ &= 0 + \partial_1 k_{1\star}^{-1} j_{1\star}\iota_{2\star}(x_2) \\ &= \partial_1 k_{1\star}^{-1} k_{1\star}(x_2) = \partial_1(x_2) = u. \end{aligned}$$

This proves the claim.

Thus, the Mayer-Vietoris sequence is exact. One can *easily* prove the naturality of the sequence with respect to maps of triads (Check!). □

5.2 The Suspension Isomorphism

Given a space, recall the *(unreduced) cone* on X is

$$CX = \frac{X \times [0,1]}{X \times \{0\}},$$

and the *(unreduced) suspension* of X is

$$\Sigma X = \frac{CX}{X \times \{1\}}.$$

It is well known that $CX \simeq \star$, i.e, the cone is always contractible. Consequently we have the following.

Proposition 5.3: (*Cone on a Space Is Homologically Trivial*)

Given a space X, we have CX is homologically trivial, i.e, $\tilde{H}_n(CX) = 0$ for all n.

Let us now prove a fundamental property of (ordinary) homology theory.

Theorem 5.4: (*Suspension Isomorphism*)

Given a space X, there exists a natural isomorphism $\tilde{H}_n(\Sigma X) \cong \tilde{H}_{n-1}(X)$.

Proof : Consider $A := q\left(X \times \left[0, \frac{3}{4}\right]\right), B := q\left(X \times \left[\frac{1}{4}, 1\right]\right) \subset \Sigma X$, where $q : X \times [0,1] \to \Sigma X$ is the quotient map. It is easy to see that $(\Sigma X; A, B)$ is a proper triad. Now,

$$\Sigma X = A \cup B, A \cong CX \simeq \star, B \cong CX \simeq \star, A \cap B \cong X \times \left[\frac{1}{4}, \frac{3}{4}\right] \simeq X.$$

From the reduced version of the Mayer-Vietoris sequence, we immediately see that $\Delta : \tilde{H}_n(\Sigma X) \to \tilde{H}_{n-1}(A \cap B)$ is an isomorphism. Since $A \cap B$ deformation ratracts onto X, we have the claim. $\square$

Corollary 5.5: (*Reduced Homology of Sphere*)

The reduced homology of the n-sphere is $\tilde{H}_k(S^n) = \begin{cases} 0, & k \neq n \\ H_0(\star), & k = n \end{cases}$.

Proof : We apply Theorem 5.4. Since $S^n = \Sigma S^{n-1}$, by induction, we only need compute the homology of S^0, which is just two points. By Proposition 3.1, we have $H_n(S^0) = H_n(\star) \oplus H_n(\star)$. Then, by the dimension axiom, it follows that $\tilde{H}_k(S^0) = \begin{cases} 0, & k \neq 0 \\ H_0(\star), & k = 0 \end{cases}$. $\square$

5.3 Relative Mayer-Vietoris Sequence

Let us now consider an arbitrary proper triad $(X; X_1, X_2)$. We have the following diagram.

$$\begin{tikzcd}
 & & H_n(X, X_1 \cap X_2) \arrow[dll, "h_{2\star}"'] \arrow[drr, "h_{1\star}"] \arrow[dd, "j_\star"] & & \\
H_n(X, X_2) \arrow[drr, "m_{2\star}"'] \arrow[dddd, bend right, "\delta_2"'] & & & & H_n(X, X_1) \arrow[dll, "m_{1\star}"] \arrow[dddd, bend left, "\delta_1"] \\
 & & H_n(X, X_1 \cup X_2) \arrow[dll, "\partial_1"'] \arrow[drr, "\partial_2"] \arrow[dd, "\partial"] & & \\
H_{n-1}(X_1 \cup X_2, X_1) & & & & H_{n-1}(X_1 \cup X_2, X_2) \\
 & & H_{n-1}(X_1 \cup X_2, X_1 \cap X_2) \arrow[ull, "d_{1\star}"'] \arrow[urr, "d_{2\star}"] \arrow[dd, "\iota_\star"] & & \\
H_{n-1}(X_2, X_1 \cap X_2) \arrow[uu, "\kappa_{1\star}", "\cong"'] \arrow[urr, "\iota_{2\star}"] \arrow[drr, "n_{1\star}"'] & & & & H_{n-1}(X_1, X_1 \cap X_2) \arrow[uu, "\kappa_{2\star}"', "\cong"] \arrow[ull, "\iota_{1\star}"'] \arrow[dll, "n_{2\star}"] \\
 & & H_{n-1}(X, X_1 \cap X_2) & &
\end{tikzcd}$$

The colored arrows are part of the corresponding long exact sequences of different triples. The isomorphims $\kappa_{1\star}, \kappa_{2\star}$ follows from the proper triad $(X_1 \cup X_2; X_1, X_2)$. In particular, we can apply Lemma 5.1. Let us define a sequence

$$\cdots \longrightarrow H_n(X, X_1 \cap X_2) \xrightarrow{\psi} H_n(X, X_1) \oplus H_n(X, X_2) \xrightarrow{\varphi} H_n(X, X_1 \cup X_2) \xrightarrow{\Delta} H_{n-1}(X, X_1 \cap X_2) \longrightarrow \cdots$$

where the maps are as follows:

$$\begin{aligned}
\psi(u) &= (h_{1\star}(u), -h_{2\star}(u)), && u \in H_n(X, X_1 \cap X_2) \\
\varphi(v_1, v_2) &= m_{1\star}(v_1) + m_{2\star}(v_2), && v_1 \in H_n(X, X_1), v_2 \in H_n(X, X_2) \\
\Delta(w) &= -n_{1\star} k_{1\star}^{-1} \partial_1(w) = n_{2\star} k_{2\star}^{-1} \partial_2(w), && w \in H_n(X, X_1 \cup X_2).
\end{aligned}$$

Theorem 5.6: (*Relative Mayer-Vietoris Sequence*)

Given a proper triad $(X; X_1, X_2)$, there exists a long exact sequence

$$\cdots \longrightarrow H_n(X, X_1 \cap X_2) \xrightarrow{\psi} H_n(X, X_1) \oplus H_n(X, X_2) \xrightarrow{\varphi} H_n(X, X_1 \cup X_2) \xrightarrow{\Delta} H_{n-1}(X, X_1 \cap X_2) \longrightarrow \cdots$$

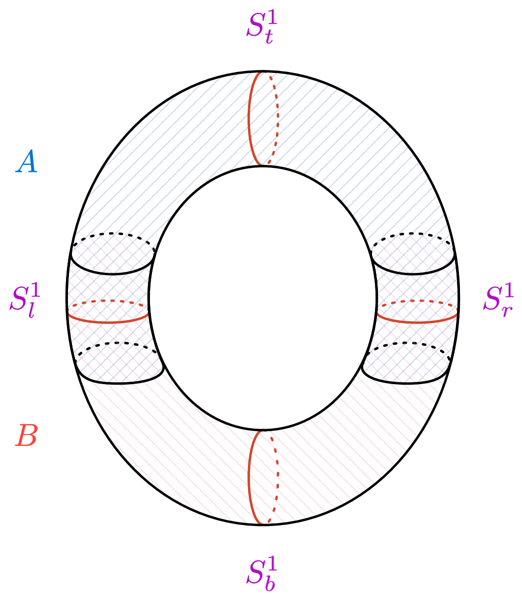
known as the *relative Mayer-Vietoris sequence*, which is moreover natural with respect to morphism of triads.

Day 6 : 6th February, 2026

`homology of` T^2 `– singular homology – singular homology of point – categorical digression : kernel, cokernel, abelian category – snake lemma`

6.1 Homology of Torus $T^2 = S^1 \times S^1$ (with $\mathbb{Z}$ coefficients)

Suppose $H_\star$ is a homology theory with $H_0(\star) = \mathbb{Z}$. Let us now compute the homology groups of the torus $T^2 = S^1 \times S^1$ using the Mayer-Vietoris sequence (Theorem 5.2). Consider the following decomposition $T^2 = A \cup B$.



Observe that $A \cong B \simeq S^1$, and $A \cap B \simeq S^1 \sqcup S^1$. Then, by Corollary 4.2, we have $H_1(S^1) = \mathbb{Z} = H_0(S^1)$, and all other homology of S^1 is zero. By Proposition 3.1, we have $H_\star(A \cap B) = H_\star(S^1) \oplus H_\star(S^1) = \mathbb{Z} \oplus \mathbb{Z}$ for $\star = 0, 1$, and 0 otherwise. It is easy to obeserve that $(A, A \cap B) \hookrightarrow (T^2, B)$ and $(B, A \cap B) \hookrightarrow (T^2, A)$ satisfy the hypothesis for excision, and thus, $(T^2;, A, B)$ is a proper triad.

Let us consider the diagrams

$$\begin{tikzcd} S^1_l \arrow[r, hook] \arrow[d, equal] & A \cap B \arrow[d, hook] & S^1_r \arrow[l, hook'] \arrow[d, equal] \\ S^1_t \arrow[r, hook] & A & S^1_t \arrow[l, hook'] \end{tikzcd} \qquad \begin{tikzcd} S^1_l \arrow[r, hook] \arrow[d, equal] & A \cap B \arrow[d, hook] & S^1_r \arrow[l, hook'] \arrow[d, equal] \\ S^1_b \arrow[r, hook] & B & S^1_b \arrow[l, hook'] \end{tikzcd}$$

which commute up to homtopy. Consequently, we can identify the map $\psi : H_\star(A \cap B) \to H_\star(A) \oplus H_\star(B)$ from the Mayer-Vietoris sequence as

$$\psi(x, y) = (x + y, -x - y).$$

Note that ψ is an injective map. Let us consider the exact sequence

$$\cdots \to \underbrace{H_2(A) \oplus H_2(B)}_{0} \to H_2(T^2) \to \underbrace{H_1(A \cap B)}_{\mathbb{Z} \oplus \mathbb{Z}} \xrightarrow{\psi} \underbrace{H_1(A) \oplus H_1(B)}_{\mathbb{Z} \oplus \mathbb{Z}} \to \ldots.$$

As $\psi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is injective, we have

$$H_2(T^2) = \ker(\psi) = \{(x, y) \mid x = -y\} \cong \mathbb{Z}.$$

Next, we compute $H_1(T^2)$. Note that $\varphi : H_\star(A) \oplus H_\star(B) \to H_\star(T^2)$ is given by

$$\varphi(x,y) = x - y.$$

Now, using the *reduced* Mayer-Vietories sequence, we have

$$\cdots \to \underbrace{\tilde{H}_1(A) \oplus \tilde{H}_1(B)}_{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\varphi} \tilde{H}_1(T^2) \to \underbrace{\tilde{H}_0(A \cap B)}_{\mathbb{Z}} \xrightarrow{\psi} \underbrace{\tilde{H}_0(A) \oplus \tilde{H}_0(B)}_{0} \to \cdots$$

We have $\operatorname{im}\varphi = \mathbb{Z}$, and thus, injectivity of ψ gives the short exact sequence

$$0 \to \underbrace{\operatorname{im}(\varphi)}_{\mathbb{Z}} \to H_1(T^2) \to \mathbb{Z} \to 0.$$

This implies $H_1(T^2) = \mathbb{Z}\,\oplus$. Finally, again the reduced Mayer-Vietoris sequence gives the exact sequence,

$$\cdots \to \underbrace{\tilde{H}_0(A) \oplus \tilde{H}_0(B)}_{0} \to \tilde{H}_0(T^2) \to \underbrace{\tilde{H}_{-1}(A \cap B)}_{0} \to \cdots$$

Hence, $H_0(T^2) = \tilde{H}_0(T^2) \oplus H_0(\star) = 0 \oplus \mathbb{Z} = \mathbb{Z}$. It is easy to see, $H_k(T^2) = 0$ for $k \geq 3$ from the Mayer-Vietoris sequence. Hence, we have computed

$$H_k(T^2) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z} \oplus \mathbb{Z}, & k = 1 \\ \mathbb{Z}, & k = 2 \\ 0, & k > 2. \end{cases}$$

6.2 Singular Homology Theory

As a first step in defining the singular homology, let us begin with n-simplex.

Definition 6.1: (*n-simplex*)

The standard (or geometric) *n-simplex* is defined as

$$\Delta^n := \left\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum t_i = 1\right\}$$

Denoting $e_i := (0, \ldots, 1, \ldots, 0)$ to be the i^{th} standard unit vector in $\mathbb{R}^{n+1}$, we have Δ^n is the convex hull of $\{e_0, \ldots, e_n\}$

In the language of category theory, the collection $\{\Delta_n\}$ is a *cosimplicial space*. Explicitly, recall from Example 1.3, the simplicial category $\mathbf{\Delta}$. We have a covariant functor $\Delta : \mathbf{\Delta} \to \mathbf{Top}$ given by $\Delta(n) = \Delta^n$. To describe the morphisms, consider a (weakly) order preserving map $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$. Then, we have

$$\begin{aligned} \Delta(\alpha) : \Delta^m &\longrightarrow \Delta^n \\ \sum_{i=0}^m t_i e_i &\mapsto \sum_{i=0}^m t_i e_{\alpha(i)}. \end{aligned}$$

In $\mathbf{\Delta}$, there are some special maps. For $0 \leq i \leq n$, we have the *face maps* $\delta_i^n : [n-1] \to [n]$ given by

$$
\begin{tikzcd}[column sep=small]
{[n-1]} \arrow[d, "\delta_i^n"'] & : & 0 \arrow[d] & 1 \arrow[d] & \cdots & i-1 \arrow[d] & i \arrow[dr] & i+1 \arrow[dr] & & \cdots & n-1 \arrow[dr] & \\
{[n]} & & 0 & 1 & & i-1 & {\color{red} i} & i+1 & i+2 & \cdots & & n
\end{tikzcd}
$$

That is, δ_i^n is the map that *misses* i. On the other hand, for $0 \le i \le n$, we have the *degeneracy maps* $\sigma_i^n : [n+1] \to [n]$ given by

$$
\begin{tikzcd}[column sep=small]
{[n+1]} \arrow[d, "\sigma_i^n"'] & : & 0 \arrow[d] & 1 \arrow[d] & \cdots & i \arrow[d] & i+1 \arrow[dl] & i+2 \arrow[dl] & \cdots & n+1 \arrow[dl] \\
{[n]} & & 0 & 1 & & {\color{red} i} & i+1 & \cdots & n &
\end{tikzcd}
$$

That is σ_i^n is the map that repeats i twice. It is a well-known fact that any morphism of $\mathbf{\Delta}$ can be written as finite composition of these δ_i^n and σ_j^m. These maps satisfy some identities, known as the *simplicial identities*:

$$
\begin{aligned}
\delta_j^{n+1} \circ \delta_i^n &= \delta_i^{n+1} \circ \delta_{j-1}^n, & i < j, \\
\sigma_j^{n+1} \circ \sigma_i^n &= \sigma_i^{n+1} \circ \sigma_{j+1}^n, & i \le j, \\
\sigma_j^{n-1} \circ \delta_i^n &= \begin{cases} \delta_i^{n-1} \circ \sigma_{j-1}^n, & i < j, \\ \mathrm{Id}_{[n-1]}, & i = j, j+1, \\ \delta_{i-1}^{n-1} \circ \sigma_j^n, & i > j+1. \end{cases}
\end{aligned}
$$

Denote the face and degeneracy maps

$$
\begin{aligned}
d_i^n := \Delta(\delta_i^n) : \Delta^{n-1} &\longrightarrow \Delta^n, \\
\sum t_j e_j &\longmapsto \sum t_j e_{\delta_i^n(j)},
\end{aligned}
\qquad
\begin{aligned}
s_i^n := \Delta(\sigma_i^n) : \Delta^{n+1} &\longrightarrow \Delta^n \\
\sum t_j e_j &\longmapsto \sum t_j e_{\sigma_i^n(j)}.
\end{aligned}
$$

Definition 6.2: (*Singular Chain Complex*)

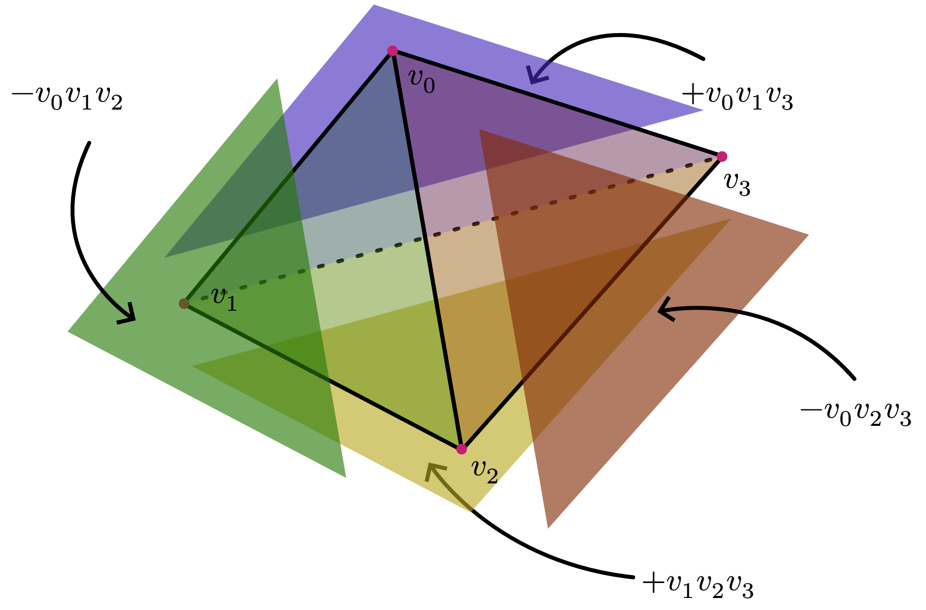
Given a space X, a *singular n-simplex* is a continuous map $\sigma : \Delta^n \to X$. The *singular chian complex* $S_\bullet(X)$ of X consists of the following data.

- The abelian group $S_n(X)$ freely generated by all singular n-simplicies. For $n < 0$, we have $S_n(X) = 0$.
- The boundary maps $\partial_n : S_n(X) \to S_{n-1}(X)$ defined on a generator $\sigma : \Delta^n \to X$ via

$$
\partial_n(\sigma) := \sum_{i=1}^n \sigma \circ d_i^n,
$$

  and then extended $\mathbb{Z}$-linearly.

Any element of $S_n(X)$ is called a *singular n-chain*, an element of $\ker(\partial_n) \subset S_n(X)$ is called a *singular n-cycle*, an element of $\mathrm{im}(\partial_{n+1}) \subset S_n(X)$ is called a *singular n-boundary*.



Visual representaion of the boundary map : $\partial_2(\Delta^3)$ is the signed sum of the faces

Exercise 6.3: (*Singular Chain Complex is a Functor*)

Check that $S_\bullet : \mathrm{Top} \to \mathrm{Ch}$ is a functor, where Ch is the category of chain complexes and chain maps.

Hint : Given a map $f : X \to Y$, define $S_n(f)(\sigma) = f \circ \sigma$ for a singular n-simplex $\sigma : \Delta^n \to X$, and extend linearly. Check that $S_\bullet(f) : S_\bullet(X) \to S_\bullet(Y)$ is a chain map. Then, check that $S_\bullet(g \circ f) = S_\bullet(g) \circ S_\bullet(f)$ and $S_\bullet(\mathrm{Id}_X) = \mathrm{Id}_{S_\bullet(X)}$.

Proposition 6.4: (*Boundary in Singular Chain Complex*)

$\partial_n \circ \partial_{n+1} = 0$, and thus, $(S_\bullet(X), \partial)$ is indeed a chain complex.

Proof : We need to check on generators only. Consider $\sigma : \Delta^{n+1} \to X$. Then we compute

$$\begin{aligned}
\partial_n \partial_{n+1}(\sigma) &= \partial_n \left(\sum_{j=0}^{n+1} (-1)^j \sigma \circ d_j^{n+1} \right) = \sum_{j=0}^{n+1} (-1)^j \partial_n \left(\sigma \circ d_j^{n+1} \right) = \sum_{j=0}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} \left(\sigma \circ d_j^{n+1} \circ d_i^n \right) \\
&= \sum_{i<j} (-1)^{i+j} \left(\sigma d_j^{n+1} d_i^n \right) + \sum_{j \le i} (-1)^{i+j} \left(\sigma d_j^{n+1} d_i^n \right) \\
&= \sum_{i<j} (-1)^{i+j} \left(\sigma d_i^{n+1} d_{j-1}^n \right) + \sum_{j \le i} (-1)^{i+j} \left(\sigma d_j^{n+1} d_i^n \right) \\
&= \sum_{i \le j'} (-1)^{i+j'+1} \left(\sigma d_i^{n+1} d_{j'}^n \right) + \sum_{j \le i} (-1)^{i+j} \left(\sigma d_j^{n+1} d_i^n \right) \\
&= -\sum_{i \le j} (-1)^{i+j} \left(\sigma d_i^{n+1} d_j^n \right) + \sum_{i \le j} (-1)^{i+j} \left(\sigma d_i^{n+1} d_j^n \right) \\
&= 0.
\end{aligned}$$

Hence, $\partial_n \circ \partial_{n+1} = 0$. Consequently, $(S_\bullet(X), \partial)$ is a chain complex. $\square$

Definition 6.5: (*Homology of a Chain Complex*)

Given any chain complex $C_\bullet = (C_n, \partial)$, the *$n^{\text{th}}$-homology group* is defined as the quotient

$$H_n(C_\bullet) = \frac{\ker(\partial_n : C_n \to C_{n-1})}{\operatorname{im}(\partial_{n+1} : C_{n+1} \to C_n)}.$$

Remark 6.6: (*Cochain Complex and Cohomology*)

We shall also consider the following case

$$C^\bullet : \cdots \to C^n \xrightarrow{\partial^n} C^{n+1} \to \cdots$$

such that $\partial^{n+1} \circ \partial^n = 0$. We say $C^\bullet$ is a *cochain complex*, and the *n^{th}-cohomology group* of $C^\bullet$ is defined as the quotient

$$H^n(C^\bullet) = \frac{\ker(\partial^n : C^n \to C^{n+1})}{\operatorname{im}(\partial^{n-1} : C^{n-1} \to C^n)}.$$

Observe that given a cochain complex $(C^\bullet, \partial^\bullet)$, we have a chain complex $(D_\bullet, \partial_\bullet)$ defined by $D_n := C^{-n}$, and $\partial_n = \partial^{-n}$.

Exercise 6.7: (*Homology is an Additive Functor*)

Given a chain map $f_\bullet : C_\bullet \to D_\bullet$, define

$$H_n(f_\bullet)([x]) = [f_n(x)], \quad [x] \in H_n(C).$$

Verify that this is well-defined, and $H_n : \mathrm{Ch} \to \mathrm{Ab}$ is a functor. Similarly, H^n is a functor as well from the category of cochain complexes to abelian groups. Moreover, verify that H_n is an *additive* functor, i.e, given $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$, we have $H_n(af_\bullet + bg_\bullet) = aH_n(f_\bullet) + bH_n(g_\bullet)$, for scalars a, b (i.e, $a, b \in R = \mathbb{Z}$)

Definition 6.8: (*Singular Homology*)

Given a space X, the *singular n-homology* is the n^{th}-homology group of the singular chain complex $S_\bullet(X)$, and it is dentoed as $H_n(X)$.

Remark 6.9: (*Homology is a Functor*)

It follows from Exercise 6.3 and Exercise 6.7 that $H_n : \mathrm{Top} \to \mathrm{Ab}$ is a functor as it is a composition of two functors.

Let us compute an example!

Proposition 6.10: (*Singular Homology of a Singleton*)

$H_n(\star) = \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0. \end{cases}$

<u>**Proof**</u> **:** Observe that for each n, there exists a unique map $\sigma_n : \Delta^n \to \star$, namely the constant map. Thus, we have $S_n(X) = \mathbb{Z}$ for $n \geq 0$. Let us figure out the boundary maps. For $n > 0$, we have

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i (\sigma_n \circ d_i^n) = \sum_{i=0}^{n} (-1)^i (\sigma_{n-1}) = \begin{cases} \sigma_{n-1}, & n \text{ is even} \\ 0, & n \text{ is odd.} \end{cases}$$

Thus, the singular n-complex $S_\bullet(\star)$ is given as

$$\cdots \to \underbrace{S_4}_{\mathbb{Z}} \xrightarrow{\mathrm{Id}} \underbrace{S_3}_{\mathbb{Z}} \xrightarrow{0} \underbrace{S_2}_{\mathbb{Z}} \xrightarrow{\mathrm{Id}} \underbrace{S_1}_{\mathbb{Z}} \xrightarrow{0} \underbrace{S_0}_{\mathbb{Z}} \to 0.$$

Immediately we get $H_n(\star) = \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0. \end{cases}$ □

The above shows that singular homology satisfies the dimension axiom from the Eilenberg-Steenrod axioms.

6.3 A Categorical Digression : Product, Coproduct, Kernel, Cokernel

In this section, we re-interpret some notions from algebra in category theoretic language. If it seems too abstract, you can skip it completely!

Definition 6.11: (*Product and Initial Object*)

Let $\mathcal{C}$ be a category, and $\{A_i\}_{\{i \in I\}}$ be a (possibly empty!) collection of objects of $\mathcal{C}$. The *product* is then defined via the following universal property.

> The product is an object $X \in \mathcal{C}$ and a collection of maps $\pi_i : X \to A_i$, such that given any other object $Y \in \mathcal{C}$ and any other collection of maps $f_i : Y \to A_i$, there exists a **unique** map $f : Y \to X$ such that $f_i = \pi_i \circ f$. We have the following diagram.

$$\begin{tikzcd} Y \arrow[d, dashed, "\exists! f"'] \arrow[dr, bend left, "f_i"] & \\ X \arrow[r, "\pi_i"'] & A_i \end{tikzcd}$$

An *initial object* in $\mathcal{C}$ is a product of an empty collection.

Since product is defined via an universal property, if it exists, it is unique up to unique isomorphism. We have an alternative, perhaps more useful definition of the initial object.

Exercise 6.12: (*Initial Object*)

Show that the initial object of a category $\mathcal{C}$ (if exists) can be equivalently defined as an object X such that given any object $Y \in \mathcal{C}$, there exists a unique morphism $X \to Y$.

Example 6.13: (*Examples of Product and Initial Objects*)

Here a few examples, that one should verify!

- Set : product is the Cartesian product of sets, and initial object is the emptyset.
- Group : product is the direct product, and initial object is the group with one element.
- Top : product is the Cartesian product of the underlying sets equipped with the product topology, and initial object is the emptyset.
- $\mathrm{Top}_\star$: product is the *smash product*, i.e, $\prod(X_i, x_i) = \frac{\prod X_i}{\bigvee X_i}$, and initial object is the singleton.

The dual notion to product is the coproduct.

Definition 6.14: (*Coproduct and Final Object*)

Let $\mathcal{C}$ be a category, and $\{A_i\}_{\{i\in I\}}$ be a (possibly empty!) collection of objects of $\mathcal{C}$. The *coproduct* is then defined via the following universal property.

> The coproduct is an object $X \in \mathcal{C}$ and a collection of maps $\iota_j : A_j \to X$, such that given any other object $Y \in \mathcal{C}$ and any other collection of maps $f_i : A_j \to Y$, there exists a **unique** map $f : X \to Y$ such that $f_i = f \circ \iota_j$. We have the following diagram.

$$\begin{tikzcd} A_j \arrow[r, "\iota_j"] \arrow[d, "f_j"'] & X \arrow[dl, dashed, bend left, "\exists! f"] \\ Y & \end{tikzcd}$$

A *final object* in $\mathcal{C}$ is a coproduct of an empty collection.

Again, coproduct (if exists) is unique up to unique isomorphism. Moreover, we have the following useful characterization of final objects.

Exercise 6.15: (*Final Object*)

Show that the final object of a category $\mathcal{C}$ (if exists) can be equivalently defined as an object X such that given any object $Y \in \mathcal{C}$, there exists a unique morphism $Y \to X$.

Example 6.16: (*Examples of Coproduct and Final Objects*)

- Set : coproduct is the disjoint union of sets, and final object is the singleton set.
- Ab : coproduct is the direct sum of Abelian groups, and final object is the group with one element.
- Group : coproduct is the *free product* of groups, and final object is the group with one element.
- Top : coproduct is the disjoint union of the underlying sets equipped with the disjoint union topology, and final object is the singleton.
- $\mathrm{Top}_\star$: coproduct is the wedge product of spaces, and final object is the singleton.

We now define the notion of (pre)additive categories, our motivation is the category Ab of Abelian groups.

Definition 6.17: (*Pre-additive Category*)

A category $\mathcal{C}$ is called an *pre-additive category* if for any $X, Y \in \mathcal{C}$ we have $\hom_{\mathcal{C}}(X, Y)$ is an Abelian group, and moreover, the composition

$$\hom_{\mathcal{C}}(X, Y) \times \hom_{\mathcal{C}}(Y, Z) \to \hom_{\mathcal{C}}(X, Z)$$

is bilinear. A functor $F : \mathcal{C} \to \mathcal{D}$ between pre-additive categories is called an *additive functor* if $F : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(F(X), F(Y))$ is a group homomorphism for all $X, Y \in \mathcal{C}$.

Proposition 6.18: (*Zero Object*)

In a pre-additive category $\mathcal{C}$, an object $\star$ is an initial object if and only if it is a final object, whence it is called a *zero object*

<u>Proof</u> : Suppose $\star \in \mathcal{C}$ is an initial object. Given any $X \in \mathcal{C}$, there exists a unique map $e : \star \to X$. Now, $\hom_{\mathcal{C}}(X, \star)$ is an Abelian group, and hence, we have the zero map $0_X : X \to \star$. If possible, suppose $f : X \to \star$ is another map. As $\mathcal{C}$ is a category, we have $\mathrm{Id}_\star : \star \to \star$, and as $\mathcal{C}$ is pre-additive, we have $0 : \star \to \star$. Then, by uniqueness, we have $\mathrm{Id}_\star = 0$. But then,

$$f = f \circ \mathrm{Id}_\star = f \circ 0 = 0_X,$$

where the last equality follows from the bilinearity of the composition. Thus, there exists a unique map $X \to \star$, which makes $\star$ into a final object. Similar argument works for the other direction as well. $\square$

In fact, one can generalize the above. In a pre-additive category, any finite product is a coproduct, and coversely, any finite coproduct is a product, whence they are called *biproduct*. Let $A, B \in \mathcal{C}$ be two objects in a pre-additive category. Suppose the product $A \oplus B \in \mathcal{C}$ exists. We have maps

$$A \overset{\pi_A}{\twoheadleftarrow} A \oplus B \overset{\pi_B}{\twoheadrightarrow} B.$$

From universal property of product, we get unique maps in the following diagrams.

$$\begin{tikzcd}
A \arrow[dr, dashed, "\iota_A"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "\mathrm{Id}_A"'] & & \\
& A \oplus B \arrow[r, "\pi_B"'] \arrow[d, "\pi_A"] & B \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[dr, dashed, "\iota_B"] \arrow[drr, bend left, "\mathrm{Id}_B"] \arrow[ddr, bend right, "0"'] & & \\
& A \oplus B \arrow[r, "\pi_B"'] \arrow[d, "\pi_A"] & B \\
& A &
\end{tikzcd}$$

Observe that,

$$\pi_A \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \mathrm{Id}_A \circ \pi_A + 0 \circ \pi_B = \pi_A + 0 = \pi_A,$$

and

$$\pi_B \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = 0 \circ \pi_A + \mathrm{Id}_B \circ \pi_B = 0 + \pi_B = \pi_B.$$

Then, from the universal property of the product, we have

$$\iota_A \circ \pi_A + \iota_B \circ \pi_B = \mathrm{Id}_{A \oplus B}.$$

Next, we check that

$$A \xhookrightarrow{\iota_A} A \oplus B \xhookleftarrow{\iota_B} B$$

is the coproduct. Indeed, for any $f : A \to C$ and $g : B \to C$, consider the map

$$h = f \circ \pi_A + g \circ \pi_B : A \oplus B \to C.$$

Then,

$$h \circ \iota_A = f \circ \pi_A \circ \iota_A + g \circ \pi_B \circ \iota_A = f \circ \mathrm{Id}_A + g \circ 0 = f + 0 = f,$$

and similarly,

$$h \circ \iota_B = f \circ \pi_A \circ \iota_B + g \circ \pi_B \circ \iota_B = f \circ 0 + g \circ \mathrm{Id}_B = 0 + g = g.$$

Now, consider $\theta : A \oplus B \to C$ be any map satisfying $\theta \circ \iota_A = f, \theta \circ \iota_B = g$. Then,

$$h = f \circ \pi_A + g \circ \pi_B = \theta \circ \iota_A \circ \pi_A + \theta \circ \iota_B \circ \pi_B = \theta \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \theta \circ \mathrm{Id}_{A \oplus B} = \theta.$$

Hence, $h : A \oplus B$ is the unique such map. This justifies that $A \xhookrightarrow{\iota_A} A \oplus B \xhookleftarrow{\iota_B} B$ is indeed a coproduct. Similarly, we can show that any finite coproduct is also a product.

Conversely, suppose we have objects A, B, C and maps $\iota_A : A \to C, \iota_B : B \to C, \pi_A : C \to A, \pi_B : C \to B$, satisfying

$$\pi_A \circ \iota_A = \mathrm{Id}_A, \pi_A \circ \iota_B = 0, \pi_B \circ \iota_A = 0, \pi_B \circ \iota_B = \mathrm{Id}_B, \iota_A \circ \pi_A + \iota_B \circ \pi_B = \mathrm{Id}_C.$$

Then, it follows that $A \xtwoheadleftarrow{\pi_A} C \xtwoheadrightarrow{\pi_B} B$ is a product, and $A \xhookrightarrow{\iota_A} C \xhookleftarrow{\iota_B} B$ is a coproduct.

Definition 6.19: (*Additive Category*)

A pre-additive category is called an *additive category* if it admits all finite products (hence all finite coproducts). In particular, the initial object exists (which is also the final object), and is called the *zero object* of the category. Given any two objects $X, Y \in \mathcal{C}$, we have the unique *zero map* $0 : X \to Y$ between them.

Proposition 6.20:

An additive functor (Definition 6.17) between additive categories preserve finite products. In particular, zero object is mapped to the zero object.

Proof : Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories. Firstly, let us show that $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$. Note that $\mathrm{Id}_{0_{\mathcal{C}}} = 0$. Since F is an additive functor, we have $\mathrm{Id}_{F(0_{\mathcal{C}})} = F\big(\mathrm{Id}_{0_{\mathcal{C}}}\big) = F(0) = 0$. That is, the identity map of $F(0_{\mathcal{C}})$ is the zero map. But then $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$.

Next, let $A, B \in \mathcal{C}$. We need to show that $F(A \oplus B) = F(A) \oplus F(B)$. Recall that $A \oplus B$ is both the product and coproduct. Thus, we have maps $\iota_A : A \to A \oplus B, \iota_B : B \to A \oplus B$ and $\pi_A : A \oplus B \to A, \pi_B : A \oplus B \to B$, which moreover satisfy

$$\pi_A \circ \iota_A = \mathrm{Id}_A, \pi_A \circ \iota_B = 0, \pi_B \circ \iota_A = 0, \pi_B \circ \iota_B = \mathrm{Id}_B, \iota_A \circ \pi_A + \iota_B \circ \pi_B = \mathrm{Id}_{A \oplus B}.$$

But then applying F we get the same relations for the object $F(A \oplus B)$. Hence, $F(A \oplus B)$ is the biproduct of $F(A)$ and $F(B)$. Inductively, it follows taht F takes any finite biproduct to a biproduct. $\square$

Example 6.21: (*Examples of Additive Category*)

Category of Abelian groups (and more generally, category of R-modules for a commutative ring R) is an additve category. The category Ch of chain complexes of Abelian groups (or even modules) is also an additive category.

Exercise 6.22:

Let R be a commutative ring with 1. Let $\mathcal{C}$ be the one-object category with hom-set R. Verify that $\mathcal{C}$ is a pre-additive category, which is not additive.

The goal is to now define (co)kernels, and relate them to monic and epic maps.

Definition 6.23: (*Kernel and Cokernel*)

Let $\mathcal{C}$ be an additive category, and $f : X \to Y$ is a map.

- The *kernel* of f is a map $\iota : \ker(f) \to X$ with $f \circ \iota = 0$, satisfying the universal property :

$$\begin{tikzcd} \ker(f) \arrow[r, "\iota"] & X \arrow[r, "f"] & Y \\ Z \arrow[u, dashed, "\exists!\tilde{g}"] \arrow[ur, "g"'] \arrow[urr, bend right, "0"'] & & \end{tikzcd}$$

given any map $g : Z \to X$ with $f \circ g = 0$, there exists a unique map $\tilde{g} : Z \to \ker(f)$ such that $\iota \circ \tilde{g} = g$.

- The *cokernel* of f is a map $\pi : X \to \operatorname{coker}(f)$ with $\pi \circ f = 0$, satisfying the universal property :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[drr, bend right, "0"'] & Y \arrow[r, "\pi"] \arrow[dr, "h"'] & \operatorname{coker}(f) \arrow[d, dashed, "\exists!\tilde{h}"] \\ & & W \end{tikzcd}$$

given any map $h : X \to W$ with $h \circ f = 0$, there exists a unique map $\tilde{h} : \operatorname{coker}(f) \to W$ such that $\tilde{h} \circ \pi = h$.

In the category of Abelian groups (or modules), the kernel of a map is indeed the 0-set. For a homomorphism $f : X \to Y$ of Abelian groups, we have $\operatorname{coker}(f) = \frac{B}{\operatorname{im}(f)}$.

Remark 6.24: (*Categorical Image and Coimage*)

Since we have defined (co)kernel as a map, we can now define the (co)kernel of them as well. This leads to the following definition : given a map $f : X \to Y$, the *image* is defined as $\ker(\operatorname{coker}(f))$ and the *coimage* is defined as $\operatorname{coker}(\ker(f))$. One can show that there is a natural map $\operatorname{coim}(f) \to \operatorname{im}(f)$, and the *first isomorphism theorem* states that this map is an isomorphism. In the category of Abelian groups, the categorical image matches with the set-theoretic image (which is already a subgroup), but in the category of groups one needs to take the *normal completion* of the set-theoretic image (which is only a subgroup).

Let us now define injective and surjective maps, and relate them kernel and cokernel.

Definition 6.25: (*Monic and Epic Maps*)

Let $f : X \to Y$ be a map in a category $\mathcal{C}$.

- f is called *monic* (or *monomorphism* or *injective*) if given any two maps $g_1, g_2 : Z \to X$ with $f \circ g_1 = f \circ g_2$, we have $g_1 = g_2$.
- f is called *epic* (or *epimorphism* or *surjective*) if given any two maps $h_1, h_2 : Y \to W$ with $h_1 \circ f = h_2 \circ f$, we have $h_1 = h_2$.

In the category Set, the above definition boils down to the usual definition of injective and surjective set maps. The categorical definition has the advantage that we do not need to the objects to be a set. Thus, for example, we can readily define monic/epic chain maps.

Caution 6.26: (*Monic, Epic and Iso*)

Suppose, $f : X \to Y$ admits a right inverse, i.e, there is a map $s : Y \to X$ such that $s \circ f = \mathrm{Id}_X$. Then, f is necessarily monic. But the converse may not be true! Consider the map $f : \mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2: clearly f is monic, but admits no right inverse. Similarly, any map admitting a left inverse is epic, but not conversely. By definition, an isomorphism admits both left and right inverse, and hence, is both monic and epic. In the category of commutative rings, the inclusion $\mathbb{Z} \to \mathbb{Q}$ is both monic and epic, but it is **not** an isomorphism!

Definition 6.27: (*Abelian Category*)

A category $\mathcal{C}$ is called an *Abelian category* if the following holds.

- $\mathcal{C}$ is an additive category (i.e, each $\hom$ is an Abelian groups, composition is bilinear, and all finite (co)products exists including the zero object).
- Any map in $\mathcal{C}$ has kernel and cokernel (this is sometimes called a *pre-Abelian category*).
- Every monic map is a kernel of some map, and every epic map is a cokernel of some map (this is called having *nomal monic* and *conormal epic* maps).

The category of modules over a commutative ring is an Abelian category, and so is the category of chain complexes of such modules! In the literature, following Grothendieck, additional axioms (which deals with arbitrary (co)products) are added to the definition of an Abelian category. In an Abelian category, a map which is both monic and epic is necessarily an iso.

Exercise 6.28: (*Monic and Epic maps in an Abelian Category*)

Let $f : X \to Y$ be a map in an Abelian category $\mathcal{C}$. Verify the following.

- f is monic if and only if $\ker(f) = 0$.
- $\iota : \ker(f) \to X$ is monic, and thus, $\ker(\ker(f)) = 0$.
- f is epic if and only if $\operatorname{coker}(f) = 0$.
- $\pi : Y \to \operatorname{coker}(f)$ is epic, and thus, $\operatorname{coker}(\operatorname{coker}(f)) = 0$.
- f is an iso $\Leftrightarrow$ f is both monic and epic $\Leftrightarrow$ $\ker(f) = 0 = \operatorname{coker}(f)$.

Note that by $\ker(f) = 0$ (and similarly, $\operatorname{coker}(f) = 0$), one should understand that the object $\ker(f)$ is the zero object, and the map $\ker(f) \to X$ is (necessarily) the 0 map.

6.4 Snake Lemma

After all the abstract nonsense, let us summarize the takeaway of the previous section! Suppose $f : X \to Y$ is a morphism of modules. Then, there exists an exact sequence

$$0 \to \ker(f) \hookrightarrow X \xrightarrow{f} Y \twoheadrightarrow \operatorname{coker}(f) \to 0.$$

Given a chain map $f_\bullet : C_\bullet \to D_\bullet$, kernel, cokernel, and image of $f_\bullet$ can be computed degree-wise, with naturally induced boundary maps. In particular, we have

$$0 \to P_\bullet \xrightarrow{f_\bullet} Q_\bullet \xrightarrow{g_\bullet} R_\bullet \to 0$$

is a short exact sequence of chain maps precisely when

$$0 \to P_n \xrightarrow{f_n} Q_n \xrightarrow{g_n} R_n \to 0$$

is a short exact sequence for each n. Note that any module A (and in particular 0), can be realized as a chain complex (*concentrated at degree* 0) by putting A at the 0^{th} place, and putting 0 everywhere else (and necessarily with 0 as boundary maps). Moreover, given any chain map $f_\bullet : C_\bullet \to D_\bullet$, we have an exact sequence

$$0 \to \ker(f)_\bullet \to C_\bullet \xrightarrow{f_\bullet} D_\bullet \to \operatorname{coker}(f)_\bullet \to 0$$

of chain complexes and maps.

We now state one of the most important results in homological algebra!

Lemma 6.29: (*Snake Lemma!*)

Consider the following commutative diagram

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' &
\end{tikzcd}$$

where the two rows are exact. Then, there exists an exact sequence

$$\ker(f) \to \ker(a) \to \ker(b) \to \ker(c) \xrightarrow{\partial} \mathrm{coker}(a) \to \mathrm{coker}(b) \to \mathrm{coker}(c) \to \mathrm{coker}(g'),$$

where ∂ is called the *boundary map*. Moreover, the sequence is natural.

Proof : We have the following diagram

$$\begin{tikzcd}
 & \ker(a) \arrow[r] \arrow[d] & \ker(b) \arrow[r] \arrow[d] & \ker(c) \arrow[d] \arrow[dddll, rounded corners, "\partial"] & \\
\ker(f) \arrow[r] \arrow[ur, bend left] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] \arrow[d] & C' \arrow[r] \arrow[d] & \mathrm{coker}(g') \\
 & \mathrm{coker}(a) \arrow[r] & \mathrm{coker}(b) \arrow[r] & \mathrm{coker}(c) \arrow[ur, bend right] &
\end{tikzcd}$$

The blue arrows can be induced naturally. Note that the colored arrows look like a snake! Let us define the map ∂. Say, $x \in \ker(c) \subset C$. Since g is surjective, we have some $y \in B$ such that $x = g(y)$. By commutativity, $g'b(y) = cg(y) = c(x) = 0 \Rightarrow b(y) \in \ker(g') = \mathrm{im}(f')$. Thus, there is some $z \in A'$ such that $f'(z) = b(y)$. Let us define $\partial(x) = [z] = z + \mathrm{im}(a)$. We need to show that ∂ is well-defined. Suppose, we have some $y' \in B$ such that $g(y') = c$. Then, $g'b(y') = cg(y') = 0 \Rightarrow b(y') \in \ker(g') = \mathrm{im}(f')$ and so, $b(y') = f'(z')$ for some $z' \in A'$. We have, $g(y - y') = g(y) - g(y') = x - x = 0 \Rightarrow y - y' \in \ker(g) = \mathrm{im}(f)$, and so $y - y' = f(w)$ for some $w \in A$. Now, $f'(z - z') = b(y - y') = bf(w) = f'a(w)$. As f' is injective, we have $z - z' = a(w)$. But then $[z] = [z']$ in $\mathrm{coker}(a)$. This proves that ∂ is well-defined. A similar argument shows that ∂ is a homomorphism. The exactness of the sequence is left as an exercise!

As for naturaility, we consider a commutative diagram of the form

$$
\begin{tikzcd}
&&& A_2 \arrow[rr, "f"] \arrow[dd, "a_2"] && B_2 \arrow[rr, "g"] \arrow[dd, "b_2"] && C_2 \arrow[rr] \arrow[dd, "c_2"] && 0 \\
&& A_1 \arrow[ur, red] \arrow[rr, "f"] \arrow[dd, "a_1"] && B_1 \arrow[ur, red] \arrow[rr, "g"] \arrow[dd, "b_1"] && C_1 \arrow[ur, red] \arrow[rr] \arrow[dd, "c_1"] && 0 & \\
& 0 \arrow[rr] && A_2' \arrow[rr, "f'"] && B_2' \arrow[rr, "g'"] && C_2' && \\
0 \arrow[rr] && A_1' \arrow[ur, red] \arrow[rr, "f'"'] && B_1' \arrow[ur, red] \arrow[rr, "g'"'] && C_1' \arrow[ur, red] &&&
\end{tikzcd}
$$

where all the rows are exact. Then, the red arrwos determine a commutative diagram of the corresponding 8-term exact sequences. Again, the proof is a diagram chasing argument, and left as an exercise! □

Day 7 : 8th February, 2026

`long exact sequence from short exact sequence of chain complexes — relative singular homology — long exact sequence of singular homology — chain homotopy invariance — singular homology of contractible space`

7.1 Long Exact Sequence in Homology

So far we have been working with only Abelian groups, which are nothing but $\mathbb{Z}$-modules. More generally, we can fix a commutative ring R, and work with R-modules. Generalizing even further, we can consider any arbitrary *Abelian category* $\mathcal{A}$, where 0-maps and taking quotients makes sense (see Definition 6.27). Then, we have the chain complex of objects from $\mathcal{A}$, denoted as $\mathrm{Ch}(\mathcal{A})$. The definition of n^{th}-homology group makes sense for $\mathrm{Ch}(\mathcal{A})$ as well. Homological algebra is the study of the (co)homology of (co)chain complex over some Abelian category.

Theorem 7.1: (*Long Exact Sequence of Homology of Chain Complex*)

Given a short exact sequence $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \to 0$ of chain complexes, there exists a long exact sequence of homology groups :

$$\cdots \to H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{\iota_\star} H_n(B) \xrightarrow{j_\star} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots,$$

which is natural with respect to maps of short exact sequences of chain complexes.

Proof : We consider part of the diagram

$$\begin{tikzcd}
 & \ker \partial_n^A \arrow[d, hook] & \ker \partial_n^B \arrow[d, hook] & \ker \partial_n^C \arrow[d, hook] & \\
0 \arrow[r] & A_n \arrow[r, "\iota_n"] \arrow[d, "\partial_n^A"'] & B_n \arrow[r, "j_n"] \arrow[d, "\partial_n^B"'] & C_n \arrow[r] \arrow[d, "\partial_n^C"'] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\iota_{n-1}"] \arrow[d, two heads] & B_{n-1} \arrow[r, "j_{n-1}"] \arrow[d, two heads] & C_{n-1} \arrow[r] \arrow[d, two heads] & 0 \\
 & \operatorname{coker} \partial_n^A & \operatorname{coker} \partial_n^B & \operatorname{coker} \partial_n^C &
\end{tikzcd}$$

Since both the rows are exact, applying the snake lemma (Lemma 6.29), we have the exact sequence

$$0 \to \ker \partial_n^A \to \ker \partial_n^B \to \ker \partial_n^C \to \operatorname{coker} \partial_n^A \to \operatorname{coker} \partial_n^B \to \operatorname{coker} \partial_n^C \to 0.$$

This holds for each n. In particular, we consider the following diagram

$$\begin{tikzcd}
 & \operatorname{coker} \partial_{n+1}^A \arrow[r] \arrow[d, "\delta_n^A"'] & \operatorname{coker} \partial_{n+1}^B \arrow[r] \arrow[d, "\delta_n^B"'] & \operatorname{coker} \partial_{n+1}^C \arrow[r] \arrow[d, "\delta_n^C"'] & 0 \\
0 \arrow[r] & \ker \partial_{n-1}^A \arrow[r] & \ker \partial_{n-1}^B \arrow[r] & \ker \partial_{n-1}^C &
\end{tikzcd}$$

By above, the rows are exact. The maps $\delta_n^A, \delta_n^B, \delta_n^C$ are induced from $\partial_n^A, \partial_n^B, \partial_n^C$ respectively. Thus, the diagram is commutative. Observe that $\ker \delta_n^A = H_n(A)$ and $\operatorname{coker} \delta_n^A = H_{n-1}(A)$, and similarly for δ_n^B, δ_n^C. Indeed, we have the diagram which might be of help

$$\begin{tikzcd}
A_{n+1} \arrow[r, "\partial_{n+1}^A"] & A_n \arrow[r, "\partial_n^A"] \arrow[d, two heads] & A_{n-1} \arrow[r, "\partial_{n-1}^A"] & A_{n-2} \\
H_n(A) \arrow[r, hook] & \text{coker } \partial_{n+1}^A \arrow[r, "\delta_n^A"'] \arrow[ur, "\widetilde{\partial_n^A}"] & \ker \partial_{n-1}^A \arrow[u, hook] \arrow[r, two heads] & H_{n-1}(A)
\end{tikzcd}$$

Hence, again applying the snake lemma, we have an exact sequence

$$H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C).$$

Chasing the diagram carefully, shows that the connecting maps are induced by $\iota_\bullet$ and $j_\bullet$. Pasting these exact sequences together, we get the long exact sequence in homology. Since snake lemma produces *natural* exact sequences, again a diagram chase shows that the long exact sequence is natural. $\square$

Remark 7.2: (*The Boundary Map*)

Let us describe the boundary map explicitly. Consider part of the diagram

$$\begin{tikzcd}
 & & B_n \arrow[r, two heads, "j_n"] \arrow[dd, "\partial_n^B"'] & C_n & \\
 & & & y \arrow[r, mapsto] \arrow[dd, mapsto] & x \arrow[dd, mapsto] \\
A_{n-1} \arrow[rr, hook, "\iota_n"] & & B_{n-1} & & \\
 & z \arrow[rr, mapsto] & & \partial_n^B(y) \arrow[r, mapsto] & 0
\end{tikzcd}$$

Suppose $\alpha \in H_n(C)$ is represented as $\alpha = [x]$ for some $x \in C_n$, with $\partial(x) = 0$. Then, it follows that $\partial(\alpha) = [z]$. The well-definedness is a consequence of the snake lemma. Thus, we can 'define' $\partial([x]) = [\iota_n^{-1}\partial_n^B j_n^{-1}(x)]$.

7.2 Relative Singular Homology and Long Exact Sequence

Let us now extend our definition of singular homology to pairs of spaces (X, A) with $A \subset X$. Given such a pair (X, A), observe that any singular n-simplex $\sigma : \Delta^n \to A$ is naturally an n-simplex in X via the inclusion map $\iota : A \hookrightarrow X$. Thus, we have an *injective* map $S_\bullet(\iota) : S_\bullet(A) \hookrightarrow S_\bullet(X)$. Let us define the cokernel

$$S_n(X, A) := \frac{S_n(X)}{S_n(A)}.$$

Observe that $S_n(X, A)$ is an Abelian group freely generated by singular n-simplexes $\sigma : \Delta^n \to X$ such that $\mathrm{im}(\sigma)$ is not *completely* contained in A. We have the well-defined *relative* boundary map

$$\begin{aligned}
\partial_n^{X,A} : S_n(X, A) &\longrightarrow S_{n-1}(X, A) \\
\left[\sum a_\sigma \sigma\right] &\longmapsto \sum a_\sigma [\partial_n^X(\sigma)].
\end{aligned}$$

In particular, if we have an n-chain in X whose boundary is compleletly contained in A, then its relative boundary is zero. It follows that $S_\bullet(X, A)$ is a chain complex, which is called the *relative singular chain complex*.

Exercise 7.3: (*Relative Singular Chain Complex is Functorial*)

Verify that $S_\bullet : \mathrm{TopPair} \to \mathrm{Ch}$ is a functor, and we can identify $S_n(X) = S_n(X, \emptyset)$ (which justifies the same notation). Moreover, $j : X = (X, \emptyset) \to (X, A)$ induces the quotient map $S_\bullet(X) \to S_\bullet(X, A)$.

The homology groups

$$H_n(X, A) := H_n(S_n(X, A))$$

are called the *relative singular homology groups* of the pair (X, A). By Exercise 6.7 and Exercise 7.3, it follows that the relative singular homology groups are functors as well. In particular, given $f : (X, A) \to (Y, B)$, we have $H_n(f) = H_n(S_\bullet(f))$.

We have a short exact sequence of chain complexes

$$0 \to S_\bullet(A) \xhookrightarrow{\iota_\bullet} S_\bullet(X) \xrightarrow{j_\bullet} S_\bullet(X, A) \to 0,$$

where $\iota : A \hookrightarrow X$ and $j : X \hookrightarrow (X, A)$ are the space level maps.

Theorem 7.4: (*Long Exact Sequence of Singular Homology*)

There exists a natural long exact sequence

$$\cdots \to H_n(A) \xrightarrow{\iota_\star} H_n(X) \xrightarrow{j_\star} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots \to H_0(A) \xrightarrow{\iota_\star} H_0(X) \xrightarrow{j_\star} H_0(X, A) \to 0.$$

Proof : The proof is immediate from Theorem 7.1. □

Theorem 7.4 justifies that singular homology satisfies the exactness axiom.

7.3 Chain Homotopy Invariance

Let us recall the notion of *chain maps* $f_\bullet : C_\bullet \to D_\bullet$ (Definition 1.13), which is given a by a collection of maps $f_n : C_n \to D_n$ such that $f_{n-1} \circ \partial_n^C = \partial_n^D \circ f_n$ holds.

Given a chain complex $C_\bullet$, the *shift (or translation)* of $C_\bullet$ by degree k is the chain complex $C[k]_\bullet$ defined as $C[k]_n = C_{n+k}$ and $\partial_n^{C[k]} = (-1)^k \partial_{n+k}^C$. We shall see later why the sign $(-1)^k$ appears in the definition. A *degree k chain map* $C_\bullet \to D_\bullet$ is a chain map $C_\bullet \to D[k]_\bullet$. In particular, a chain map $C_\bullet \to D_\bullet$ has degree 0.

Exercise 7.5: (*Shift Functor*)

Let $\Sigma^k : \mathrm{Ch} \to \mathrm{Ch}$ be the functor that shifts a chain complex by degree k. Check that $\Sigma^k \circ \Sigma^l = \Sigma^{k+l}$ and $\Sigma^0 = \mathrm{Id}$. In particular, Σ^k is an *isomorphism* with inverse Σ^{-k}.

Definition 7.6: (*Chain Map and Chain Homotopy*)

Given two chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$, a *chain homotopy* between them is a degree 1 map $h_\bullet : C_\bullet \to D_{\bullet+1}$ such that $f - g = \partial \circ h + h \circ \partial$ holds. That is, we have the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, bend right, "f_{n+1}"'] \arrow[d, bend left, "g_{n+1}"] & C_n \arrow[r, "\partial^C_n"] \arrow[dl, "h_n"'] \arrow[d, bend right, "f_n"'] \arrow[d, bend left, "g_n"] & C_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"] \arrow[d, bend right, "f_{n-1}"'] \arrow[d, bend left, "g_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"'] & D_n \arrow[r, "\partial^D_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

where, we have $f_n - g_n = h_{n-1} \circ \partial^C_n + \partial^D_{n+1} \circ h_n$

Exercise 7.7: (*Chain Homotopy is an Equivalence Relation*)

Check that the chain homotopy is an equivalence relation on the collection of all chain maps $C_\bullet \to D_\bullet$.

Proposition 7.8: (*Homology of Chain Homotopic Maps*)

Suppose $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are homotopic chain maps. Then, they induce the same map between homology groups.

Proof : Suppose $f - g = \partial h + h\partial$ for a chain homotopy $h : C_\bullet \to D_{\bullet+1}$. Since homology is an additive functor (Exercise 6.7), we only need to show that $H_n(\partial h + h\partial) = 0$. For a class $[\alpha] \in H_n(C)$, we have $\partial(\alpha) = 0$. Recall, $[\alpha] = \alpha + \operatorname{im} \partial$ Then, it follows

$$H_n(\partial h + h\partial)[\alpha] = [(\partial h + h\partial)(\alpha)] = [\partial h(\alpha)] = 0.$$

Thus, $H_n(f) = H_n(g)$ follows for all n. $\square$

7.4 Singular Homology of Contractible Space

Before proving the homotopy invariance for singular homology, let us start with a contractible space, say, X. Fix a point $x_0 \in X$. Let us define a chain map $\varepsilon_\bullet : S_\bullet(X) \to S_\bullet(X)$ by setting $\varepsilon_n = 0$ for $n \neq 0$, and

$$\varepsilon_0\Big(\sum a_\sigma \sigma\Big) = \Big(\sum a_\sigma\Big)\sigma_0,$$

where $\sigma_0 : \Delta^0 \to X$ is the 0-simplex that maps to x_0.

Proposition 7.9:

$\varepsilon_\bullet : S_\bullet(X) \to S_\bullet(X)$ is a chain map.

Proof : The only nontrivial part is the commutativity of the diagram

$$\begin{tikzcd}
S_1(X) \arrow[r, "d_1"] \arrow[d, "\varepsilon_1 = 0"'] & S_0(X) \arrow[d, "\varepsilon_0"] \\
S_1(X) \arrow[r, "d_1"'] & S_0(X)
\end{tikzcd}$$

Suppose $\sigma : \Delta^1 \to X$ is a singular 1-simplex. Then,

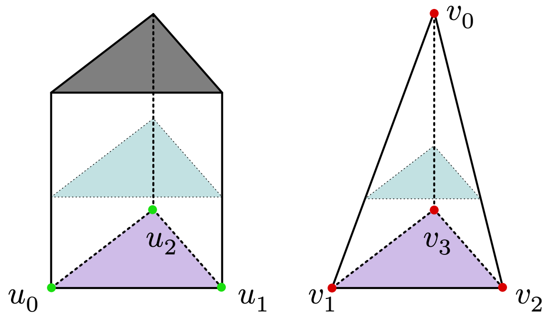
$$\varepsilon_0 d_1(\sigma) = \varepsilon_0(\sigma \circ d_0 - \sigma \circ d_1) = (1-1)\sigma_0 = 0.$$

Thus, $\varepsilon_0 d_1 = 0 = d_1 0$. This proves that $\varepsilon_\bullet$ is a chain map. $\square$

Now, suppose $h : X \times [0,1] \to X$ is a hometopy from Id_X to the constant map $c_{x_0} : X \to X$ which maps everything to a point $x_0 \in X$. Using h, we define a chain homotopy between $\varepsilon_\bullet$ and the identity chain map. Firstly, consider the map

$$\begin{aligned} q_n : \Delta^n \times [0,1] &\longrightarrow \Delta^{n+1} \\ ((t_0, ..., t_n), s) &\longmapsto (s, (1-s)t_0, ..., (1-s)t_n) \end{aligned}$$

Since $s + \sum_{i=0}^n (1-s)t_i = s + (1-s) = 1$, the map is well-defined, and clearly, it is continuous. As q_n is a surjective map between compact T^2 spaces, we have q_n is a quotient map.



The map $q_2 : \Delta^2 \times [0,1] \to \Delta^3$

In fact, it follows that q_n identifies $\Delta^n \times \{1\}$ to the vertex $v_0 = (1, 0, ..., 0) \in \Delta^{n+1}$, and keeps every other point as it is. Note that q_0 is the identity map $[0,1] \to [0,1]$. Now, given a singular n-simplex $\sigma : \Delta^n \to X$, consider the diagram

$$\begin{tikzcd} \Delta^n \times [0,1] \arrow[r, "\sigma \times \mathrm{Id}"] \arrow[d, "q_n"] & X \times [0,1] \arrow[r, "h"] & X \\ \Delta^{n+1} \arrow[urr, dashed, bend right, "s_n(\sigma)"'] & & \end{tikzcd}$$

Since $h(X \times \{1\}) = \{x_0\}$, it follows from the universal property of a quotient map that there exists a unique continuous map $s_n(\sigma) : \Delta^{n+1} \to X$ such that

$$s_n(\sigma) \circ q_n = h \circ \left(\sigma \times \mathrm{Id}_{[0,1]}\right).$$

Extending linearly, we have a map $s_n : S_n(X) \to S_{n+1}(X)$.

Now, observe that for $0 < i \leq n+1$ we have the diagram

$$
\begin{tikzcd}
\Delta^{n-1}\times[0,1] \arrow[r, "d^n_{i-1}\times \mathrm{Id}"] \arrow[d, "q_{n-1}"'] & \Delta^n\times[0,1] \arrow[r, "\sigma\times \mathrm{Id}"] \arrow[d, "q_n"] & X\times[0,1] \arrow[r, "h"] & X \\
\Delta^n \arrow[r, "d_i^{n+1}"'] \arrow[urrr, dashed, bend right=40, "s_{n-1}(\sigma\circ d^n_{i-1})"'] & \Delta^{n+1} \arrow[urr, dashed, "s_n(\sigma)"'] & &
\end{tikzcd}
$$

Hence, $s_n(\sigma)d_i^{n+1} = s_{n-1}(\sigma d^n_{i-1})$ holds for $0 < i \leq n+1$. Clearly, $s_n(\sigma)d_0^{n+1} = \sigma$.

Proposition 7.10:

$s_\bullet : S_\bullet(X) \to S_{\bullet+1}(X)$ is a chain homotopy betwen Id and $\varepsilon_\bullet$.

Proof : For a 0-simplex $\sigma : \Delta^0 \to X$, we have

$$\partial(s\sigma) = (s\sigma)d_0 - (s\sigma)d_1 = \sigma - \sigma_0,$$

since $h(\sigma(0), 1) = x_0$. For a singular n-simplex $\sigma : \Delta^n \to X$ with $n \geq 1$, we compute

$$
\begin{aligned}
\partial(s\sigma) &= \sum_{i=0}^{n+1} (-1)^i (s\sigma)d_i^{n+1} = (s\sigma)d_0^{n+1} - \sum_{i=1}^{n+1} (-1)^{i-1}(s\sigma)d_i^{n+1} \\
&= \sigma - \sum_{i=1}^{n+1} (-1)^{i-1} s(\sigma d^n_{i-1}) \\
&= \sigma - s\left(\sum_{i=0}^{n} (-1)^i \sigma d_i^n\right), \text{by replacing } i \leftrightarrow i-1, \text{and by linearity of } s \\
&= \sigma - s(\partial\sigma) \\
\Rightarrow \sigma - \underbrace{\varepsilon_n(\sigma)}_{0} &= \partial(s\sigma) + s(\partial\sigma).
\end{aligned}
$$

Thus, we have $\mathrm{Id} - \varepsilon = \partial s + s\partial$. In other words, $s_\bullet$ is a chain homotopy between Id and $\varepsilon_\bullet$ $\square$

Theorem 7.11: (*Singular Homology of Contractible Space*)

Let X be a contracitble space. Then, $H_n(X) = 0$ for all $n \neq 0$.

Proof : Suppose $h : X \times [0,1] \to X$ is a homotopy that takes Id_X to the constant map c_{x_0} for some point $x_0 \in X$. Then, as above, construct the chain map $\varepsilon_\bullet : S_\bullet(X) \to S_\bullet(X)$. Since by Proposition 7.10 Id is chain homotopic to ε, an application of Proposition 7.8 it follows that they give the same map in homology. But homology is an additive functor. Hence, $\mathrm{Id} = H_n(\mathrm{Id}) = H_n(\varepsilon) = 0$ for $n \neq 0$. The only way identity map can be the zero map if the object is zero. Thus, $H_n(X) = 0$ for $n \neq 0$. $\square$

Exercise 7.12: (0^{th}-*Homology and Path Component*)

Show that $H_0(X)$ is freely generated by the path components of X. In particular, $H_0(X) = \mathbb{Z}$ for a path connected space.

Day 8 : 10th February, 2026

`homotopy invariance of singular homology — barycentric subdivision`

8.1 Homotopy Invariance of Singular Homology

Our goal is to show that homotopic maps induce the same homomorphism of singular homology groups. Given a space X, consider the maps

$$\begin{aligned} \xi^X : X &\to X \times [0,1] \\ x &\mapsto (x,0). \end{aligned} \qquad \begin{aligned} \eta^X : X &\to X \times [0,1] \\ x &\mapsto (x,1). \end{aligned}$$

They are clearly natural in X, and hence, induce natural chain maps

$$\xi^X_\bullet, \eta^X_\bullet : S_\bullet(X) \to S_\bullet(X \times [0,1]).$$

In particular, $\xi_\bullet, \eta_\bullet : S_\bullet(\_) \Rightarrow S_\bullet(\_ \times [0,1])$ are natural transformations.

Proposition 8.1:

The chain maps $\xi_\bullet, \eta_\bullet$ are naturally chain homotopic.

Proof : We need to construct maps $s^X_n : S_n(X) \to S_{n+1}(X \times [0,1])$, such that it is a chain homotopy:

$$\eta^X_n - \xi^X_n = \partial s^X_n + s^X_{n-1}\partial.$$

Moreover, we need it to be natural: for any $f : X \to Y$, we require

$$S_{n+1}(f \times \mathrm{Id}) \circ s^X_n = s^Y_n \circ S_n(f).$$

We construct s^X_n inductively.

- For a 0-simplex $\sigma : \Delta^0 \to X$, let us define $s^X_0(\sigma) : \Delta^1 = \Delta^0 \times [0,1] \to X \times [0,1]$ by

  $$s^X_0(\sigma)(t) = (\sigma(0), t).$$

  It follows that

  $$\partial\big(s^X_0(\sigma)\big) = s^X_0(\sigma)d_0 - s^X_0(\sigma)d_1 = \eta^X_0\sigma - \xi^X_0\sigma.$$

  Naturality is apparent from the definition.
- Inductively assume that we have constructed s^X_k for $k < n$, for all space X. We construct s^X_n. The identity map $\mathrm{Id} : \Delta^n \to \Delta^n$ is a singular n-simplex, denote $\iota_n \in S_n(\Delta^n)$ to be the corresponding element. In order to define s^X_n, we require $s^{\Delta^n}_n$ to satisfy

  $$\partial s^{\Delta^n}_n(\iota_n) = \eta^{\Delta^n}_n(\iota_n) - \xi^{\Delta^n}_n(\iota_n) - s^{\Delta^n}_{n-1}(\partial\iota_n),$$

  where the right-hand-side is defined by induction. We have the following diagram

$$
\begin{tikzcd}
S_n(\Delta^n) \arrow[r, "\partial"] \arrow[d, shift right, "\eta_n^{\Delta^n}"'] \arrow[d, shift left, "\xi_n^{\Delta^n}"] & S_{n-1}(\Delta^n) \arrow[r, "\partial"] \arrow[dl, "s_{n-1}^{\Delta^n}"'] \arrow[d, shift right, "\eta_{n-1}^{\Delta^n}"'] \arrow[d, shift left, "\xi_{n-1}^{\Delta^n}"] & S_{n-2}(\Delta^n) \arrow[dl, "s_{n-2}^{\Delta^n}"] \\
S_n(\Delta^n \times I) \arrow[r, "\partial"'] & S_{n-1}(\Delta^n \times I) &
\end{tikzcd}
$$

Observe that

$$
\begin{aligned}
&\partial\big(\eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial\iota_n)\big) \\
&= \eta_{n-1}^{\Delta^n}(\partial\iota_n) - \xi_{n-1}^{\Delta^n}(\partial\iota_n) - \partial\big(s_{n-1}^{\Delta^n}(\partial\iota_n)\big), \quad \text{since } \xi, \eta \text{ are chain maps} \\
&= \eta_{n-1}^{\Delta^n}(\partial\iota_n) - \xi_{n-1}^{\Delta^n}(\partial\iota_n) - \big(\eta_{n-1}^{\Delta^n}(\partial\iota_n) - \xi_{n-1}^{\Delta^n}(\partial\iota_n) - s_{n-2}^{X}(\partial\partial\iota_n)\big), \\
&\qquad\qquad\qquad\qquad \text{as } s_{n-1}^{\Delta^n}, s_{n-2}^{\Delta^n} \text{ are part of the chain homotopy} \\
&= 0, \quad \text{as } \partial\partial\iota_n = 0.
\end{aligned}
$$

In other words, the RHS is an n-cycle. Since $\Delta^n \times [0,1]$ is contractible, by Theorem 7.11, the RHS is a boundary. In particular, we can *choose* some singular $(n+1)$-chain $a \in S_{n+1}(\Delta^n \times I)$ such that

$$\partial a = \eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial\iota_n).$$

Set, $s_n^{\Delta^n}(\iota_n) = a$. Then, for any space X and any singular n-simplex $\sigma : \Delta^n \to X$, set

$$s_n^X(\sigma) = S_{n+1}(\sigma \times \mathrm{Id})(a).$$

Let us verify the required conditions. We have the diagram

$$
\begin{tikzcd}
& & & S_n(\Delta^n) \arrow[rr, "\partial"] \arrow[dl, "S_n(\sigma)"'] \arrow[ddll, "s_n^{\Delta^n}"'] \arrow[dd, shift right, "\eta_n^{\Delta^n}"'] \arrow[dd, shift left, "\xi_n^{\Delta^n}"] & & S_{n-1}(\Delta^n) \arrow[dl, "S_{n-1}(\sigma)"'] \arrow[ddll, "s_n^{\Delta^n}"'] \arrow[dd, shift right, "\eta_{n-1}^{\Delta^n}"'] \arrow[dd, shift left, "\xi_{n-1}^{\Delta^n}"] \\
& & \overset{\sigma}{S_n(X)} \arrow[rr, "\partial"] \arrow[ddll, "s_n^X"'] \arrow[dd, shift right, "\eta_n^X"'] \arrow[dd, shift left, "\xi_n^X"] & & S_{n-1}(X) \arrow[ddll, "s_{n-1}^X"'] \arrow[dd, shift right, "\eta_n^X"'] \arrow[dd, shift left, "\xi_{n-1}^X"] & \\
& \overset{a}{S_{n+1}(\Delta^n \times I)} \arrow[rr, "\partial"] \arrow[dl, "S_{n+1}(\sigma \times \mathrm{Id})"'] & & S_n(\Delta^n \times I) \arrow[rr, "\partial"] \arrow[dl, "S_n(\sigma \times \mathrm{Id})"'] & & S_{n-1}(\Delta^n \times I) \arrow[dl, "S_{n-1}(\sigma \times \mathrm{Id})"'] \\
S_{n+1}(X \times I) \arrow[rr, "\partial"] & & S_n(X \times I) \arrow[rr, "\partial"] & & S_{n-1}(X \times I) &
\end{tikzcd}
$$

We compute

$$
\begin{aligned}
\partial s_n^X(\sigma) &= \partial S_{n+1}(\sigma \times \mathrm{Id})(a) = S_n(\sigma \times \mathrm{Id})(\partial a) \\
&= S_n(\sigma \times \mathrm{Id})\big(\eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial\iota_n)\big) \\
&= \eta_n^X S_n(\sigma)(\iota_n) - \xi_n^X S_n(\sigma)(\iota_n) - s_{n-1}^X S_{n-1}(\sigma)(\partial\iota_n), \\
&\qquad\qquad \text{as } \xi^X, \eta^X \text{ are natural transformations, and } s_{n-1} \text{ is natural} \\
&= \eta_n^X S_n(\sigma)(\iota_n) - \xi_n^X S_n(\sigma)(\iota_n) - s_{n-1}^X \partial S_n(\sigma)(\iota_n) \\
&= \eta_n^X \sigma - \xi_n^X \sigma - s_{n-1}^X \partial\sigma.
\end{aligned}
$$

Thus, s_n^X satisfies the chain homotopy condition. Next, consider a continuous map $f : X \to Y$. We have the diagram

$$\begin{tikzcd}
{} & \overset{\sigma}{S_n(X)} \arrow[rr, "S_n(f)"] \arrow[dl, "s_n^X"'] & & \overset{f\circ\sigma}{S_n(Y)} \arrow[dl, "s_n^Y"'] \\
S_{n+1}(X\times I) \arrow[rr, "S_n(f\times \mathrm{Id})"'] & & S_{n+1}(Y\times I) & \\
& \underset{a}{S_{n+1}(\Delta^n\times I)} \arrow[ul, bend left, "S_{n+1}(\sigma\times \mathrm{Id})"] \arrow[ur, bend right, "S_{n+1}((f\circ\sigma)\times \mathrm{Id})"'] & &
\end{tikzcd}$$

We compute

$$\begin{aligned}
S_{n+1}(f\times \mathrm{Id})s_n^X(\sigma) &= S_{n+1}(f\times \mathrm{Id})S_{n+1}(\sigma\times \mathrm{Id})(a) \\
&= S_{n+1}((f\circ\sigma)\times \mathrm{Id})(a) \\
&= s_n^Y(f\circ\sigma) \\
&= s_n^Y S_n(f)(\sigma).
\end{aligned}$$

This proves naturality of s_n

Hence, inductively, we have a natural chain homotopy $s_\bullet : \xi_\bullet \simeq \eta_\bullet$. □

Note that the chain homotopy obtained in Proposition 8.1 is not unique since we made a choice at each induction step, but the homotopy is still natural.

Theorem 8.2: (*Singular Homology is Homotopy Invariant*)

Suppose $f, g : (X, A) \to (Y, B)$ are homotopic maps. Then, $H_n(f) = H_n(g) : H_n(X, A) \to H_n(Y, B)$ for all n.

Proof : Suppose $h : (X, A) \times I \to (Y, B)$ is a homtopy $f \simeq g$. We have the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\eta^A", "\xi^A"'] \arrow[d, hook, "\iota"'] & A\times[0,1] \arrow[r, "h|_{A\times[0,1]}"] \arrow[d, hook, "\iota\times\mathrm{Id}"'] & B \arrow[d, hook] \\
X \arrow[r, "\eta^X", "\xi^X"'] & X\times[0,1] \arrow[r, "h"'] & Y
\end{tikzcd}$$

where, $\xi^X(x) = (x, 0), \eta^X(x) = (x, 1), \xi^A(a) = (a, 0), \eta^A(a) = (a, 1)$. By Proposition 8.1, we have a natural chain homotopy $s_\bullet^X : S_\bullet(\eta^X) \simeq S_\bullet(\xi^X)$ and $s_\bullet^A : S_\bullet(\eta^A) \simeq S_\bullet(\xi^A)$. Naturality implies

$$S_{n+1}(\iota \times \mathrm{Id}) \circ s_n^A = s_n^X \circ S_n(\iota).$$

Hence, we can define $s_n^{X,A} : S_n(X, A) \to S_{n+1}(X \times [0, 1], A \times [0, 1])$, which is clearly a chain homtopy $S_\bullet(\eta^{X,A}) \simeq S_\bullet(\xi^{X,A})$. Then,

$$S_n(g) - S_n(f) = S_n(h \circ \eta^{X,A}) - S_n(h \circ \xi^{X,A}) = S_n(h)(S_n(\eta^{X,A}) - S_n(\xi^{X,A}))$$

$$= S_n(h) \circ \left(\partial s_n^{X,A} + s_{n-1}^{X,A}\partial\right) = \partial\left(S_{n+1}(h)s_n^{X,A}\right) + \left(S_n(h)s_{n-1}^{X,A}\right)\partial.$$

Thus, $\zeta_n = S_{n+1}(h)s_n^{X,A} : S_n(X) \to S_{n+1}(Y)$ is a chain homtopy $S_\bullet(g) \simeq S_\bullet(f)$. Hence, by Proposition 7.8, we have $H_n(f) = H_n(g)$. □

8.2 Barycentric Subdivision

Our next goal is to show that singular homology satisfy the exicion axiom. For that we first need to construct *barycentric subdivision* of simplicies.

Let $D \subset \mathbb{R}^n$ be a convex set. Given points $v_0, \dots, v_p \in D$, the *affine singular p-simplex* is defined as

$$\begin{aligned} \sigma = s\left[v_0, \dots, v_p\right] &: \Delta^p \to D \\ \sum_{i=0}^{p} \lambda_i e_i &\mapsto \sum_{i=0}^{p} \lambda_i v_i, \end{aligned}$$

where e_i are the standard unit vectors of $\mathbb{R}^{p+1}$. Note that convexitty of D implies that σ is well-defined. The *barycenter* of σ is defined as

$$\sigma^\beta := \frac{1}{p+1}\sum_{i=0}^{p} v_i.$$

In particular, we shall be interested in the identity map $\iota_p : \Delta^p \to \Delta^p$ and the corresponding barycenter ι_p^β.

Now, D being convex, is contractible. In fact, for each $v \in D$, we have a contracting homotopy

$$\begin{aligned} H_v : D \times [0,1] &\to D \\ (u,t) &\mapsto (1-t)u + tv. \end{aligned}$$

Using the cone construction from earlier, we then get a chain homotopy $S_\bullet(D) \to S_{\bullet+1}(D)$ (Proposition 7.10). Let us denote this chain homotopy as

$$\begin{aligned} S_p(D) &\to S_{p+1}(D) \\ \sigma &\mapsto v \cdot \sigma. \end{aligned}$$

In particular, the affine simplex $\sigma = \left[v_0, \dots, v_p\right]$ is mapped to $v \cdot \sigma := \left[v, v_0, \dots, v_p\right]$. Recall from Proposition 7.10 that

$$\partial(v \cdot \sigma) = \begin{cases} \sigma - v \cdot \partial\sigma, & p > 0, \\ \sigma - \varepsilon(\sigma)v, & p = 0, \end{cases}$$

where $\varepsilon : S_0(D) \to \mathbb{Z}$ is given by $\varepsilon(\sum n_\sigma \sigma) = \sum n_\sigma$.

Let us now inductively define an operator $\mathcal{B}_p = \mathcal{B}_p(X) : S_p(X) \to S_p(X)$ for any space X, called the *barycentric subdivision*. For any $\sigma : \Delta^p \to X$ define,

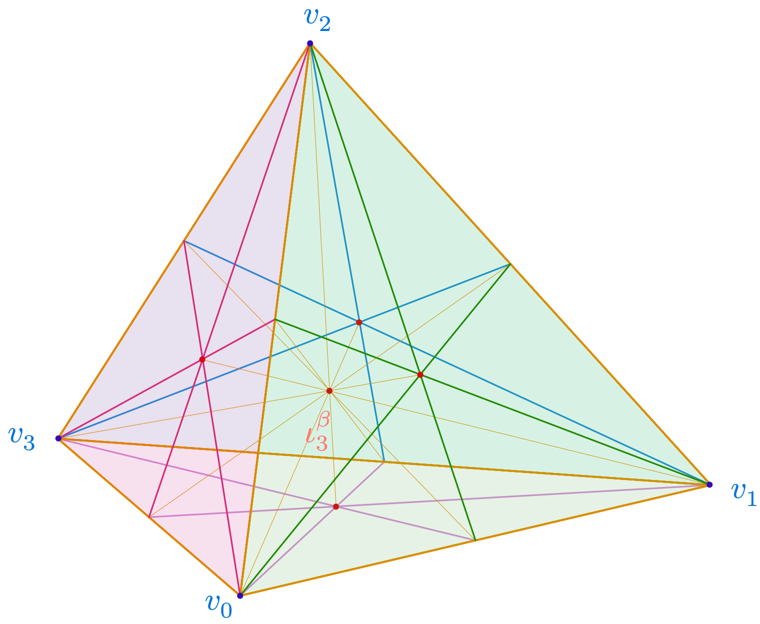
$$\mathcal{B}_p^X(\sigma) = S_p(\sigma)\mathcal{B}_p^{\Delta^p}(\iota_p),$$

where

$$\mathcal{B}_p^{\Delta^p}(\iota_p) = \begin{cases} i_0, & p = 0 \\ \iota_p^\beta \cdot \mathcal{B}_{p-1}^{\Delta^p}(\partial\iota_p), & p > 0. \end{cases}$$

Clearly this is well-defined.

Let us visualize the barycentric subdivision of ι_3, the identity simplex of Δ^3.



Barycentric sub-division of Δ^3

On each fact, we have the corresponding barycenter. For the 1-simplex $[0, 1]$, the barycenter is the just midpoint. The sub-dvision operator is taking the *cone* over the simplices of the barycentric sub-division of the lower-dimensional face. You can see that there are 24 (check!) new Δ^3 after subdividing it once.

Day 9 : 13th February, 2026

`chain homotopy of barycentric subdivision – iterated barycentric subdivision – excision in singular homology – additivity in singular homology – Hurewicz homomorphism`

9.1 Barycentric Subdivision and Chain Homtopy

Intuitively, barycentric subdivision divides a given chain into smaller pieces with matching boundary, so that it should not change the boundary, and have no effect on the homology. We show that this operator is in fact a natrual chain map, which is naturally chain homotopic to the identity.

Proposition 9.1: (*Barycentric Subdivision is a Natural Chain Map*)

The barycentric subdivision operator $\mathcal{B}^X_\bullet : S_\bullet(X) \to S_\bullet(X)$ is a chain map, which is natural in X. Moreover, it is naturally chain homotopic to the identity map.

Proof : Given any $f : X \to Y$, we have

$$S_n(f)\mathcal{B}^X_n(\sigma) = S_n(f)S_n(\sigma)\mathcal{B}^{\Delta^p}_n(\iota_n) = S_n(f\circ\sigma)\mathcal{B}^{\Delta^n}_n(\iota_n) = \mathcal{B}^Y_n(f\circ\sigma) = \mathcal{B}^Y_n S_n(f)(\sigma),$$

which proves the naturality. Next, we check

$$\partial\mathcal{B}^{\Delta^p}_p(\iota_p) = \partial(\iota^\beta_p \cdot \mathcal{B}^{\Delta^p}_{p-1})(\partial\iota_p) = \mathcal{B}^{\Delta^p}_{p-1}(\partial\iota_p) - \iota^\beta_p\cdot\partial\partial\iota_p = \mathcal{B}^{\Delta^p}_{p-1}(\partial\iota_p), \quad p > 0,$$

and $\partial\mathcal{B}^{\Delta^0}_0(\iota_0) = \partial(\iota_0) = 0$. Then, for any $\sigma : \Delta^p \to X$ we have

$$\begin{aligned}\mathcal{B}^X_{p-1}\partial\sigma &= \mathcal{B}^X_{p-1}\partial S_p(\sigma)\iota_p = \mathcal{B}^X_{p-1}S_{p-1}(\sigma)(\partial\iota_p) = S_{p-1}(\sigma)\mathcal{B}^{\Delta_p}_{p-1}(\partial\iota_p)\\ &= S_{p-1}(\sigma)\partial\mathcal{B}^{\Delta_p}_{p-1}(\iota_p) = \partial S_p(\sigma)\mathcal{B}(\iota_p) = \partial\mathcal{B}(\sigma).\end{aligned}$$

This proves that $\mathcal{B}^X_\bullet$ is a chain map.

Next, we construct a natural chain homotopy $\mathrm{Id} \simeq \mathcal{B}^X_\bullet$. We define $\mathcal{T}^X_n : S_n(X) \to S_{n+1}(X)$ inductively. For any $\sigma : \Delta^n \to X$, set $\mathcal{T}^X_n(\sigma) = S_{n+1}(\sigma)\mathcal{T}^{\Delta^n}_n(\iota_n)$, where we have

$$\mathcal{T}^{\Delta^n}_n(\iota_n) = \iota^\beta_n\cdot\big(\iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n)\big).$$

Assume inductively that $\mathcal{T}^X_p$ is natural, and part of the chain homotopy for $p < n$. Then we compute

$$\begin{aligned}\partial\mathcal{T}^{\Delta^n}_n(\iota_n) &= \partial\big(\iota^\beta_n\cdot\big(\iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n)\big)\big)\\ &= \iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n) - \iota^\beta_n\cdot\big(\partial\iota_n - \partial\mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n)\big)\\ &= \iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n) - \iota^\beta_n\cdot\big(\mathcal{B}^{\Delta^n}_{n-1}(\partial\iota_n) + \mathcal{T}^{\Delta^n}_{n-2}(\partial\partial\iota_n)\big), \quad \text{by induction}\\ &= \iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n) - \iota^\beta_n\cdot\mathcal{B}^{\Delta^n}_{n-1}(\partial\iota_n), \quad \text{as } \partial\partial\iota_n = 0\\ &= \iota_n - \mathcal{B}^{\Delta^n}_n(\iota_n) - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n), \quad \text{by definition of } \mathcal{B}^{\Delta^n}_n(\iota_n).\end{aligned}$$

Now, for any $\sigma : \Delta^n \to X$ we have

$$\begin{aligned}\partial\mathcal{T}^X_n(\sigma) &= \partial S_{n+1}(\sigma)\mathcal{T}^{\Delta^n}_n(\iota_n) = S_n(\sigma)\partial\mathcal{T}^{\Delta^n}_n(\iota_n)\\ &= S_n(\sigma)(\iota_n) - S_n(\sigma)\mathcal{B}^{\Delta^n}_n(\iota_n) - S_n(\sigma)\mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n)\\ &= \sigma - \mathcal{B}^X_n(\sigma) - \mathcal{T}^X_{n-1}S_{n-1}(\sigma)(\partial\iota_n), \text{by definition and by naturality}\end{aligned}$$

$$= \sigma - \mathcal{B}_n^X(\sigma) - \mathcal{T}_{n-1}^X \partial S_n(\sigma)(\iota_n)$$
$$= \sigma - \mathcal{B}_n^X(\sigma) - \mathcal{T}_{n-1}^X \partial\sigma.$$

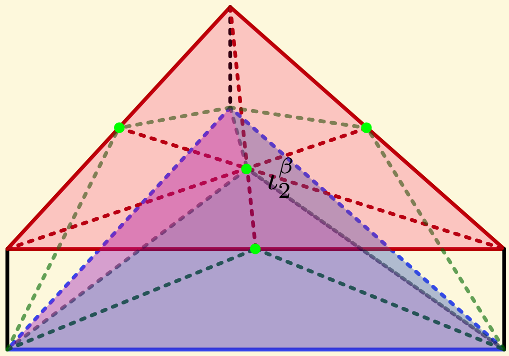
Thus, $\mathcal{T}_n^X$ is part of the required chain homotopy. Also, for any $f : X \to Y$, we have

$$S_{n+1}(f)\mathcal{T}_n^X(\sigma) = S_{n+1}(f)S_{n+1}(\sigma)\mathcal{T}_n^{\Delta^n}(\iota_n) = S_{n+1}(f\sigma)\mathcal{T}_n^{\Delta^n}(\iota_n) = \mathcal{T}_n^Y(f\sigma) = \mathcal{T}_n^Y S_n(f)(\sigma),$$

which proves the naturality. Hence, by induction, we see that $\mathcal{T}_\bullet^X : \mathrm{Id} - \mathcal{B}_\bullet^X$ is a chain homotopy, which is natural in X. $\square$

Remark 9.2:

Here is the geometric intution behidn the chain homotopy constructed in Proposition 9.1. Given the n -simplex $\iota_n : \Delta^n \to \Delta^n$, the homotopy can be thought of as a homtopy $\Delta^n \times [0,1] \to \Delta^n$, with ι_n on $\Delta^n \times \{0\}$ and $\mathcal{B}_n^{\Delta^n}(\iota_n)$ on $\Delta^n \times \{1\}$.



The chain homotopy for $\iota_2 : \Delta^2 \to \Delta^2$ with $\mathcal{B}_2^{\Delta^2}(\iota_2)$.

We can do this by coning over all the simplices of $\Delta^n \times \{0\} \cup \partial\Delta^n \times [0,1]$ with the barycenter of $\Delta^n \times \{1\}$ as the coning point.

9.2 Iterated Barycentric Subdivision

Given a space X, consider $\mathcal{U}$ to be a collection of subsets such that $X = \cup_{U \in \mathcal{U}} \mathring{U}$, i.e, interiors of the sets from $\mathcal{U}$ form an open cover of X. A singular n-simplex $\sigma : \Delta^n \to X$ is called *$\mathcal{U}$-small* if $\sigma(\Delta^n) \subset U$ for some $U \in \mathcal{U}$. Note that the simplicies appearing in the boundary of an $\mathcal{U}$-small simplex is again $\mathcal{U}$-small. Hence, we can define $S_\bullet^{\mathcal{U}}(X) \subset S_\bullet(X)$ to be the chain complex generated by $\mathcal{U}$-small singular simplices, with (the restriction of) the usual boundary map. The corresponding homology is denoted as $H_\bullet^{\mathcal{U}}(X)$. Our goal is to show that $H_\bullet^{\mathcal{U}}(X)$ is isomorphic to $H_\bullet(X)$. We achieve this by representing a homology class by a chain of $\mathcal{U}$-small simplices, by iterating the barycentric subdivision finitely many times.

Lemma 9.3: (*Diameter of Affine Singular Simplex*)

Let $v_0, ..., v_p \in \mathbb{R}^n$. Then, $\mathcal{B}_p[v_0, ..., v_p]$ is a linear combination of affine simplices, each with diameter at most $\frac{p}{p+1}$ $\operatorname{diam}[v_0, ..., v_p] = \frac{p}{p+1}\max_{i,j}\|v_i - v_j\|$, where we consider the Euclidean norm.

Proof : First, let us observe that $\operatorname{diam}\ [v_0, ..., v_p] = \max_{i,j}\|v_i - v_j\|$. Let $x, y \in [v_0, ..., v_p]$, and say $x = \sum \lambda_i v_i$ with $\sum \lambda_i = 1$ and $\lambda_i \geq 0$. Then,

$$\|x - y\| = \left\|\sum \lambda_i v_i - \sum \lambda_i y\right\| = \left\|\sum \lambda_i (v_i - y)\right\| \leq \sum \lambda_i \|v_i - y\| \leq \max_i \|v_i - y\|$$

In particular, taking $y = v_j$, we have $\|x - v_j\| \leq \max_i\|v_i - v_j\|$. But then for arbitrary x, y we have $\|x - y\| \leq \max_i\|v_i - y\| \leq \max_{i,j}\|v_i - v_j\|$. Thus, $\operatorname{diam}[v_0, ..., v_p] \leq \max_{i,j}\|v_i - v_j\|$. Clearly the diameter is attained as well, which shows the equality.

Next, we inductively show that diameter of each simplex in $\mathcal{B}_p[v_0, ..., v_p]$ is at most $\frac{p}{p+1}\max_{i,j}\|v_i - v_j\|$. It is trivial for $p = 0$. We have

$$\mathcal{B}_p[v_0, ..., v_p] = S_p(\sigma)\left(\iota_p^\beta \cdot \mathcal{B}_{p-1}(\partial \iota_p)\right) = \sum (-1)^i \sigma^\beta \cdot \mathcal{B}_{p-1}[v_0, ..., \hat{v}_\imath, ..., v_p],$$

where $\sigma = [v_0, ..., v_p]$ is the affine singular simplex. By induction, the simplices in $\mathcal{B}_{p-1}[v_0, ..., \hat{v}_\imath, ..., v_p]$ has diameter at most

$$\frac{p-1}{p}\operatorname{diam}[v_0, ..., \hat{v}_\imath, ..., v_p] \leq \frac{p-1}{p}\operatorname{diam}\ [v_0, ..., v_p] < \frac{p}{p+1}\operatorname{diam}\ [v_0, ..., v_p]$$

. The simplices in $\mathcal{B}_p[v_0, ..., v_p]$ has vertices σ^β and the vertices from $\mathcal{B}_{p-1}[v_0, ..., \hat{v}_\imath, ...v_p]$. Let us compute $\sup\{\|\sigma^\beta - x\| \mid x \in [v_0, ..., v_p]\}$. Say $x = \sum \lambda_i v_i \in [v_0, ..., v_p]$. Then, $\|\sigma^\beta - x\| \leq \max_i\|\sigma^\beta - v_i\|$. Now,

$$\begin{aligned} \|\sigma^\beta - v_j\| &= \left\|\frac{1}{p+1}\left(\sum v_i\right) - v_j\right\| = \frac{1}{p+1}\left\|\sum(v_i - v_j)\right\| \\ &\leq \frac{1}{p+1}\sum\|v_i - v_j\| \leq \frac{p}{p+1}\max_{i,j}\|v_i - v_j\| \\ &= \frac{p}{p+1}\operatorname{diam}[v_0, ..., v_p]. \end{aligned}$$

The claim then follows by induction. □

Lemma 9.4: (*Iterated Barycentric Subdivision*)

The inclusion of chain complex $S_\bullet^{\mathcal{U}}(X) \to S_\bullet(X)$ induces an isomorphism in homology.

Proof : Suppose $\sigma : \Delta^n \to X$ is a singular n-simplex. Then, consider the collection $\mathcal{U}_\sigma = \left\{\sigma^{-1}\left(\mathring{U}\right) \mid U \in \mathcal{U}\right\}$, which is an open cover of the compact metric space Δ^n. Then, there exists a Lebesgue number, say, $\delta > 0$ for this covering. Suppose $k = k(\sigma)$ is a natural number such that $\left(\frac{n}{n+1}\right)^k \operatorname{diam}[e_0, ..., e_n] < \delta$. Then, it follows from Lemma 9.3 that each of the simplices appearing in $\mathcal{B}^{\circ k}[e_0, ..., e_n]$ is contained in one of the sets from $\mathcal{U}_\sigma$. But then $\mathcal{B}^{\circ k}(\sigma)$ is $\mathcal{U}$-small by naturality.

Next, using Proposition 9.1, let us show that $\mathcal{B}_\bullet^{\circ k} \simeq \operatorname{Id}$. Indeed, consider

$$\mathcal{T}^k := \sum_{i=0}^{k-1} \mathcal{T}\mathcal{B}^{\circ i}.$$

Then, we have

$$\partial\mathcal{T}^k + \mathcal{T}^k\partial = \sum \partial\mathcal{T}\mathcal{B}^{\circ i} + \mathcal{T}\mathcal{B}^{\circ i}\partial = \sum(\partial\mathcal{T} + \mathcal{T}\partial)\mathcal{B}^{\circ i} = \sum(\mathrm{Id} - \mathcal{B})\mathcal{B}^{\circ i} = \mathrm{Id} - \mathcal{B}^{\circ k}.$$

Thus, $\mathcal{T}^k : \mathrm{Id} \simeq \mathcal{B}^{\circ k}$ is a chain homotopy, which is natural by construction.

Finally, let us show that the induced map $\Theta : H_\bullet^{\mathcal{U}}(X) \to H_\bullet(X)$ is an isomorphism.

- **Injectivity** : Suppose $\Theta(\alpha) = 0$ for some $\alpha \in H_n^{\mathcal{U}}(X)$. Say, $\alpha = [a]$ for some n-cycle $a \in S_n^{\mathcal{U}}(X)$. Then, $a = \partial b$ for some $b \in S_{n+1}(X)$. By using Lemma 9.3 to each of the simplices appearing in b, we have some k such that $\mathcal{B}^{\circ k}(b)$ is $\mathcal{U}$-small. Then, $b - \mathcal{B}^{\circ k}(b) = \partial\mathcal{T}^k(b) + \mathcal{T}^k\partial b = \partial\mathcal{T}^k(b) + \mathcal{T}^k(a) \Rightarrow \partial b - \partial\mathcal{B}^{\circ k}(b) = 0 + \partial\mathcal{T}^k(a) \Rightarrow a = \partial b = \partial\big(\mathcal{T}^k(a) + \mathcal{B}^{\circ k}(b)\big)$. Now, by construction, $\mathcal{T}$ takes $\mathcal{U}$-small chain to $\mathcal{U}$-small ones, and hence, $\mathcal{T}^k(a) \in S_{n+1}^{\mathcal{U}}(X)$. Clearly $\mathcal{B}^{\circ k}(b) \in S_{n+1}^{\mathcal{U}}(X)$. Thus, a is an $\mathcal{U}$-small boundary. In other words, $\alpha = [a] = 0$ in $H_n^{\mathcal{U}}(X)$. This shows, Θ is injective.
- **Surjectivity** : Next, suppose $a \in S_n(X)$ is a cycle representing a class in $H_n(X)$. Then, there exists some k such that $\mathcal{B}^{\circ k}(a)$ is an $\mathcal{U}$-small n-chain. As $\mathcal{B}^{\circ k}$ is a chain map, it follows that $\mathcal{B}^{\circ k}(a)$ is a cycle. Also, $a - \mathcal{B}^{\circ k}(a) = \partial\mathcal{T}^k(a) + \mathcal{T}^k(\partial a) = \partial\mathcal{T}^k(a)$ implies that a is homologous to the cycle $\mathcal{B}^{\circ k}(a)$, which represents a class in $H_n^{\mathcal{U}}(X)$. Clearly, $\theta[\mathcal{B}^{\circ k}(a)] = [a]$, proving the surjectivity.

Thus, $\Theta : H_\bullet^{\mathcal{U}}(X) \to H_\bullet(X)$ is an isomorphism. □

Let us also show the same as above for a pair (X, A). Denote $\mathcal{U} \cap A := \{U \cap A \mid U \in \mathcal{U}\}$. Clearly, the interiors of sets of $\mathcal{U} \cap A$ covers A. Let us defien

$$S_\bullet^{\mathcal{U}}(X, A) := \frac{S_\bullet^{\mathcal{U}}(X)}{S_\bullet^{\mathcal{U}\cap A}(A)},$$

which is clearly a chain complex. Denote the homology as $H_\bullet^{\mathcal{U}}(X, A)$

Lemma 9.5: (*Barycentrinc Subdivision in Relative Homology*)

The inclusion map $\theta : S_\bullet^{\mathcal{U}}(X, A) \to S_\bullet(X, A)$ induces an isomorphism $\Theta : H_\bullet^{\mathcal{U}}(X, A) \to H_\bullet(X, A)$.

Proof : We have a commutative diagram of short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet^{\mathcal{U}\cap A}(A) \arrow[r] \arrow[d, "\theta_A"'] & S_\bullet^{\mathcal{U}}(X) \arrow[r] \arrow[d, "\theta_X"'] & S_\bullet^{\mathcal{U}}(X, A) \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & S_\bullet(A) \arrow[r] & S_\bullet(X) \arrow[r] & S_\bullet(X, A) \arrow[r] & 0
\end{tikzcd}$$

By Theorem 7.1, we have the commutative diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_n^{\mathcal{U}\cap A}(A) \arrow[r] \arrow[d, "H_n(\theta_A)"'] & H_n^{\mathcal{U}}(X) \arrow[r] \arrow[d, "H_n(\theta_X)"'] & H_n^{\mathcal{U}}(X, A) \arrow[r] \arrow[d, "H_n(\theta)"'] & H_{n-1}^{\mathcal{U}\cap A}(A) \arrow[r] \arrow[d, "H_{n-1}(\theta_A)"'] & H_{n-1}^{\mathcal{U}}(X) \arrow[r] \arrow[d, "H_{n-1}(\theta_X)"'] & \cdots \\
\cdots \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

By Lemma 9.4, 4-out-of-5 vertical arrows are isomorphisms. Hence, by the 5-lemma, it follows that $H_n(\theta)$ is an isomorphism as well. $\square$

9.3 Excision in Singular Homology

We are now in a position to prove the excision theorem in singular homology.

Theorem 9.6: (*Exicision*)

Given a pair (X, A) and a subset $B \subset A$, the inclusion

$$\iota : (X \setminus B, A \setminus B) \hookrightarrow (X, A)$$

induces an isomorphism in homology, provided $\overline{B} \subset \mathring{A}$ holds.

Proof : Let us get an equivalent statement first. Set

$$U := A, \quad V := X \setminus B.$$

Then, $\overline{B} \subset B \Rightarrow X \setminus \overline{B} \subset X \setminus B = V \Rightarrow X \setminus \overline{B} \subset \mathring{V}$. Since $\overline{B} \subset \mathring{A} = \mathring{U}$, it follows that the interiors of U, V cover X. Thus, we can now apply Lemma 9.4 for the collection $\mathcal{U} := \{U, V\}$.

Observe that $S_\bullet^{\mathcal{U}}(X) = S_\bullet(U) + S_\bullet(V)$, as an $\mathcal{U}$-small simplex is either contained in U or in V (and possibly in both). Also, $S_\bullet(U) \cap S_\bullet(V) = S_\bullet(U \cap V)$. Note that

$$\frac{S_n(V)}{S_n(U \cap V)} = \frac{S_n(V)}{S_n(U) \cap S_n(V)} \simeq \frac{S_n(U) + S_n(V)}{S_n(U)} = \frac{S_n^{\mathcal{U}}(X)}{S_n(U)}.$$

The isomorphism in the middle is the second isomorphism theorem. In particular, it is induced by the natural inclusion $S_n(V) \subset S_n(U) + S_n(V)$. Observe that in Lemma 9.4, the barycentric subdivision and the chain homotopy (and hence, their iterates), takes chains in $S_\bullet(A)$ to itself. In particular, passing to the quotient, we see that $\frac{S_\bullet^{\mathcal{U}}(X)}{S_\bullet(U)} \to \frac{S_\bullet(X)}{S_\bullet(U)}$ induces isomorphism in homology. Alternatively, observe that $S_\bullet^{\mathcal{U} \cap A}(A) = S_\bullet(A)$, and so, we can use Lemma 9.5 for $S_\bullet^{\mathcal{U}}(X, A) = \frac{S_n^\bullet(X)}{S_\bullet(U)}$. Hence, we have isomorphism $H_\bullet(V, U \cap V) \to H_\bullet(X, U)$.

Translating back to our original notaion, we have $V = X \setminus B, U \cap V = A \cap (X \setminus B) = A \setminus B$. Hence, we have the exicision isomorphism $H_\bullet(X \setminus B, A \setminus B) \to H_\bullet(X, A)$. $\square$

9.4 Additivity of Singular Homology

As proved in Proposition 3.1, we immediately have that singular homology is finitely additve. But it is more than that!

Theorem 9.7: (*Homology of Path Components*)

Given a space X, we have $H_n(X) = \bigoplus_{P \in \pi_0(X)} H_n(P)$, where $\pi_0(X)$ is the set of path components of X. In particular, singular homology is additive (Definition 3.3).

Proof : Since Δ^n is path connected, and n-simplex is contained in a path component. Thus, we have a natural decomposion $S_n(X) = \bigoplus_{P \in \pi_0(X)} S_n(P)$. Clearly the boundary map restricts to each summand. Then, kernel and image splits as well. We have, $H_n(X) = \bigoplus H_n(P)$.

Given a disjoint union $X = \sqcup X_\alpha$, each X_α is union of path components. In particular, the above argument works and gives $H_\bullet(X) = \bigoplus H_\bullet(X_\alpha)$. This shows that singular homology is additive. $\square$

Thus, when computing the singular homology, we might as well assume that the space is path connected.

9.5 Hurewicz Homomorphism

Since $\Delta^1 = \{(t_0, t_1) \in \mathbb{R}^2 \mid t_0 + t_1 = 1, t_0 \geq 0, t_1 \geq 0\}$ is essentially the interval, a singular 1-simplex is nothing but a path in the space X. Given $\sigma : \Delta^1 \to X$, let us consider the path

$$\begin{aligned} P(\sigma) : [0,1] &\to X \\ t &\mapsto \sigma(1-t, t). \end{aligned}$$

Conversely, any path $\gamma : [0,1] \to X$ can be thought of as a 1-simplex $S(\gamma)(t_0, t_1) = \gamma(t_1)$. Clearly, $S \circ P = \mathrm{Id} = P \circ S$. The standard $\frac{1}{2}$-concatenation of paths then gives a concatenation of 1-simplices as $\sigma, \tau : \Delta^1 \to X$, with $\sigma(0,1) = \tau(1,0)$ via $\sigma \star \tau = S(P(\sigma) \star P(\tau))$. Explicitly, we have

$$(\sigma \star \tau)(t_0, t_1) = \begin{cases} \sigma(2t_0 - 1, 2t_1), & 0 \leq t_1 \leq \frac{1}{2} \\ \tau(2t_0, 2t_1 - 1), & \frac{1}{2} \leq t_1 \leq 1. \end{cases}$$

We define a 2-simplex $\omega : \Delta^2 \to X$ by

$$\omega(t_0, t_1, t_2) = (\sigma \star \tau)\left(t_0 + \frac{t_1}{2}, \frac{t_1}{2} + t_2\right).$$

Then, one can check that

$$\partial\omega = \omega|_{t_0=0} - \omega|_{t_1=0} + \omega|_{t_2=0} = \tau - \sigma \star \tau + \sigma.$$

Indeed, for $t_0 = 0$, we have $t_1 + t_2 = 1 \Rightarrow \frac{t_1}{2} + t_2 = 1 - \frac{t_1}{2}$. Clearly, $1 - \frac{t_1}{2} \leq \frac{1}{2} \Rightarrow t_1 \geq 1$ is not possible. Thus,

$$\omega|_{t_0=0} = (\sigma \star \tau)\left(\frac{t_1}{2}, 1 - \frac{t_1}{2}\right) = \tau(t_1, 1 - t_1) = \tau(t_1, t_2).$$

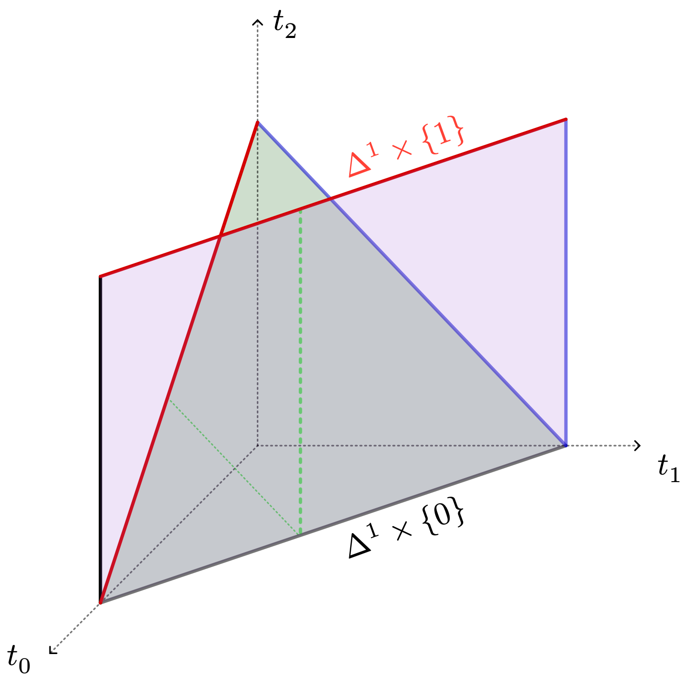
Similarly, $\omega|_{t_2=0} = \sigma$. That $\omega|_{t_1=0} = \sigma \star \tau$ is evident from the equation. But then we have

$$[\sigma \star \tau] = [\sigma] + [\tau] \in \frac{S_1(X)}{B_1(X)},$$

where $B_1(X) = \mathrm{im}(\partial : S_2(X) \to S_1(X))$.

Now, consider the quotient map

$$\begin{aligned} q : \Delta^1 \times [0,1] &\to \Delta^2 \\ (t_0, t_1, s) &\mapsto (t_0, (1-s)t_1, st_1). \end{aligned}$$



The identification $\Delta^1 \times [0,1] \to \Delta^2$.

Suppose $h : \Delta^1 \times [0,1] \to X$ is an end-point preserving homotopy between two paths. It follows that h passes to the quotient, and gives a 2-simplex, say, α. Moreover,

$$\partial\alpha = c - h_1 + h_0,$$

where c is the constant 1-simplex given by restricting h to $(\Delta^1 \circ d_0) \times [0,1]$, and $h_0 := h|_{\Delta^1 \times \{0\}}, h_1 := h|_{\Delta^1 \times \{1\}}$ are the paths. Since a constant 1-simplex is clearly a boundary, it follows that $[h_0] = [h_1]$ in $\frac{S_1(X)}{B_1(X)}$.

Now, clearly a loop is an 1-cycle, and thus gives a homology class. Moreover, homotopic loops (with fixed basepoint) gives the same homology class. Thus, for any fixed $x_0 \in X$, we have obtained a well-defined map

$$\begin{aligned} \eta' : \pi_1(X, x_0) &\to H_1(X) \\ [\sigma] &\mapsto [\sigma], \end{aligned}$$

which is moreover a group homomorphism. Recall that in general $\pi_1(X, x_0)$ is non-Abelian, whereas $H_1(X)$ is always Abelian.

Definition 9.8: (*Abelianization*)

Given a group G, the *commutator group* $[G, G]$ is defined as the normal subgroup generated by all commutators $[x, y] = xyx^{-1}y^{-1}$ for all $x, y \in G$. The group $G^{\text{ab}} := \frac{G}{[G,G]}$ is called the *Abelianization* of the group G.

Exercise 9.9: (*Universal Property of Abelianization*)

Verify that Abelianization is the left adjoint to the forgetful functor $\text{Ab} \to \text{Grp}$.

In particular, $\eta' : \pi_n(X, x_0) \to H_n(X)$ naturally defines a (unique) group homomorphism

$$\eta : \pi_1(X, x_0)^{\text{ab}} \to H_1(X).$$

This map is called the *Hurewicz homomorphism*.

Remark 9.10:

More generally, Hurewicz homomorphism can be defined in a similar way as $\eta : \pi_n(X, x_0) \to H_n(X)$, and even as $\pi_n(X, A, a_0) \to H_n(X, A)$ for $a_0 \in A \subset X$. Hurewicz homomorphism is natural, and commutes with the suspension map.

Day 10 : 17th February, 2026

`Hurewicz isomorphism – reduced singular homology – cofibration – signular homology of cofiber – cellular decomposition`

10.1 Hurewicz Isomorphism

Let us now show that the Hurewicz map is an isomorphism for path connected spaces.

Theorem 10.1: (*Hurewicz Isomorphism*)

If X is a path connected space with a basepoint x_0, then the Hurewicz map $\eta : \pi_1(X, x_0)^{\mathrm{ab}} \to H_1(X)$ is an isomorphism.

Proof : For each $x \in X$, fix a path $u_x : x_0 \to x$, with $\overline{u}_x$ as the reversed pat $x \to x_0$. Assume u_{x_0} to be the constant path c_{x_0}. Next, for any $\sigma : \Delta^1 \to X$, denote $\sigma_0 = \sigma d_0 = \sigma(0,1)$ and $\sigma_1 = \sigma d_1 = \sigma(1,0)$. Then, $\left(u_{\sigma_1} \star \sigma\right) \star \overline{u}_{\sigma_0}$ is a loop at x_0. Let us define the linear map

$$\begin{aligned} \xi' : S_1(X) &\to \pi_1(X, x_0)^{\mathrm{ab}} \\ \sigma &\mapsto \left[\left(u_{\sigma_1} \star \sigma\right) \star \overline{u}_{\sigma_0}\right]. \end{aligned}$$

This is possible since $\pi_1(X, x_0)^{\mathrm{ab}}$ is Abelian. Now, for any 2-simplex $\omega : \Delta^2 \to X$, denote the faces $\omega_j = \omega d_j : \Delta^1 \to X$, and then observe that $\omega_2 \star \omega_0 \simeq \omega_1$. Also, it is easy to see that

$$\begin{aligned} \xi'(\omega_1) &= \left[\left(u_{\omega_{11}} \star \omega_1\right) \star \overline{u}_{\omega_{10}}\right] = \left[\left(u_{\omega_{11}} \star (\omega_2 \star \omega_0)\right) \star \overline{u}_{\omega_{10}}\right] \\ &= \left[\left(\left(u_{\omega_{21}} \star \omega_2\right) \star \overline{u}_{\omega_{20}}\right) \star \left(\left(u_{\omega_{01}} \star \omega_0\right) \star \overline{u}_{\omega_{00}}\right)\right], \text{since } \overline{u}_{\omega_{20}} \star u_{\omega_{01}} \simeq c_{x_0} \\ &= \left[\left(u_{\omega_{21}} \star \omega_2\right) \star \overline{u}_{\omega_{20}}\right] + \left[\left(u_{\omega_{01}} \star \omega_0\right) \star \overline{u}_{\omega_{00}}\right] \\ &= \xi'(\omega_2) + \xi'(\omega_0) \end{aligned}$$

Hence, $\xi'(\partial\omega) = 0$ which implies that ξ' factors as a map $\xi : \frac{S_1(X)}{B_1(X)} \to \pi_1(X)^{\mathrm{ab}}$. Clearly, $\xi \circ \eta = \mathrm{Id}$, and hence, η is injective. To show that η is surjective, let $\alpha \in H_1(X)$ be represented by some 1-cycle $a := \sum n_\sigma \sigma$ with $\partial a = 0$. By linearity, we have a map $u : S_0(X) \to S_1(X)$. Then, for any $\sigma : \Delta^1 \to X$, we have $u(\partial\sigma) = u(\sigma d_0 - \sigma d_1) = u_{\sigma_0} - u_{\sigma_1} = u_{\sigma_0} + \overline{u}_{\sigma_1}$. In particular, we have

$$\eta\left(\sum n_\sigma \xi(\sigma)\right) = \sum n_\sigma\left(\left[u_{\sigma_1}\right] + [\sigma] + \left[\overline{u}_{\sigma_1}\right]\right) = \sum n_\sigma([\sigma] + [u\partial\sigma]) = \alpha + u([\partial a]) = \alpha,$$

as $\partial a = 0$. This implies η is surjective. □

Exercise 10.2:

Compute the homology groups of $S_g := \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g}$, the torus with g-holes.

Hint : Recall, $\pi_1(S_g) = \langle a_1, b_1, \cdots, a_g, b_g \mid \prod [a_i b_i] = 1\rangle$ by Van Kampen's theorem.

Remark 10.3: (*The Hurewicz Isomorphism*)

The more general Hurewicz isomorphism states the following. Suppose X is a $(n-1)$-connected, i.e, $\pi_i(X) = 0$ for $i < n$. Then, the Hurewicz map $h : \pi_n(X) \to H_n(X)$ is an isomorphism. As the Hurewicz map commutes with the suspension map, it follows moreover that $h : \pi_{n+1}(X) \to H_{n+1}(X)$ is an epimorphism. The isomorphism lets us inductively show that

$$\pi_i(S^n) = \begin{cases} 0, & i < n \\ \mathbb{Z}, & i = n. \end{cases}$$

In general, computing the higher homotopy groups of the sphere is a nontrivial problem.

10.2 Reduced Singular Homology

Suppose X is a space, and $c : X \to \star$ is the constant map. Recall, the reduced homology is defined as the kernel $\tilde{H}_n(X) := \ker\left(H_n(X) \xrightarrow{H_n(c)} H_n(\star)\right)$. Let us interpret it in a different way for singular homology. Consider the chain map $\varepsilon_\bullet : S_\bullet(X) \to S_\bullet(\star)$ given by $\varepsilon_n = 0$ for $n \neq 0$, and

$$\varepsilon_0\left(\sum n_\sigma \sigma\right) = \left(\sum n_\sigma\right)\sigma_0,$$

where $\sigma_0 : \Delta^0 \to \star$ is the unique 0-simplex.

Exercise 10.4:

Using Proposition 6.10, verify that $H_\bullet(c) = H_\bullet(\varepsilon) : B_\bullet(X) \to H_\bullet(\star)$.

Let us now define a chain homotopy $s_n : S_n(X) \to S_{n+1}(\star)$ by setting $s_n(\sigma) = (-1)^n \sigma_{n+1}$ on generators and extending linearly. Here, $\sigma_{n+1} : \Delta^n \to \star$ is the unique $(n+1)$-simplex. Note that

$$\begin{aligned} s\partial(\sigma) + \partial s(\sigma) &= s\left(\sum_0^n (-1)^i \sigma d_i\right) + (-1)^n \partial\sigma_{n+1} \\ &= (-1)^{n-1}\sum_0^n (-1)^i \sigma_n + (-1)^n \sum_0^{n+1} (-1)^i \sigma_n \\ &= (-1)^n\left[-\sum_0^n (-1)^i \sigma_n + \sum_0^{n+1} (-1)^i \sigma_n\right] \\ &= (-1)^n (-1)^{n+1}\sigma_n = -\sigma_n = \varepsilon_n(\sigma) - S_n(c)(\sigma). \end{aligned}$$

Then, $H_\bullet(\varepsilon) = H_\bullet(c)$ follows.

Definition 10.5: (*Augmented Singular Chain Complex*)

Given a space X, the *augmented singular chain complex* is defined as the chain complex $S_2(X) \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, where $\varepsilon(\sum n_\sigma \sigma) = \sum n_\sigma$. This map ε is called the *augmentation map*.

Observe that considering $\mathbb{Z}$ as a chain complex concentrated at degree 0, we can look at the augmentation map as a chain map by setting it zero every nozero degree.

Exercise 10.6:

Verify that the reduced singular homology is the homology of the augmented singular chain complex.

10.3 Cofibration and Homology

Let us recall the notion of cofibration.

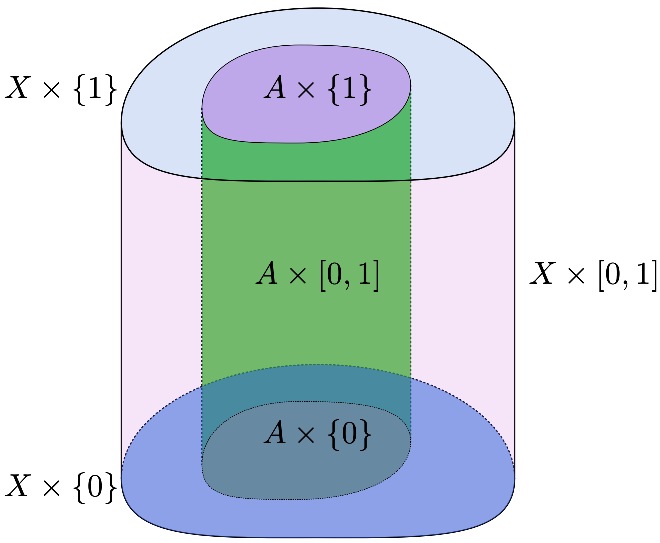
Definition 10.7: (*Cofibration*)

A map $f : A \to X$ is called a *cofibration* if it has the *homotopy extension property* against any space, i.e, given a homotopy $h : A \times [0,1] \to Y$ and a map $F : X \to Y$ with $F \circ f = h|_{A\times 0}$, there exists a homotopy $H : X \times [0,1] \to Y$ such that $H|_{X\times\{t\}} \circ f = h|_{X\times\{t\}}$. In other words, the commuting diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & Y^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "F"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

admits a solution H which commutes with other maps.

The homotopy extension is easier to visualize if we only consider an inclusion map $\iota : A \hookrightarrow X$.



Homotopy extension for $\iota : A \hookrightarrow X$

Most of the times, we will only concern ourselves about an inclusion being a cofibration. In fact, given any function $f : A \to X$, consider the *mapping space*

$$M_f := X \cup_f (A \times [0,1]),$$

where we identify $(a,0) \sim f(a)$. It is easy to see that M_f (strongly) deformation retracts onto X by collapsing $A \times [0,1]$. On the other hand, we have the inclusion $\iota : A \hookrightarrow M_f$ given by $a \mapsto (a,1)$. We then have a *homtopy* commutative diagram

$$\begin{tikzcd} & M_f \\ A \arrow[ur, hook, "\iota"] \arrow[r, "f"'] & X \arrow[u, hook, "j"', "\text{s.d.r}"] \end{tikzcd}$$

Exercise 10.8: (*Mapping Cylinder is a Cofibration*)

Given any map $f : A \to X$, show that $\iota : A \to M_f$ is a cofibration.

Let us now observe a few easy topological consequences about cofibraions.

Lemma 10.9: (*Composition of Cofibration*)

Composition of cofibrations is again a cofibration.

Proof : Suppose $f : X \to Y, g : Y \to Z$ are cofibrations. Let us consider the homotopy extension diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"] \arrow[dd, bend right, "g \circ f"'] & W^I \arrow[dd, "\mathrm{ev}_0"] \\ Y \arrow[d, "g"] & \\ Z \arrow[r, "F"] & W \end{tikzcd}$$

We first get a lift $H_1 : Y \to W^I$ such that $\mathrm{ev}_0 \circ H_1 = F \circ g$ and $H_1 \circ f = h$. Then, we get another lift $H_2 : Z \to W^I$ such that $H_2 \circ g = H_1$ and $\mathrm{ev}_0 \circ H_2 = F$. Then, we have $H_2 \circ (g \circ f) = H_1 \circ f = h$, and $\mathrm{ev}_0 \circ H_2 = F$. Thus, H_2 is the required extension, proving that $g \circ f$ is a cofibration. $\square$

Next, we consider pushout of a cofibration. Let us now recall the defintion first.

Definition 10.10: (*Pushout of Spaces*)

Given continuous maps $X \xleftarrow{f} A \xrightarrow{g} Y$, the *pushout* is defined via the following universal property.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "p"] \arrow[ddr, bend left, "\varphi"] & \\ X \arrow[r, "q"'] \arrow[drr, bend right, "\psi"'] & Z \arrow[dr, dashed, "\exists! h"] & \\ & & W \end{tikzcd}$$

Given any map $\varphi : Y \to W, \psi : X \to W$ with $\varphi \circ g = \psi \circ f$, there exists a unique map $h : Z \to W$ such that $h \circ p = \varphi, h \circ q = \psi$.

Explicitly, one can construct the pushout as the identification space $Z = (X \sqcup Y)/_\sim$, where $\sim$ is the smallest equivalence relation such that $f(a) \sim g(a)$ for $a \in A$.

Example 10.11:

Given a map $f : A \to X$, the mapping cone $C(f)$ is the pushout of the diagram $CA \hookleftarrow A \xrightarrow{f} X$. The pushout of the sphere as the boundary $D^n \hookleftarrow S^{n-1} \hookrightarrow D^n$ is the sphere S^n.

Lemma 10.12: (*Pushout of a Cofibration*)

Suppose $f : A \to X$ is a cofibration. Then, any pushout of f is again a cofibration.

__**Proof**__ **:** Consider a pushout diagram and a homtopy extension problem for the middle arrow

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "f"'] & Z \arrow[r, "h"] \arrow[d, "g"] & Y^I \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r, "q"'] & W \arrow[r, "F"'] & Y
\end{tikzcd}$$

As f is a cofibration, we have a lift

$$\begin{tikzcd}
A \arrow[r, "h \circ p"] \arrow[d, "f"'] & Y^I \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r, "F \circ q"'] \arrow[ur, dashed, "G"] & Y
\end{tikzcd}$$

Next, we use the pushout to get a unique map

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "f"'] & Z \arrow[d, "g"] \arrow[ddr, bend left, "h"] & \\
X \arrow[r, "q"] \arrow[drr, bend right, "G"'] & W \arrow[dr, dashed, "H"] & \\
 & & Y^I
\end{tikzcd}$$

We claim that H is the solution in

$$\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d, "g"'] & Y^I \arrow[d, "\mathrm{ev}_0"] \\
W \arrow[r, "F"'] \arrow[ur, dashed, "H"] & Y
\end{tikzcd}$$

We alredy have $H \circ g = h$ from the pushout diagram. Next, we again look at the pushout

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "f"'] & Z \arrow[d, "g"] \arrow[ddr, bend left, "\mathrm{ev}_0 \circ h"] & \\
X \arrow[r, "q"] \arrow[drr, bend right, "\mathrm{ev}_0 \circ G"'] & W \arrow[dr, dashed, "\mathrm{ev}_0 \circ H", "F"'] & \\
& & Y
\end{tikzcd}$$

It follows that both $\mathrm{ev}_0 \circ H$ and F solves the pushout diagram. Hence, by the uniqueness, we have $\mathrm{ev}_0 \circ H = F$. This proves that H is a homotopy extension, as required. Thus, the pushout $g : Z \to W$ is a cofibration. $\square$

Lemma 10.13: (*Contractible Cofibered Subspace*)

Suppose $A \subset X$ is contractible, and the inclusion $\iota : A \hookrightarrow X$ is a cofibration. Then, the quotient $q : X \to X/A$ is a homotopy equivalence.

Proof : Since A is contractible, there exists a homotopy $h : A \times [0,1] \to A$ such that $h|_{A \times \{0\}} = \mathrm{Id}_A$ and $h|_{A \times \{1\}} = \{a_0\}$ for some $a_0 \in A$. As $\iota : A \hookrightarrow X$ is a cofibration, we have a homotopy extension, $H : X \times [0,1] \to X$ with $H_0 := H|_{X \times \{0\}} = \mathrm{Id}_X$. Observe that $H_1 := H|_{X \times \{1\}} : X \to X$ maps A to a point $\{a_0\}$, and thus, we have a map $r : X/A \to X$ such that $r \circ q = H_1$. By construction, $H : r \circ q \simeq \mathrm{Id}_X$ is a homotopy. On the other hand, the homotopy $H_t : X \to X$ maps $H_t(A) \subset A$, and hence, passing to quotient we have the homotopy $\tilde{H}_t : X/A \to X/A$. Clearly $\tilde{H}_0 = \mathrm{Id}_{X/A}$. Also, $qr([x]) = qrq(x) = qH_1(x) = \tilde{H}_1 q(x) = \tilde{H}_1([x])$, i.e, $\tilde{H}_1 = q \circ r$. Thus, $\tilde{H} : q \circ r \simeq \mathrm{Id}_{X/A}$ is a homotopy. This proves that $q : X \to X/A$ is a homotopy equivalence. $\square$

Theorem 10.14: (*Homology of Quotient by a Cofibered Subspace*)

Suppose $A \subset X$ is a subspace such that the inclusion $\iota : A \hookrightarrow X$ is a cofibration. Then, the quotient map $q : X \to X/A$ induces an isomorphism $H_n(X, A) \to H_n(X/A, \star) \cong \tilde{H}_n(X/A)$.

Proof : Conider the mapping $C(\iota)$ which is given as the pushout

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook, "\iota"'] & CA \arrow[d, "j"] \\
X \arrow[r] & C(\iota)
\end{tikzcd}$$

As ι is a cofibration, by Lemma 10.12, we have j is a cofibration. But CA is contractible. Hence, by Lemma 10.13, we have $q : C(\iota) \to C(\iota)/CA = X/A$ is a homotopy equivalence. Now, the inclusion map $(X, A) \to (C(\iota), CA)$ induces an isomorphism in homology since we can first excise out the coning point, and then apply a deformation retract. Also, $(C(\iota), CA) \to (X/A, \star)$ induces and isomorphism as the quotient map is a homotopy equivalence. Hence, composing, it follows that the quotient $(X, A) \to (X/A, \star)$ induces an isomorphism in homology.

The isomorphism $H_n(X/A, \star) \cong \tilde{H}_n(X/A)$ follows from applying the quotient map $(Y, \emptyset) \to (Y/\emptyset, \emptyset/\emptyset) = (Y, \star)$ for any space Y. $\square$

Example 10.15: (*CW pair*)

Let X be a CW-complex, and $A \subset X$ a subcomplex. Then the inclusion $A \hookrightarrow X$ is a cofibration. In fact, if X is a locally finite CW complex (i.e, only finitely many cells of each dimension), then we can drop the assumption that A is a sub-complex and only require that $A \subset X$ is closed and a CW complex on its own.

10.4 Cellular Decomposition

Let us recall the definition of a (relative) CW complex. Given a pair (X, A), we say X is obtained from by *attaching an n-cell* if there is a pushout diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "\varphi"] \arrow[d, hook] & A \arrow[d, hook, "\iota"] \\ D^n \arrow[r, "\Phi"'] & X \end{tikzcd}$$

Since the inclusion $S^{n-1} \hookrightarrow D^n$ is trivially a cofibration, it follows from Lemma 10.12 that $\iota : A \hookrightarrow X$ is also a cofibration. The map $\varphi : S^{n-1} \to A$ is called the *attaching map* for the n-cell, and Φ is called the *characteristic map*. Note that A is necessarily closed in X, and Φ is a homeomorphism of the interior $\mathring{D}^n$ onto $X \setminus A$. Indeed, X is homeomorphic to the mapping cone $C(\varphi) = A \cup_\varphi \underbrace{C(S^{n-1})}_{D^n}$.

Example 10.16:

The n-sphere is obtained from the point $\{\star\}$ by attaching an n-cell, where the attaching map is the constant map. On the other hand, we can also realize the n-sphere from the n-disc by attaching a single n-cell, where the attaching map is the identity map on the boundary. The real projective space $\mathbb{RP}^n$ is obtained from $\mathbb{RP}^{n-1}$ by attaching an n-cell. The complex projective space $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a $2n$-cell.

Remark 10.17:

Suppose X is a T_2-space with a closed subspace $A \subset X$. Let $\Phi : D^n \to X$ be continuous map, which restricts to a homeomorphism of $\mathring{D}^n$ onto $X \setminus A$. Then, X is obtained from A by attaching an n-cell, via the attaching map $\varphi := \Phi|_{S^{n-1} = \partial D^n}$.

More generally, we can attach multiple n-cells at once. Given a pair (X, A), we say X is obtained from A by *attaching n-cells* if there exists a pushout diagram

$$\begin{tikzcd}
\bigsqcup_{\alpha\in J} S^{n-1}_\alpha \arrow[r, "\varphi"] \arrow[d, hook] & A \arrow[d, hook, "\iota"] \\
\bigsqcup_{\alpha\in J} D^n_\alpha \arrow[r, "\Phi"'] & X
\end{tikzcd}$$

Here, J is an arbitrary (possibly empty) indexing set. The restriction $\varphi_\alpha := \varphi|_{S^{n-1}_\alpha}$ is called the *attaching map* for the n-cell $E^n_\alpha := \mathring{D}^n_\alpha$. It follows that A is closed in X, and Φ induces a homeomorphism of $\sqcup E^n_\alpha$ with $X \setminus A$, in particular $X \setminus A$ is a union of components, each being an n-cell. In case $J = \emptyset$, we have $X = A$.

Exercise 10.18:

Suppose X is obtained from A by attachin n-cells, with attaching maps $\{\varphi_\alpha : S^{n-1}_\alpha \to A\}_{\alpha\in J}$. Show that a map $f : A \to Y$ extends to a map $F : X \to Y$ if and only if each $f \circ \varphi_\alpha$ is null-homotopic (whence the extension is determined by the choice of null-homotopies).

Definition 10.19: (*CW Complex*)

Let $A \subset X$ be given. A *CW decomposition* of the pair (X, A) is a sequence of subspaces

$$A = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X,$$

such that the following holds.

1. $X = \cup_{i\geq 0} X^i$, with the colimit topology (i.e, a map $f : X \to Y$ is continuous if and only if $f|_{X^n} : X^n \to X$ is continuous for all n).
2. For each $n \geq 0$, the space X^n is obtained from X^{n-1} by attaching n-cells.

The pair (X, A) equipped with a CW decomposition is called a *relative CW complex*. When $A = \emptyset$, we say X is a *CW complex*. The space X^n is called the *n-skeleton* of the CW complex. We say (X, A) is a *finite CW complex* (resp. *countable CW complex*) of there are only finitely (resp. countably) many cells attached. We say (X, A) is a *locally finite CW complex* if for each n, only finitely many n-cells are attached.

Exercise 10.20:

Suppose (X, A) is a relative CW complex. Show that the inclusion $A \hookrightarrow X$ is a cofibration. More generally, given a CW complex X, and a subcomplex $Y \subset X$ (i.e, Y is closed and consists of cells of X), the inclusion $Y \hookrightarrow X$ is a cofibration.

Hint : Use Lemma 10.12 to the cofibration $\sqcup S^{n-1} \hookrightarrow \sqcup D^n$, and observe that the composition of cofibrations is again a cofibration (Lemma 10.9). Inductively extend the homotopy $X^n \times [0, 1]$ to $X^{n+1} \times [0, 1]$. The final homotopy is continuous as the topology on X is a colimit topology.

In the above definition, there is no restriction on the topology on A. If A is a T_2-space, then X is a T_2-space, moreover the topology on X is given by the colimit topology with respect to the family consisting of A and the closure of cells. This leads to the following definition.

Definition 10.21: (*Whitehead Complex*)

A *Whitehead complex* is a space X, along with a *cell decomposition* $\{e_\lambda \mid \lambda \in \Lambda\}$, where each e_λ is homeomorphic to $\mathring{D}^n$ for some n, such that the following holds.

1. X is a T_2-space.
2. For each n-cell, there is a map $\Phi_\lambda : D^n \to X$ such that $\Phi|_{\mathring{D}^n} : \mathring{D}^n \to e_\lambda$ is a homeomorphism, and Φ maps the boundary ∂D^n into X^{n-1}, which is the union of cells of dimension $\leq n-1$.
3. (**C**losure Finiteness) The closure $\overline{e}_\lambda$ of each cell intersects only a finitely many cells.
4. (**W**eak Topology) The topology on X is induced by the colimit topology of the family $\{\overline{e}_\lambda \mid \lambda \in \Lambda\}$, i.e, $U \subset X$ is open if and only if $U \cap \overline{e}_\lambda$ is open for all $\lambda \in \Lambda$.

It follows that any Whitehead complex carries a CW decomposition, and conversely, a CW complex is a Whitehead complex.

Remark 10.22:

Note that the phrase "X is a CW complex" technically means that "X is a space equipped with a CW decomposition, and the corresponding colimit topology". There are spaces which does not admit any CW decomposition, e.g., the Hawaiian earring. To see this, one first proves that a space admitting a CW decomposition must be locally contractible, but Hawaiian earring is not.

Definition 10.23: (*Cellular Map*)

Let $(X, A), (Y, B)$ be two relative CW complexes. A map $f : (X, A) \to (Y, B)$ is called a *cellular map* if $f(X^n) \subset Y^n$ holds for each $n \geq -1$. Given a CW complex X, a subspace $A \subset X$ is called a CW subcomplex if A is a CW complex and the inclusion map $A \hookrightarrow X$ is cellular.

We have the following important theorem, proof of which follows from inductively constructing the map cell-by-cell.

Theorem 10.24: (*Cellular Approximation Theorem*)

Let X, Y be CW complexes. Then, any map $f : X \to Y$ is homotopic to a cellular map $g : X \to Y$. Suppose $B \subset X$ is a subcomplex, i.e, B is a CW complex and $\iota B \hookrightarrow X$ is cellular. If $f|_B : B \to Y$ is already cellular, then the homotopy $f \simeq g$ can be chosen to relative to B, i.e, the homotopy stays constant on B.

Next, recall the notion of weak homotopy equivalence.

Definition 10.25: (*Weak Homotopy Equivalence*)

A map $f : X \to Y$ is called a *weak homotopy equivalence* if

- $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is group isomorphism for each $n \geq 1$ and each $x \in X$, and
- $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.

Example 10.26: (*Digital Circle*)

Consider the four point space $X = \{N, E, W, S\}$ with the following topology

$$\{\emptyset, \{N\}, \{S\}, X\}.$$

This space is *not* T_2. Let us consider a map $f : S^1 \to X$ which maps the open upper semicirlce (resp. lower semicircle) to N (resp. to S), and the two remaining points to E and W respectively. Clearly, f is a continuous map. Surjectivity of f shows that X is a path connected space. Moreover, one can construct a contractible universal cover of X, and show that $\pi_1(X) = \mathbb{Z}$ and $\pi_k(X) = 0$ for $k > 1$. It follows that $f : S^1 \to X$ is a weak homotopy equivalence. The space X is called the *digital circle*. Note that f does not admit a homotopy inverse, since any $g : X \to S^1$ is necessarily a constant map (as the image must be path connected and at most finitely many points).

By the *Whitehead theorem*, we have that a weak homtopy equivalence between CW complexes is a homotopy equivalence. In general, we have the following theorem.

Theorem 10.27: (*CW Approximation Theorem*)

Given any space Y, there exists a CW complex X and weak homotopy equivalence $\alpha : X \to Y$, which is called a *CW approximation* of Y. Moreover, if $f : Y_1 \to Y_2$ is a map, and $\alpha_i : X_i \to Y_i$ are CW approximations for $i = 1, 2$, then there exists a (cellular) map $g : X_1 \to X_2$, defined unique up to homotopy, such that $\alpha_2 \circ g \simeq f \circ \alpha_1$.

Note that CW approximation is applicable for any space, but since we cannot in general invert (even up to homotopy) a weak homotopy equivalence, this does not let us replace a space. However, when computing algebraic invariants like homotopy or (co)homology, it is good enough to replace any space by a CW approximation.

Remark 10.28: (*CW Type*)

A space X is said to be of *CW type* if X is homotopy equivalent to a space Y, where Y admits a CW decomposition. We have noted that the Hawaiian earring does not admit a CW decomposition, moreover it does not have CW type. Consider the *Hedgehog space*

$$X := \left\{ re^{\iota\theta} \mid 0 \leq r \leq 1, \theta \in \mathbb{Q} \right\} \subset \mathbb{C}$$

which is a dense collection of spokes. Again, X is not locally contractible at any point other than the origin, and hence, X does not admit a CW decomposition. On the other hand, X is contractible, and so, it is homotopy equivalent to a CW complex (the singleton). So, X is of CW type. It is easy to see that Whitehead theorem generalizes to CW tpye : a weak homotopy equivalence between spaces of CW type is a homotopy equivalence.