Course notes for

Algebraic Topology II (KSM4E02)

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Day 9 : 13th February, 2026

`chain homotopy of barycentric subdivision – iterated barycentric subdivision – excision in singular homology – additivity in singular homology – Hurewicz homomorphism`

9.1 Barycentric Subdivision and Chain Homtopy

Intuitively, barycentric subdivision divides a given chain into smaller pieces with matching boundary, so that it should not change the boundary, and have no effect on the homology. We show that this operator is in fact a natrual chain map, which is naturally chain homotopic to the identity.

Proposition 9.1: (*Barycentric Subdivision is a Natural Chain Map*)

The barycentric subdivision operator $\mathcal{B}^X_\bullet : S_\bullet(X) \to S_\bullet(X)$ is a chain map, which is natural in X. Moreover, it is naturally chain homotopic to the identity map.

Proof : Given any $f : X \to Y$, we have

$$S_n(f)\mathcal{B}^X_n(\sigma) = S_n(f)S_n(\sigma)\mathcal{B}^{\Delta^p}_n(\iota_n) = S_n(f\circ\sigma)\mathcal{B}^{\Delta^n}_n(\iota_n) = \mathcal{B}^Y_n(f\circ\sigma) = \mathcal{B}^Y_n S_n(f)(\sigma),$$

which proves the naturality. Next, we check

$$\partial\mathcal{B}^{\Delta^p}_p(\iota_p) = \partial\big(\iota^\beta_p \cdot \mathcal{B}^{\Delta^p}_{p-1}\big)(\partial\iota_p) = \mathcal{B}^{\Delta^p}_{p-1}(\partial\iota_p) - \iota^\beta_p \cdot \partial\partial\iota_p = \mathcal{B}^{\Delta^p}_{p-1}(\partial\iota_p), \quad p > 0,$$

and $\partial\mathcal{B}^{\Delta^0}_0(\iota_0) = \partial(\iota_0) = 0$. Then, for any $\sigma : \Delta^p \to X$ we have

$$\begin{aligned}\mathcal{B}^X_{p-1}\partial\sigma = \mathcal{B}^X_{p-1}\partial S_p(\sigma)\iota_p &= \mathcal{B}^X_{p-1}S_{p-1}(\sigma)\big(\partial\iota_p\big) = S_{p-1}(\sigma)\mathcal{B}^{\Delta_p}_{p-1}\big(\partial\iota_p\big)\\ &= S_{p-1}(\sigma)\partial\mathcal{B}^{\Delta_p}_{p-1}\big(\iota_p\big) = \partial S_p(\sigma)\mathcal{B}\big(\iota_p\big) = \partial\mathcal{B}(\sigma).\end{aligned}$$

This proves that $\mathcal{B}^X_\bullet$ is a chain map.

Next, we construct a natural chain homotopy $\mathrm{Id} \simeq \mathcal{B}^X_\bullet$. We define $\mathcal{T}^X_n : S_n(X) \to S_{n+1}(X)$ inductively. For any $\sigma : \Delta^n \to X$, set $\mathcal{T}^X_n(\sigma) = S_{n+1}(\sigma)\mathcal{T}^{\Delta^n}_n(\iota_n)$, where we have

$$\mathcal{T}^{\Delta^n}_n(\iota_n) = \iota^\beta_n \cdot \big(\iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n)\big).$$

Assume inductively that $\mathcal{T}^X_p$ is natural, and part of the chain homotopy for $p < n$. Then we compute

$$\begin{aligned}\partial\mathcal{T}^{\Delta^n}_n(\iota_n) &= \partial\big(\iota^\beta_n \cdot \big(\iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n)\big)\big)\\ &= \iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n) - \iota^\beta_n \cdot \big(\partial\iota_n - \partial\mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n)\big)\\ &= \iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n) - \iota^\beta_n \cdot \big(\mathcal{B}^{\Delta^n}_{n-1}(\partial\iota_n) + \mathcal{T}^{\Delta^n}_{n-2}(\partial\partial\iota_n)\big), \quad \text{by induction}\end{aligned}$$

$$
\begin{aligned}
&= \iota_n - \mathcal{T}_{n-1}^{\Delta^n}(\partial\iota_n) - \iota_n^\beta \cdot \mathcal{B}_{n-1}^{\Delta^n}(\partial\iota_n), \quad \text{as } \partial\partial\iota_n = 0 \\
&= \iota_n - \mathcal{B}_n^{\Delta^n}(\iota_n) - \mathcal{T}_{n-1}^{\Delta^n}(\partial\iota_n), \quad \text{by definition of } \mathcal{B}_n^{\Delta^n}(\iota_n).
\end{aligned}
$$

Now, for any $\sigma : \Delta^n \to X$ we have

$$
\begin{aligned}
\partial\mathcal{T}_n^X(\sigma) &= \partial S_{n+1}(\sigma)\mathcal{T}_n^{\Delta^n}(\iota_n) = S_n(\sigma)\partial\mathcal{T}_n^{\Delta^n}(\iota_n) \\
&= S_n(\sigma)(\iota_n) - S_n(\sigma)\mathcal{B}_n^{\Delta^n}(\iota_n) - S_n(\sigma)\mathcal{T}_{n-1}^{\Delta^n}(\partial\iota_n) \\
&= \sigma - \mathcal{B}_n^X(\sigma) - \mathcal{T}_{n-1}^X S_{n-1}(\sigma)(\partial\iota_n), \text{by definition and by naturality} \\
&= \sigma - \mathcal{B}_n^X(\sigma) - \mathcal{T}_{n-1}^X \partial S_n(\sigma)(\iota_n) \\
&= \sigma - \mathcal{B}_n^X(\sigma) - \mathcal{T}_{n-1}^X \partial\sigma.
\end{aligned}
$$

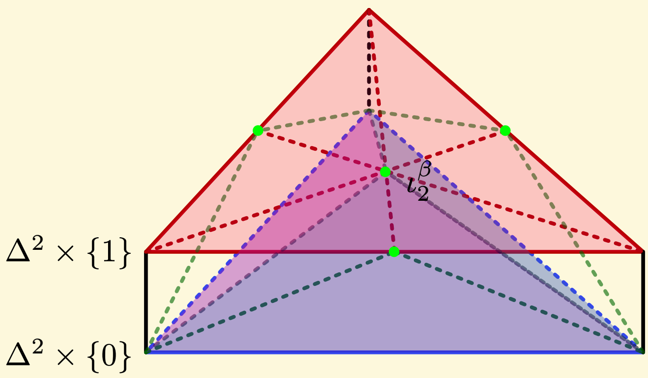
Thus, $\mathcal{T}_n^X$ is part of the required chain homotopy. Also, for any $f : X \to Y$, we have

$$
S_{n+1}(f)\mathcal{T}_n^X(\sigma) = S_{n+1}(f)S_{n+1}(\sigma)\mathcal{T}_n^{\Delta^n}(\iota_n) = S_{n+1}(f\sigma)\mathcal{T}_n^{\Delta^n}(\iota_n) = \mathcal{T}_n^Y(f\sigma) = \mathcal{T}_n^Y S_n(f)(\sigma),
$$

which proves the naturality. Hence, by induction, we see that $\mathcal{T}_\bullet^X : \mathrm{Id} - \mathcal{B}_\bullet^X$ is a chain homotopy, which is natural in X. □

Remark 9.2:

Here is the geometric intution behidn the chain homotopy constructed in Proposition 9.1. Given the n-simplex $\iota_n : \Delta^n \to \Delta^n$, the homotopy can be thought of as a homtopy $\Delta^n \times [0,1] \to \Delta^n$, with ι_n on $\Delta^n \times \{0\}$ and $\mathcal{B}_n^{\Delta^n}(\iota_n)$ on $\Delta^n \times \{1\}$.



The chain homotopy for $\iota_2 : \Delta^2 \to \Delta^2$ with $\mathcal{B}_2^{\Delta^2}(\iota_2)$.

We can do this by coning over all the simplices of $\Delta^n \times \{0\} \cup \partial\Delta^n \times [0,1]$ with the barycenter of $\Delta^n \times \{1\}$ as the coning point.

9.2 Iterated Barycentric Subdivision

Given a space X, consider $\mathcal{U}$ to be a collection of subsets such that $X = \cup_{U\in\mathcal{U}} \mathring{U}$, i.e, interiors of the sets from $\mathcal{U}$ form an open cover of X. A singular n-simplex $\sigma : \Delta^n \to X$ is called *$\mathcal{U}$-small* if $\sigma(\Delta^n) \subset U$ for some $U \in \mathcal{U}$. Note that the simplicies appearing in the boundary of an $\mathcal{U}$-small simplex is again $\mathcal{U}$-small. Hence, we can define $S_\bullet^{\mathcal{U}}(X) \subset S_\bullet(X)$ to be the chain complex generated by $\mathcal{U}$-small singular simplices, with (the restriction of) the usual boundary map. The corresponding homology is denoted as $H_\bullet^{\mathcal{U}}(X)$. Our goal is to show that $H^{\mathcal{U}}_\bullet(X)$ is isomorphic to $H_\bullet(X)$. We achieve this by representing a homology class by a chain of $\mathcal{U}$-small simplices, by iterating the barycentric subdivision finitely many times.

Lemma 9.3: (*Diameter of Affine Singular Simplex*)

Let $v_0, ..., v_p \in \mathbb{R}^n$. Then, $\mathcal{B}_p[v_0, ..., v_p]$ is a linear combination of affine simplices, each with diameter at most $\frac{p}{p+1} \operatorname{diam}[v_0, ..., v_p] = \frac{p}{p+1} \max_{i,j}\|v_i - v_j\|$, where we consider the Euclidean norm.

<u>**Proof**</u> **:** First, let us observe that $\operatorname{diam}\ [v_0, ..., v_p] = \max_{i,j}\|v_i - v_j\|$. Let $x, y \in [v_0, ..., v_p]$, and say $x = \sum \lambda_i v_i$ with $\sum \lambda_i = 1$ and $\lambda_i \geq 0$. Then,

$$\|x-y\| = \left\|\sum \lambda_i v_i - \sum \lambda_i y\right\| = \left\|\sum \lambda_i (v_i - y)\right\| \leq \sum \lambda_i \|v_i - y\| \leq \max_i \|v_i - y\|$$

In particular, taking $y = v_j$, we have $\|x - v_j\| \leq \max_i \|v_i - v_j\|$. But then for arbitrary x, y we have $\|x - y\| \leq \max_i \|v_i - y\| \leq \max_{i,j}\|v_i - v_j\|$. Thus, $\operatorname{diam}[v_0, ..., v_p] \leq \max_{i,j}\|v_i - v_j\|$. Clearly the diameter is attained as well, which shows the equality.

Next, we inductively show that diameter of each simplex in $\mathcal{B}_p[v_0, ..., v_p]$ is at most $\frac{p}{p+1} \max_{i,j}\|v_i - v_j\|$. It is trivial for $p = 0$. We have

$$\mathcal{B}_p[v_0, ..., v_p] = S_p(\sigma)\left(\iota_p^\beta \cdot \mathcal{B}_{p-1}(\partial \iota_p)\right) = \sum (-1)^i \sigma^\beta \cdot \mathcal{B}_{p-1}[v_0, ..., \hat{v}_i, ..., v_p],$$

where $\sigma = [v_0, ..., v_p]$ is the affine singular simplex. By induction, the simplices in $\mathcal{B}_{p-1}[v_0, ..., \hat{v}_i, ..., v_p]$ has diameter at most

$$\frac{p-1}{p} \operatorname{diam}[v_0, ..., \hat{v}_i, ..., v_p] \leq \frac{p-1}{p} \operatorname{diam}\ [v_0, ..., v_p] < \frac{p}{p+1} \operatorname{diam}\ [v_0, ..., v_p]$$

. The simplices in $\mathcal{B}_p[v_0, ..., v_p]$ has vertices σ^β and the vertices from $\mathcal{B}_{p-1}[v_0, ..., \hat{v}_i, ...v_p]$. Let us compute $\sup\{\|\sigma^\beta - x\| \mid x \in [v_0, ..., v_p]\}$. Say $x = \sum \lambda_i v_i \in [v_0, ..., v_p]$. Then, $\|\sigma^\beta - x\| \leq \max_i \|\sigma^\beta - v_i\|$. Now,

$$\begin{aligned} \|\sigma^\beta - v_j\| &= \left\|\frac{1}{p+1}\left(\sum v_i\right) - v_j\right\| = \frac{1}{p+1}\left\|\sum (v_i - v_j)\right\| \\ &\leq \frac{1}{p+1} \sum \|v_i - v_j\| \leq \frac{p}{p+1} \max_{i,j} \|v_i - v_j\| \\ &= \frac{p}{p+1} \operatorname{diam}[v_0, ..., v_p]. \end{aligned}$$

The claim then follows by induction. □

Lemma 9.4: (*Iterated Barycentric Subdivision*)

The inclusion of chain complex $S^{\mathcal{U}}_\bullet(X) \to S_\bullet(X)$ induces an isomorphism in homology.

<u>**Proof**</u> **:** Suppose $\sigma : \Delta^n \to X$ is a singular n-simplex. Then, consider the collection $\mathcal{U}_\sigma = \left\{\sigma^{-1}\left(\mathring{U}\right) \mid U \in \mathcal{U}\right\}$, which is an open cover of the compact metric space Δ^n. Then, there exists a Lebesgue number, say, $\delta > 0$ for this covering. Suppose $k = k(\sigma)$ is a natural number such that $\left(\frac{n}{n+1}\right)^k \operatorname{diam}[e_0, ..., e_n] < \delta$. Then, it follows from Lemma 9.3 that each of the simplices appearing in $\mathcal{B}^{\circ k}[e_0, ..., e_n]$ is contained in one of the sets from $\mathcal{U}_\sigma$. But then $\mathcal{B}^{\circ k}(\sigma)$ is $\mathcal{U}$-small by naturality.

Next, using Proposition 9.1, let us show that $\mathcal{B}_\bullet^{\circ k} \simeq \mathrm{Id}$. Indeed, consider

$$\mathcal{T}^k := \sum_{i=0}^{k-1} \mathcal{T}\mathcal{B}^{\circ i}.$$

Then, we have

$$\partial\mathcal{T}^k + \mathcal{T}^k\partial = \sum \partial\mathcal{T}\mathcal{B}^{\circ i} + \mathcal{T}\mathcal{B}^{\circ i}\partial = \sum(\partial\mathcal{T} + \mathcal{T}\partial)\mathcal{B}^{\circ i} = \sum(\mathrm{Id} - \mathcal{B})\mathcal{B}^{\circ i} = \mathrm{Id} - \mathcal{B}^{\circ k}.$$

Thus, $\mathcal{T}^k : \mathrm{Id} \simeq \mathcal{B}^{\circ k}$ is a chain homotopy, which is natural by construction.

Finally, let us show that the induced map $\Theta : H_\bullet^{\mathcal{U}}(X) \to H_\bullet(X)$ is an isomorphism.

- **Injectivity** : Suppose $\Theta(\alpha) = 0$ for some $\alpha \in H_n^{\mathcal{U}}(X)$. Say, $\alpha = [a]$ for some n-cycle $a \in S_n^{\mathcal{U}}(X)$. Then, $a = \partial b$ for some $b \in S_{n+1}(X)$. By using Lemma 9.3 to each of the simplices appearing in b, we have some k such that $\mathcal{B}^{\circ k}(b)$ is $\mathcal{U}$-small. Then, $b - \mathcal{B}^{\circ k}(b) = \partial\mathcal{T}^k(b) + \mathcal{T}^k\partial b = \partial\mathcal{T}^k(b) + \mathcal{T}^k(a) \Rightarrow \partial b - \partial\mathcal{B}^{\circ k}(b) = 0 + \partial\mathcal{T}^k(a) \Rightarrow a = \partial b = \partial\big(\mathcal{T}^k(a) + \mathcal{B}^{\circ k}(b)\big)$. Now, by construction, $\mathcal{T}$ takes $\mathcal{U}$-small chain to $\mathcal{U}$-small ones, and hence, $\mathcal{T}^k(a) \in S_{n+1}^{\mathcal{U}}(X)$. Clearly $\mathcal{B}^{\circ k}(b) \in S_{n+1}^{\mathcal{U}}(X)$. Thus, a is an $\mathcal{U}$-small boundary. In other words, $\alpha = [a] = 0$ in $H_n^{\mathcal{U}}(X)$. This shows, Θ is injective.
- **Surjectivity** : Next, suppose $a \in S_n(X)$ is a cycle representing a class in $H_n(X)$. Then, there exists some k such that $\mathcal{B}^{\circ k}(a)$ is an $\mathcal{U}$-small n-chain. As $\mathcal{B}^{\circ k}$ is a chain map, it follows that $\mathcal{B}^{\circ k}(a)$ is a cycle. Also, $a - \mathcal{B}^{\circ k}(a) = \partial\mathcal{T}^k(a) + \mathcal{T}^k(\partial a) = \partial\mathcal{T}^k(a)$ implies that a is homologous to the cycle $\mathcal{B}^{\circ k}(a)$, which represents a class in $H_n^{\mathcal{U}}(X)$. Clearly, $\theta[\mathcal{B}^{\circ k}(a)] = [a]$, proving the surjectivity.

Thus, $\Theta : H_\bullet^{\mathcal{U}}(X) \to H_\bullet(X)$ is an isomorphism. ☐

Let us also show the same as above for a pair (X, A). Denote $\mathcal{U} \cap A := \{U \cap A \mid U \in \mathcal{U}\}$. Clearly, the interiors of sets of $\mathcal{U} \cap A$ covers A. Let us defien

$$S_\bullet^{\mathcal{U}}(X, A) := \frac{S_\bullet^{\mathcal{U}}(X)}{S_\bullet^{\mathcal{U}\cap A}(A)},$$

which is clearly a chain complex. Denote the homology as $H_\bullet^{\mathcal{U}}(X, A)$

Lemma 9.5: (*Barycentrinc Subdivision in Relative Homology*)

The inclusion map $\theta : S_\bullet^{\mathcal{U}}(X, A) \to S_\bullet(X, A)$ induces an isomorphism $\Theta : H_\bullet^{\mathcal{U}}(X, A) \to H_\bullet(X, A)$.

<u>**Proof**</u> **:** We have a commutative diagram of short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet^{\mathcal{U}\cap A}(A) \arrow[r] \arrow[d, "\theta_A"'] & S_\bullet^{\mathcal{U}}(X) \arrow[r] \arrow[d, "\theta_X"'] & S_\bullet^{\mathcal{U}}(X, A) \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & S_\bullet(A) \arrow[r] & S_\bullet(X) \arrow[r] & S_\bullet(X, A) \arrow[r] & 0
\end{tikzcd}$$

By Theorem 7.1, we have the commutative diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_n^{\mathcal{U}\cap A}(A) \arrow[r] \arrow[d, "H_n(\theta_A)"'] & H_n^{\mathcal{U}}(X) \arrow[r] \arrow[d, "H_n(\theta_X)"'] & H_n^{\mathcal{U}}(X,A) \arrow[r] \arrow[d, "H_n(\theta)"'] & H_{n-1}^{\mathcal{U}\cap A}(A) \arrow[r] \arrow[d, "H_{n-1}(\theta_A)"'] & H_{n-1}^{\mathcal{U}}(X) \arrow[r] \arrow[d, "H_{n-1}(\theta_X)"'] & \cdots \\
\cdots \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

By Lemma 9.4, 4-out-of-5 vertical arrows are isomorphisms. Hence, by the 5-lemma, it follows that $H_n(\theta)$ is an isomorphism as well. □

9.3 Excision in Singular Homology

We are now in a position to prove the excision theorem in singular homology.

Theorem 9.6: (*Excision*)

Given a pair (X, A) and a subset $B \subset A$, the inclusion

$$\iota : (X \setminus B, A \setminus B) \hookrightarrow (X, A)$$

induces an isomorphism in homology, provided $\overline{B} \subset \mathring{A}$ holds.

Proof : Let us get an equivalent statement first. Set

$$U := A, \quad V := X \setminus B.$$

Then, $\overline{B} \subset B \Rightarrow X \setminus \overline{B} \subset X \setminus B = V \Rightarrow X \setminus \overline{B} \subset \mathring{V}$. Since $\overline{B} \subset \mathring{A} = \mathring{U}$, it follows that the interiors of U, V cover X. Thus, we can now apply Lemma 9.4 for the collection $\mathcal{U} := \{U, V\}$.

Observe that $S_\bullet^{\mathcal{U}}(X) = S_\bullet(U) + S_\bullet(V)$, as an $\mathcal{U}$-small simplex is either contained in U or in V (and possibly in both). Also, $S_\bullet(U) \cap S_\bullet(V) = S_\bullet(U \cap V)$. Note that

$$\frac{S_n(V)}{S_n(U \cap V)} = \frac{S_n(V)}{S_n(U) \cap S_n(V)} \cong \frac{S_n(U) + S_n(V)}{S_n(U)} = \frac{S_n^{\mathcal{U}}(X)}{S_n(U)}.$$

The isomorphism in the middle is the second isomorphism theorem. In particular, it is induced by the natural inclusion $S_n(V) \subset S_n(U) + S_n(V)$. Observe that in Lemma 9.4, the barycentric subdivision and the chain homotopy (and hence, their iterates), takes chains in $S_\bullet(A)$ to itself. In particular, passing to the quotient, we see that $\frac{S_\bullet^{\mathcal{U}}(X)}{S_\bullet(U)} \to \frac{S_\bullet(X)}{S_\bullet(U)}$ induces isomorphism in homology. Alternatively, observe that $S_\bullet^{\mathcal{U}\cap A}(A) = S_\bullet(A)$, and so, we can use Lemma 9.5 for $S_\bullet^{\mathcal{U}}(X, A) = \frac{S_n^\bullet(X)}{S_\bullet(U)}$. Hence, we have isomorphism $H_\bullet(V, U \cap V) \to H_\bullet(X, U)$.

Translating back to our original notaion, we have $V = X \setminus B, U \cap V = A \cap (X \setminus B) = A \setminus B$. Hence, we have the exicision isomorphism $H_\bullet(X \setminus B, A \setminus B) \to H_\bullet(X, A)$. □

9.4 Additivity of Singular Homology

As proved in Proposition 3.1, we immediately have that singular homology is finitely additve. But it is more than that!

Theorem 9.7: (*Homology of Path Components*)

Given a space X, we have $H_n(X) = \bigoplus_{P \in \pi_0(X)} H_n(P)$, where $\pi_0(X)$ is the set of path components of X. In particular, singular homology is additive (Definition 3.3).

Proof : Since Δ^n is path connected, and n-simplex is contained in a path component. Thus, we have a natural decomposion $S_n(X) = \bigoplus_{P \in \pi_0(X)} S_n(P)$. Clearly the boundary map restricts to each summand. Then, kernel and image splits as well. We have, $H_n(X) = \bigoplus H_n(P)$.

Given a disjoint union $X = \sqcup X_\alpha$, each X_α is union of path components. In particular, the above argument works and gives $H_\bullet(X) = \bigoplus H_\bullet(X_\alpha)$. This shows that singular homology is additive. □

Thus, when computing the singular homology, we might as well assume that the space is path connected.

9.5 Hurewicz Homomorphism

Since $\Delta^1 = \{(t_0, t_1) \in \mathbb{R}^2 \mid t_0 + t_1 = 1, t_0 \geq 0, t_1 \geq 0\}$ is essentially the interval, a singular 1-simplex is nothing but a path in the space X. Given $\sigma : \Delta^1 \to X$, let us consider the path

$$\begin{aligned} P(\sigma) : [0,1] &\to X \\ t &\mapsto \sigma(1-t, t). \end{aligned}$$

Conversely, any path $\gamma : [0,1] \to X$ can be thought of as a 1-simplex $S(\gamma)(t_0, t_1) = \gamma(t_1)$. Clearly, $S \circ P = \mathrm{Id} = P \circ S$. The standard $\frac{1}{2}$-concatenation of paths then gives a concatenation of 1-simplices as $\sigma, \tau : \Delta^1 \to X$, with $\sigma(0,1) = \tau(1,0)$ via $\sigma \star \tau = S(P(\sigma) \star P(\tau))$. Explicitly, we have

$$(\sigma \star \tau)(t_0, t_1) = \begin{cases} \sigma(2t_0 - 1, 2t_1), & 0 \leq t_1 \leq \frac{1}{2} \\ \tau(2t_0, 2t_1 - 1), & \frac{1}{2} \leq t_1 \leq 1. \end{cases}$$

We define a 2-simplex $\omega : \Delta^2 \to X$ by

$$\omega(t_0, t_1, t_2) = (\sigma \star \tau)\left(t_0 + \frac{t_1}{2}, \frac{t_1}{2} + t_2\right).$$

Then, one can check that

$$\partial \omega = \omega|_{t_0=0} - \omega|_{t_1=0} + \omega|_{t_2=0} = \tau - \sigma \star \tau + \sigma.$$

Indeed, for $t_0 = 0$, we have $t_1 + t_2 = 1 \Rightarrow \frac{t_1}{2} + t_2 = 1 - \frac{t_1}{2}$. Clearly, $1 - \frac{t_1}{2} \leq \frac{1}{2} \Rightarrow t_1 \geq 1$ is not possible. Thus,

$$\omega|_{t_0=0} = (\sigma \star \tau)\left(\frac{t_1}{2}, 1 - \frac{t_1}{2}\right) = \tau(t_1, 1 - t_1) = \tau(t_1, t_2).$$

Similarly, $\omega|_{t_2=0} = \sigma$. That $\omega|_{t_1=0} = \sigma \star \tau$ is evident from the equation. But then we have

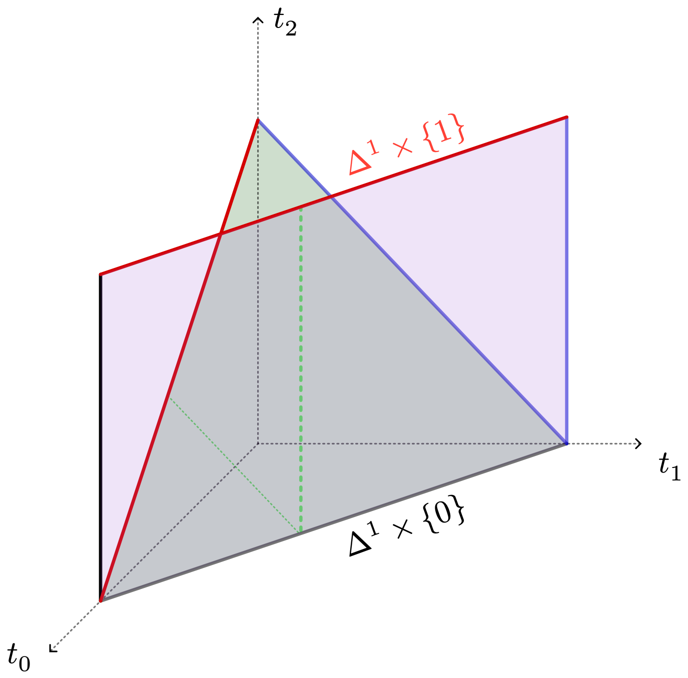
$$[\sigma \star \tau] = [\sigma] + [\tau] \in \frac{S_1(X)}{B_1(X)},$$

where $B_1(X) = \mathrm{im}(\partial : S_2(X) \to S_1(X))$.

Now, consider the quotient map

$$q : \Delta^1 \times [0,1] \to \Delta^2$$

$$(t_0, t_1, s) \mapsto (t_0, (1-s)t_1, st_1).$$



The identification $\Delta^1 \times [0,1] \to \Delta^2$.

Suppose $h : \Delta^1 \times [0,1] \to X$ is an end-point preserving homotopy between two paths. It follows that h passes to the quotient, and gives a 2-simplex, say, α. Moreover,

$$\partial \alpha = c - h_1 + h_0,$$

where c is the constant 1-simplex given by restricting h to $(\Delta^1 \circ d_0) \times [0,1]$, and $h_0 := h|_{\Delta^1 \times \{0\}}, h_1 := h|_{\Delta^1 \times \{1\}}$ are the paths. Since a constant 1-simplex is clearly a boundary, it follows that $[h_0] = [h_1]$ in $\frac{S_1(X)}{B_1(X)}$.

Now, clearly a loop is an 1-cycle, and thus gives a homology class. Moreover, homotopic loops (with fixed basepoint) gives the same homology class. Thus, for any fixed $x_0 \in X$, we have obtained a well-defined map

$$\begin{aligned} \eta' : \pi_1(X, x_0) &\to H_1(X) \\ [\sigma] &\mapsto [\sigma], \end{aligned}$$

which is moreover a group homomorphism. Recall that in general $\pi_1(X, x_0)$ is non-Abelian, whereas $H_1(X)$ is always Abelian.

Definition 9.8: (*Abelianization*)

Given a group G, the *commutator group* $[G, G]$ is defined as the normal subgroup generated by all commutators $[x, y] = xyx^{-1}y^{-1}$ for all $x, y \in G$. The group $G^{\mathrm{ab}} := \frac{G}{[G,G]}$ is called the *Abelianization* of the group G.

Exercise 9.9: (*Universal Property of Abelianization*)

Verify that Abelianization is the left adjoint to the forgetful functor $\mathrm{Ab} \to \mathrm{Grp}$.

In particular, $\eta' : \pi_n(X, x_0) \to H_n(X)$ naturally defines a (unique) group homomorphism

$$\eta : \pi_1(X, x_0)^{\mathrm{ab}} \to H_1(X).$$

This map is called the *Hurewicz homomorphism*.

Remark 9.10:

More generally, Hurewicz homomorphism can be defined in a similar way as $\eta : \pi_n(X, x_0) \to H_n(X)$, and even as $\pi_n(X, A, a_0) \to H_n(X, A)$ for $a_0 \in A \subset X$. Hurewicz homomorphism is natural, and commutes with the suspension map.