Course notes for

# Algebraic Topology II (KSM4E02)

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`homotopy invariance of singular homology — barycentric subdivision`

### 8.1 Homotopy Invariance of Singular Homology

Our goal is to show that homotopic maps induce the same homomorphism of singular homology groups. Given a space $X$, consider the maps

$$\begin{aligned} \xi^X : X &\to X \times [0,1] \\ x &\mapsto (x, 0). \end{aligned} \qquad \begin{aligned} \eta^X : X &\to X \times [0,1] \\ x &\mapsto (x, 1). \end{aligned}$$

They are clearly natural in $X$, and hence, induce natural chain maps

$$\xi^X_\bullet, \eta^X_\bullet : S_\bullet(X) \to S_\bullet(X \times [0,1]).$$

In particular, $\xi_\bullet, \eta_\bullet : S_\bullet(\_) \Rightarrow S_\bullet(\_ \times [0,1])$ are natural transformations.

**Proposition 8.1:**

The chain maps $\xi_\bullet, \eta_\bullet$ are naturally chain homotopic.

**Proof :** We need to construct maps $s^X_n : S_n(X) \to S_{n+1}(X \times [0,1])$, such that it is a chain homotopy:

$$\eta^X_n - \xi^X_n = \partial s^X_n + s^X_{n-1}\partial.$$

Moreover, we need it to be natural: for any $f : X \to Y$, we require

$$S_{n+1}(f \times \mathrm{Id}) \circ s^X_n = s^Y_n \circ S_n(f).$$

We construct $s^X_n$ inductively.

- For a $0$-simplex $\sigma : \Delta^0 \to X$, let us define $s^X_0(\sigma) : \Delta^1 = \Delta^0 \times [0,1] \to X \times [0,1]$ by

  $$s^X_0(\sigma)(t) = (\sigma(0), t).$$

  It follows that

  $$\partial\big(s^X_0(\sigma)\big) = s^X_0(\sigma)d_0 - s^X_0(\sigma)d_1 = \eta^X_0\sigma - \xi^X_0\sigma.$$

  Naturality is apparent from the definition.
- Inductively assume that we have constructed $s^X_k$ for $k < n$, for all space $X$. We construct $s^X_n$. The identity map $\mathrm{Id} : \Delta^n \to \Delta^n$ is a singular $n$-simplex, denote $\iota_n \in S_n(\Delta^n)$ to be the corresponding element. In order to define $s^X_n$, we require $s^{\Delta^n}_n$ to satisfy

$$\partial s_n^{\Delta^n}(\iota_n) = \eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial \iota_n),$$

where the right-hand-side is defined by induction. We have the following diagram

$$\begin{tikzcd}
S_n(\Delta^n) \arrow[r, "\partial"] \arrow[d, "\eta_n^{\Delta^n}"', "\xi_n^{\Delta^n}"] & S_{n-1}(\Delta^n) \arrow[r, "\partial"] \arrow[dl, "s_{n-1}^{\Delta^n}"'] \arrow[d, "\eta_{n-1}^{\Delta^n}"', "\xi_{n-1}^{\Delta^n}"] & S_{n-2}(\Delta^n) \arrow[dl, "s_{n-2}^{\Delta^n}"] \\
S_n(\Delta^n \times I) \arrow[r, "\partial"'] & S_{n-1}(\Delta^n \times I) &
\end{tikzcd}$$

Observe that

$$\begin{aligned}
&\partial\big(\eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial\iota_n)\big) \\
&= \eta_{n-1}^{\Delta^n}(\partial\iota_n) - \xi_{n-1}^{\Delta^n}(\partial\iota_n) - \partial\big(s_{n-1}^{\Delta^n}(\partial\iota_n)\big), \quad \text{since } \xi, \eta \text{ are chain maps} \\
&= \eta_{n-1}^{\Delta^n}(\partial\iota_n) - \xi_{n-1}^{\Delta^n}(\partial\iota_n) - \big(\eta_{n-1}^{\Delta^n}(\partial\iota_n) - \xi_{n-1}^{\Delta^n}(\partial\iota_n) - s_{n-2}^{X}(\partial\partial\iota_n)\big), \\
&\qquad\qquad\qquad\qquad \text{as } s_{n-1}^{\Delta^n}, s_{n-2}^{\Delta^n} \text{ are part of the chain homotopy} \\
&= 0, \quad \text{as } \partial\partial\iota_n = 0.
\end{aligned}$$

In other words, the RHS is an $n$-cycle. Since $\Delta^n \times [0,1]$ is contractible, by Theorem 7.11, the RHS is a boundary. In particular, we can *choose* some singular $(n+1)$-chain $a \in S_{n+1}(\Delta^n \times I)$ such that

$$\partial a = \eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial\iota_n).$$

Set, $s_n^{\Delta^n}(\iota_n) = a$. Then, for any space $X$ and any singular $n$-simplex $\sigma : \Delta^n \to X$, set

$$s_n^X(\sigma) = S_{n+1}(\sigma \times \mathrm{Id})(a).$$

Let us verify the required conditions. We have the diagram

$$\begin{tikzcd}
 & & & S_n(\Delta^n) \arrow[rr, "\partial"] \arrow[dl, "S_n(\sigma)"'] \arrow[ddll, "s_n^{\Delta^n}"'] \arrow[dd, "\eta_n^{\Delta^n}"', "\xi_n^{\Delta^n}"] & & S_{n-1}(\Delta^n) \arrow[dl, "S_{n-1}(\sigma)"'] \arrow[ddll, "s_n^{\Delta^n}"'] \arrow[dd, "\eta_{n-1}^{\Delta^n}"', "\xi_{n-1}^{\Delta^n}"] \\
 & & \overset{\sigma}{S_n(X)} \arrow[rr, "\partial"] \arrow[ddll, "s_n^X"'] \arrow[dd, "\eta_n^X"', "\xi_n^X"] & & S_{n-1}(X) \arrow[ddll, "s_{n-1}^X"'] \arrow[dd, "\eta_n^X"', "\xi_{n-1}^X"] & \\
 & \overset{a}{S_{n+1}(\Delta^n \times I)} \arrow[rr, "\partial"] \arrow[dl, "S_{n+1}(\sigma \times \mathrm{Id})"'] & & S_n(\Delta^n \times I) \arrow[rr, "\partial"] \arrow[dl, "S_n(\sigma \times \mathrm{Id})"'] & & S_{n-1}(\Delta^n \times I) \arrow[dl, "S_{n-1}(\sigma \times \mathrm{Id})"'] \\
S_{n+1}(X \times I) \arrow[rr, "\partial"] & & S_n(X \times I) \arrow[rr, "\partial"] & & S_{n-1}(X \times I) &
\end{tikzcd}$$

We compute

$$\begin{aligned}
\partial s_n^X(\sigma) &= \partial S_{n+1}(\sigma \times \mathrm{Id})(a) = S_n(\sigma \times \mathrm{Id})(\partial a) \\
&= S_n(\sigma \times \mathrm{Id})\big(\eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial\iota_n)\big) \\
&= \eta_n^X S_n(\sigma)(\iota_n) - \xi_n^X S_n(\sigma)(\iota_n) - s_{n-1}^X S_{n-1}(\sigma)(\partial\iota_n), \\
&\qquad\qquad \text{as } \xi^X, \eta^X \text{ are natural transformations, and } s_{n-1} \text{ is natural} \\
&= \eta_n^X S_n(\sigma)(\iota_n) - \xi_n^X S_n(\sigma)(\iota_n) - s_{n-1}^X \partial S_n(\sigma)(\iota_n)
\end{aligned}$$

$$= \eta_n^X\sigma - \xi_n^X\sigma - s_{n-1}^X\partial\sigma.$$

Thus, $s_n^X$ satisfies the chain homotopy condition. Next, consider a continuous map $f : X \to Y$. We have the diagram

$$\begin{tikzcd}
 & \overset{\sigma}{S_n(X)} \arrow[rr, "S_n(f)"] \arrow[dl, "s_n^X"'] & & \overset{f\circ\sigma}{S_n(Y)} \arrow[dl, "s_n^Y"'] \\
S_{n+1}(X\times I) \arrow[rr, "S_n(f\times \mathrm{Id})"'] & & S_{n+1}(Y\times I) & \\
 & \underset{a}{S_{n+1}(\Delta^n\times I)} \arrow[ul, bend left, "S_{n+1}(\sigma\times\mathrm{Id})"] \arrow[ur, bend right, "S_{n+1}((f\circ\sigma)\times\mathrm{Id})"'] & &
\end{tikzcd}$$

We compute

$$\begin{aligned}
S_{n+1}(f\times \mathrm{Id})s_n^X(\sigma) &= S_{n+1}(f\times\mathrm{Id})S_{n+1}(\sigma\times\mathrm{Id})(a)\\
&= S_{n+1}((f\circ\sigma)\times\mathrm{Id})(a)\\
&= s_n^Y(f\circ\sigma)\\
&= s_n^Y S_n(f)(\sigma).
\end{aligned}$$

This proves naturality of $s_n$

Hence, inductively, we have a natural chain homotopy $s_\bullet : \xi_\bullet \simeq \eta_\bullet$. ☐

Note that the chain homotopy obtained in Proposition 8.1 is not unique since we made a choice at each induction step, but the homotopy is still natural.

**Theorem 8.2:** (*Singular Homology is Homotopy Invariant*)

Suppose $f, g : (X, A) \to (Y, B)$ are homotopic maps. Then, $H_n(f) = H_n(g) : H_n(X, A) \to H_n(Y, B)$ for all $n$.

**Proof :** Suppose $h : (X, A) \times I \to (Y, B)$ is a homtopy $f \simeq g$. We have the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\eta^A", "\xi^A"'] \arrow[d, hook, "\iota"'] & A\times[0,1] \arrow[r, "h|_{A\times[0,1]}"] \arrow[d, hook, "\iota\times\mathrm{Id}"'] & B \arrow[d, hook] \\
X \arrow[r, "\eta^X", "\xi^X"'] & X\times[0,1] \arrow[r, "h"'] & Y
\end{tikzcd}$$

where, $\xi^X(x) = (x, 0), \eta^X(x) = (x, 1), \xi^A(a) = (a, 0), \eta^A(a) = (a, 1)$. By Proposition 8.1, we have a natural chain homotopy $s_\bullet^X : S_\bullet(\eta^X) \simeq S_\bullet(\xi^X)$ and $s_\bullet^A : S_\bullet(\eta^A) \simeq S_\bullet(\xi^A)$. Naturality implies

$$S_{n+1}(\iota\times\mathrm{Id})\circ s_n^A = s_n^X \circ S_n(\iota).$$

Hence, we can define $s_n^{X,A} : S_n(X, A) \to S_{n+1}(X \times [0,1], A \times [0,1])$, which is clearly a chain homtopy $S_\bullet(\eta^{X,A}) \simeq S_\bullet(\xi^{X,A})$. Then,

$$\begin{aligned} S_n(g) - S_n(f) &= S_n(h \circ \eta^{X,A}) - S_n(h \circ \xi^{X,A}) = S_n(h)\left(S_n(\eta^{X,A}) - S_n(\xi^{X,A})\right) \\ &= S_n(h) \circ \left(\partial s_n^{X,A} + s_{n-1}^{X,A} \partial\right) = \partial\left(S_{n+1}(h) s_n^{X,A}\right) + \left(S_n(h) s_{n-1}^{X,A}\right)\partial. \end{aligned}$$

Thus, $\zeta_n = S_{n+1}(h) s_n^{X,A} : S_n(X) \to S_{n+1}(Y)$ is a chain homtopy $S_\bullet(g) \simeq S_\bullet(f)$. Hence, by Proposition 7.8, we have $H_n(f) = H_n(g)$. □

## 8.2 Barycentric Subdivision

Our next goal is to show that singular homology satisfy the exicion axiom. For that we first need to construct *barycentric subdivision* of simplicies.

Let $D \subset \mathbb{R}^n$ be a convex set. Given points $v_0, \ldots, v_p \in D$, the *affine singular $p$-simplex* is defined as

$$\begin{aligned} \sigma = s\left[v_0, \ldots, v_p\right] : \Delta^p &\to D \\ \sum_{i=0}^{p} \lambda_i e_i &\mapsto \sum_{i=0}^{p} \lambda_i v_i, \end{aligned}$$

where $e_i$ are the standard unit vectors of $\mathbb{R}^{p+1}$. Note that convexitty of $D$ implies that $\sigma$ is well-defined. The *barycenter* of $\sigma$ is defined as

$$\sigma^\beta := \frac{1}{p+1} \sum_{i=0}^{p} v_i.$$

In particular, we shall be interested in the identity map $\iota_p : \Delta^p \to \Delta^p$ and the corresponding barycenter $\iota_p^\beta$.

Now, $D$ being convex, is contractible. In fact, for each $v \in D$, we have a contracting homotopy

$$\begin{aligned} H_v : D \times [0,1] &\to D \\ (u, t) &\mapsto (1-t)u + tv. \end{aligned}$$

Using the cone construction from earlier, we then get a chain homotopy $S_\bullet(D) \to S_{\bullet+1}(D)$ (Proposition 7.10). Let us denote this chain homotopy as

$$\begin{aligned} S_p(D) &\to S_{p+1}(D) \\ \sigma &\mapsto v \cdot \sigma. \end{aligned}$$

In particular, the affine simplex $\sigma = \left[v_0, \ldots, v_p\right]$ is mapped to $v \cdot \sigma := \left[v, v_0, \ldots, v_p\right]$. Recall from Proposition 7.10 that

$$\partial(v \cdot \sigma) = \begin{cases} \sigma - v \cdot \partial\sigma, & p > 0, \\ \sigma - \varepsilon(\sigma) v, & p = 0, \end{cases}$$

where $\varepsilon : S_0(D) \to \mathbb{Z}$ is given by $\varepsilon(\sum n_\sigma \sigma) = \sum n_\sigma$.

Let us now inductively define an operator $\mathcal{B}_p = \mathcal{B}_p(X) : S_p(X) \to S_p(X)$ for any space $X$, called the *barycentric subdivision*. For any $\sigma : \Delta^p \to X$ define,

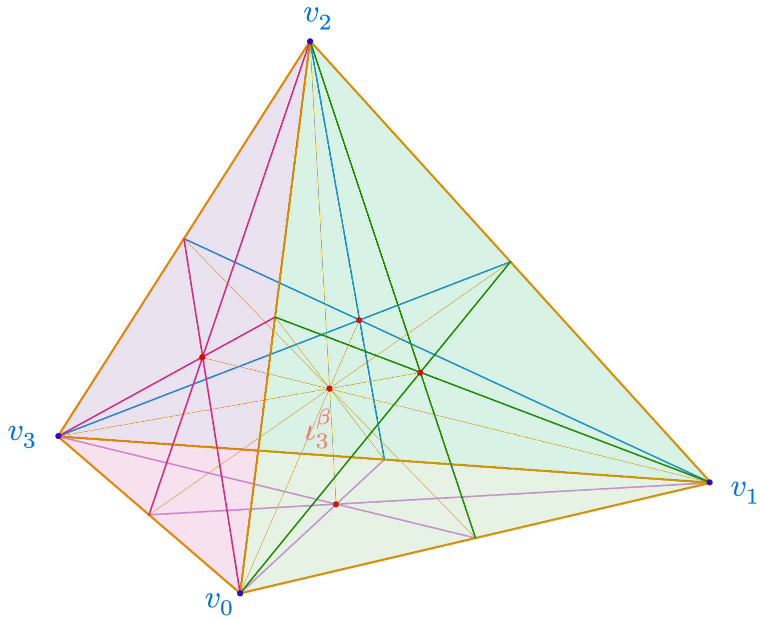
$$\mathcal{B}_p^X(\sigma) = S_p(\sigma) \mathcal{B}_p^{\Delta^p}(\iota_p),$$

where

$$
\mathcal{B}_p^{\Delta^p}(\iota_p) = \begin{cases} i_0, & p = 0 \\ \iota_p^\beta \cdot \mathcal{B}_{p-1}^{\Delta^p}(\partial \iota_p), & p > 0. \end{cases}
$$

Clearly this is well-defined.

Let us visualize the barycentric subdivision of $\iota_3$, the identity simplex of $\Delta^3$.




Barycentric sub-division of $\Delta^3$

On each fact, we have the corresponding barycenter. For the $1$-simplex $[0, 1]$, the barycenter is the just midpoint. The sub-dvision operator is taking the *cone* over the simplices of the barycentric sub-division of the lower-dimensional face. You can see that there are 24 (check!) new $\Delta^3$ after subdividing it once.