Course notes for

Algebraic Topology II (KSM4E02)

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`long exact sequence from short exact sequence of chain complexes – relative singular homology – long exact sequence of singular homology – chain homotopy invariance – singular homology of contractible space`

7.1 Long Exact Sequence in Homology

So far we have been working with only Abelian groups, which are nothing but $\mathbb{Z}$-modules. More generally, we can fix a commutative ring R, and work with R-modules. Generalizing even further, we can consider any arbitrary *Abelian category* $\mathcal{A}$, where 0-maps and taking quotients makes sense (see Definition 6.27). Then, we have the chain complex of objects from $\mathcal{A}$, denoted as $\mathrm{Ch}(\mathcal{A})$. The definition of n^{th}-homology group makes sense for $\mathrm{Ch}(\mathcal{A})$ as well. Homological algebra is the study of the (co)homology of (co)chain complex over some Abelian category.

Theorem 7.1: (*Long Exact Sequence of Homology of Chain Complex*)

Given a short exact sequence $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \to 0$ of chain complexes, there exists a long exact sequence of homology groups :

$$\cdots \to H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{\iota_\star} H_n(B) \xrightarrow{j_\star} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots,$$

which is natural with respect to maps of short exact sequences of chain complexes.

Proof : We consider part of the diagram

$$\begin{array}{ccccccccc}
 & & \ker \partial_n^A & & \ker \partial_n^B & & \ker \partial_n^C & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_n & \xrightarrow{\iota_n} & B_n & \xrightarrow{j_n} & C_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_n^A} & & \downarrow{\scriptstyle \partial_n^B} & & \downarrow{\scriptstyle \partial_n^C} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{\iota_{n-1}} & B_{n-1} & \xrightarrow{j_{n-1}} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker} \partial_n^A & & \operatorname{coker} \partial_n^B & & \operatorname{coker} \partial_n^C & &
\end{array}$$

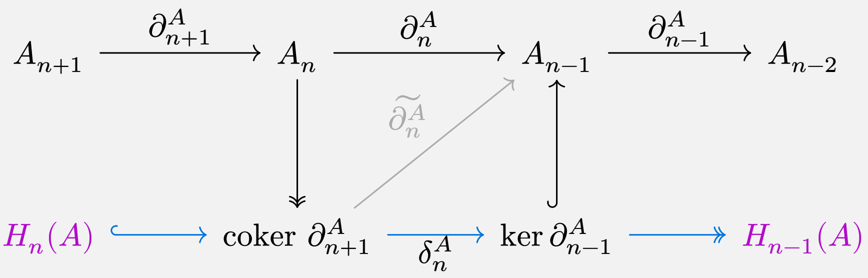
Since both the rows are exact, applying the snake lemma (Lemma 6.29), we have the exact sequence

$$0 \to \ker \partial_n^A \to \ker \partial_n^B \to \ker \partial_n^C \to \operatorname{coker} \partial_n^A \to \operatorname{coker} \partial_n^B \to \operatorname{coker} \partial_n^C \to 0.$$

This holds for each n. In particular, we consider the following diagram

$$
\begin{array}{ccccccc}
 & \text{coker } \partial_{n+1}^A & \longrightarrow & \text{coker } \partial_{n+1}^B & \longrightarrow & \text{coker } \partial_{n+1}^C & \longrightarrow 0 \\
 & \delta_n^A \downarrow & & \delta_n^B \downarrow & & \delta_n^C \downarrow & \\
0 \longrightarrow & \ker \partial_{n-1}^A & \longrightarrow & \ker \partial_{n-1}^B & \longrightarrow & \ker \partial_{n-1}^C &
\end{array}
$$

By above, the rows are exact. The maps $\delta_n^A, \delta_n^B, \delta_n^C$ are induced from $\partial_n^A, \partial_n^B, \partial_n^C$ respectively. Thus, the diagram is commutative. Observe that $\ker \delta_n^A = H_n(A)$ and $\text{coker } \delta_n^A = H_{n-1}(A)$, and similarly for δ_n^B, δ_n^C. Indeed, we have the diagram which might be of help



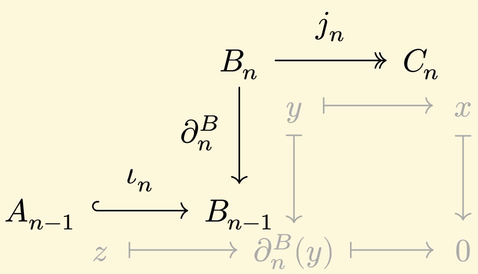
Hence, again applying the snake lemma, we have an exact sequence

$$H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C).$$

Chasing the diagram carefully, shows that the connecting maps are induced by $\iota_\bullet$ and $j_\bullet$. Pasting these exact sequences together, we get the long exact sequence in homology. Since snake lemma produces *natural* exact sequences, again a diagram chase shows that the long exact sequence is natural. $\square$

Remark 7.2: (*The Boundary Map*)

Let us describe the boundary map explicitly. Consider part of the diagram



Suppose $\alpha \in H_n(C)$ is represented as $\alpha = [x]$ for some $x \in C_n$, with $\partial(x) = 0$. Then, it follows that $\partial(\alpha) = [z]$. The well-definedness is a consequence of the snake lemma. Thus, we can 'define' $\partial([x]) = [\iota_n^{-1} \partial_n^B j_n^{-1}(x)]$.

7.2 Relative Singular Homology and Long Exact Sequence

Let us now extend our definition of singular homology to pairs of spaces (X, A) with $A \subset X$. Given such a pair (X, A), observe that any singular n-simplex $\sigma : \Delta^n \to A$ is naturally an n-simplex in X via the inclusion map $\iota : A \hookrightarrow X$. Thus, we have an *injective* map $S_\bullet(\iota) : S_\bullet(A) \hookrightarrow S_\bullet(X)$. Let us define the cokernel

$$S_n(X, A) := \frac{S_n(X)}{S_n(A)}.$$

Observe that $S_n(X, A)$ is an Abelian group freely generated by singular n-simplexes $\sigma : \Delta^n \to X$ such that $\text{im}(\sigma)$ is not *completely* contained in A. We have the well-defined *relative* boundary map

$$\partial_n^{X,A} : S_n(X,A) \longrightarrow S_{n-1}(X,A)$$
$$\left[\sum a_\sigma \sigma\right] \longmapsto \sum a_\sigma [\partial_n^X(\sigma)].$$

In particular, if we have an n-chain in X whose boundary is compleletly contained in A, then its relative boundary is zero. It follows that $S_\bullet(X,A)$ is a chain complex, which is called the *relative singular chain complex*.

Exercise 7.3: (*Relative Singular Chain Complex is Functorial*)

Verify that $S_\bullet : \mathrm{TopPair} \to \mathrm{Ch}$ is a functor, and we can identify $S_n(X) = S_n(X, \emptyset)$ (which justifies the same notation). Moreover, $j : X = (X, \emptyset) \to (X, A)$ induces the quotient map $S_\bullet(X) \to S_\bullet(X,A)$.

The homology groups

$$H_n(X,A) := H_n(S_n(X,A))$$

are called the *relative singular homology groups* of the pair (X,A). By Exercise 6.7 and Exercise 7.3, it follows that the relative singular homology groups are functors as well. In particular, given $f : (X,A) \to (Y,B)$, we have $H_n(f) = H_n(S_\bullet(f))$.

We have a short exact sequence of chain complexes

$$0 \to S_\bullet(A) \overset{\iota_\bullet}{\hookrightarrow} S_\bullet(X) \overset{j_\bullet}{\twoheadrightarrow} S_\bullet(X,A) \to 0,$$

where $\iota : A \hookrightarrow X$ and $j : X \hookrightarrow (X,A)$ are the space level maps.

Theorem 7.4: (*Long Exact Sequence of Singular Homology*)

There exists a natural long exact sequence

$$\cdots \to H_n(A) \overset{\iota_\star}{\to} H_n(X) \overset{j_\star}{\to} H_n(X,A) \overset{\partial}{\to} H_{n-1}(A) \to \cdots \to H_0(A) \overset{\iota_\star}{\to} H_0(X) \overset{j_\star}{\to} H_0(X,A) \to 0.$$

Proof : The proof is immediate from Theorem 7.1. □

Theorem 7.4 justifies that singular homology satisfies the exactness axiom.

7.3 Chain Homotopy Invariance

Let us recall the notion of *chain maps* $f_\bullet : C_\bullet \to D_\bullet$ (Definition 1.13), which is given a by a collection of maps $f_n : C_n \to D_n$ such that $f_{n-1} \circ \partial_n^C = \partial_n^D \circ f_n$ holds.

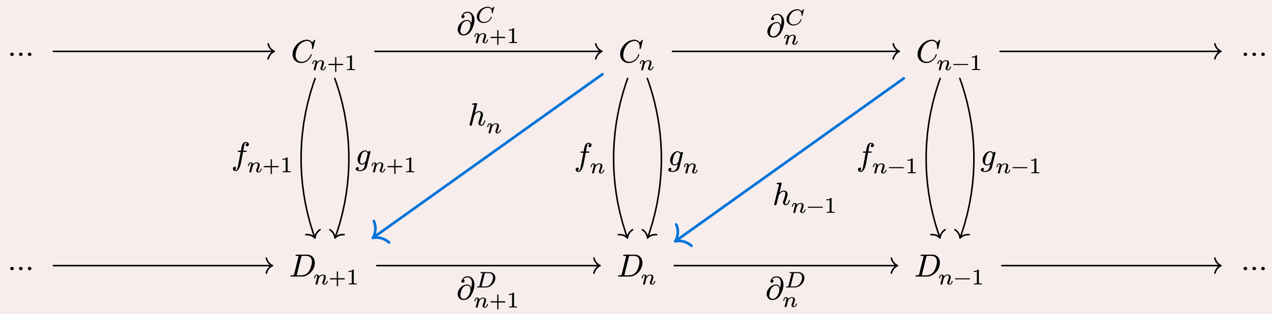
Given a chain complex $C_\bullet$, the *shift (or translation)* of $C_\bullet$ by degree k is the chain complex $C[k]_\bullet$ defined as $C[k]_n = C_{n+k}$ and $\partial_n^{C[k]} = (-1)^k \partial_{n+k}^C$. We shall see later why the sign $(-1)^k$ appears in the definition. A *degree k chain map* $C_\bullet \to D_\bullet$ is a chain map $C_\bullet \to D[k]_\bullet$. In particular, a chain map $C_\bullet \to D_\bullet$ has degree 0.

Exercise 7.5: (*Shift Functor*)

Let $\Sigma^k : \mathrm{Ch} \to \mathrm{Ch}$ be the functor that shifts a chain complex by degree k. Check that $\Sigma^k \circ \Sigma^l = \Sigma^{k+l}$ and $\Sigma^0 = \mathrm{Id}$. In particular, Σ^k is an *isomorphism* with inverse Σ^{-k}.

Definition 7.6: (*Chain Map and Chain Homotopy*)

Given two chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$, a *chain homotopy* between them is a degree 1 map $h_\bullet : C_\bullet \to D_{\bullet+1}$ such that $f - g = \partial \circ h + h \circ \partial$ holds. That is, we have the following diagram



where, we have $f_n - g_n = h_{n-1} \circ \partial^C_n + \partial^D_{n+1} \circ h_n$

Exercise 7.7: (*Chain Homotopy is an Equivalence Relation*)

Check that the chain homotopy is an equivalence relation on the collection of all chain maps $C_\bullet \to D_\bullet$.

Proposition 7.8: (*Homology of Chain Homotopic Maps*)

Suppose $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are homotopic chain maps. Then, they induce the same map between homology groups.

Proof : Suppose $f - g = \partial h + h\partial$ for a chain homotopy $h : C_\bullet \to D_{\bullet+1}$. Since homology is an additive functor (Exercise 6.7), we only need to show that $H_n(\partial h + h\partial) = 0$. For a class $[\alpha] \in H_n(C)$, we have $\partial(\alpha) = 0$. Recall, $[\alpha] = \alpha + \operatorname{im}\partial$ Then, it follows

$$H_n(\partial h + h\partial)[\alpha] = [(\partial h + h\partial)(\alpha)] = [\partial h(\alpha)] = 0.$$

Thus, $H_n(f) = H_n(g)$ follows for all n. □

7.4 Singular Homology of Contractible Space

Before proving the homotopy invariance for singular homology, let us start with a contractible space, say, X. Fix a point $x_0 \in X$. Let us define a chain map $\varepsilon_\bullet : S_\bullet(X) \to S_\bullet(X)$ by setting $\varepsilon_n = 0$ for $n \neq 0$, and

$$\varepsilon_0\Big(\sum a_\sigma \sigma\Big) = \Big(\sum a_\sigma\Big)\sigma_0,$$

where $\sigma_0 : \Delta^0 \to X$ is the 0-simplex that maps to x_0.

Proposition 7.9:

$\varepsilon_\bullet : S_\bullet(X) \to S_\bullet(X)$ is a chain map.

Proof : The only nontrivial part is the commutativity of the diagram

$$\begin{array}{ccc} S_1(X) & \xrightarrow{d_1} & S_0(X) \\ {\scriptstyle \varepsilon_1 = 0}\downarrow & & \downarrow{\scriptstyle \varepsilon_0} \\ S_1(X) & \xrightarrow[d_1]{} & S_0(X) \end{array}$$

Suppose $\sigma : \Delta^1 \to X$ is a singular 1-simplex. Then,

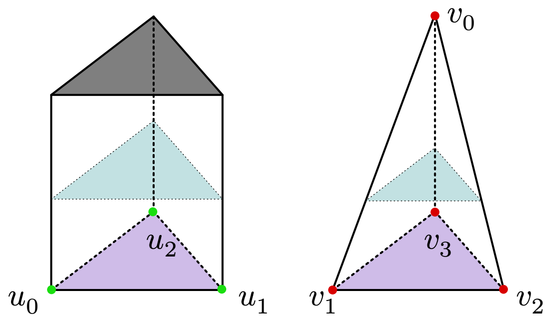
$$\varepsilon_0 d_1(\sigma) = \varepsilon_0(\sigma \circ d_0 - \sigma \circ d_1) = (1-1)\sigma_0 = 0.$$

Thus, $\varepsilon_0 d_1 = 0 = d_1 0$. This proves that $\varepsilon_\bullet$ is a chain map. $\square$

Now, suppose $h : X \times [0,1] \to X$ is a hometopy from Id_X to the constant map $c_{x_0} : X \to X$ which maps everything to a point $x_0 \in X$. Using h, we define a chain homotopy between $\varepsilon_\bullet$ and the identity chain map. Firstly, consider the map

$$\begin{aligned} q_n : \Delta^n \times [0,1] &\longrightarrow \Delta^{n+1} \\ ((t_0, \ldots, t_n), s) &\longmapsto (s, (1-s)t_0, \ldots, (1-s)t_n) \end{aligned}$$

Since $s + \sum_{i=0}^n (1-s)t_i = s + (1-s) = 1$, the map is well-defined, and clearly, it is continuous. As q_n is a surjective map between compact T^2 spaces, we have q_n is a quotient map.



The map $q_2 : \Delta^2 \times [0,1] \to \Delta^3$

In fact, it follows that q_n identifies $\Delta^n \times \{1\}$ to the vertex $v_0 = (1, 0, \ldots, 0) \in \Delta^{n+1}$, and keeps every other point as it is. Note that q_0 is the identity map $[0,1] \to [0,1]$. Now, given a singular n-simplex $\sigma : \Delta^n \to X$, consider the diagram

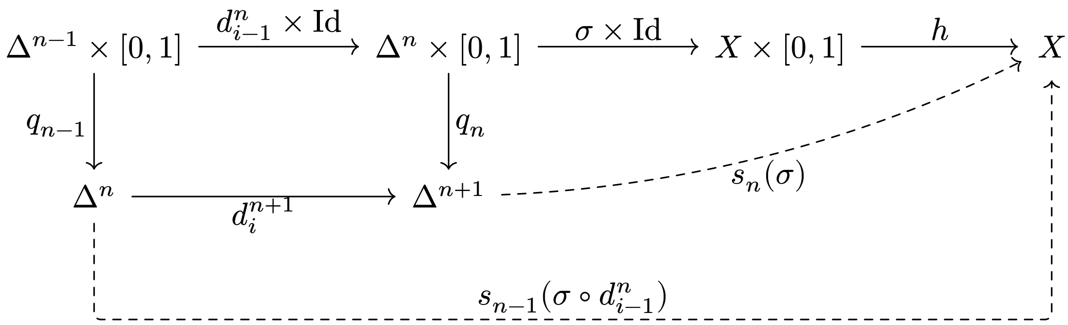
$$\begin{array}{ccccc} \Delta^n \times [0,1] & \xrightarrow{\sigma \times \mathrm{Id}} & X \times [0,1] & \xrightarrow{h} & X \\ \downarrow{\scriptstyle q_n} & & & \nearrow_{s_n(\sigma)} & \\ \Delta^{n+1} & & & & \end{array}$$

Since $h(X \times \{1\}) = \{x_0\}$, it follows from the universal property of a quotient map that there exists a unique continuous map $s_n(\sigma) : \Delta^{n+1} \to X$ such that

$$s_n(\sigma) \circ q_n = h \circ \left(\sigma \times \mathrm{Id}_{[0,1]}\right).$$

Extending linearly, we have a map $s_n : S_n(X) \to S_{n+1}(X)$.

Now, observe that for $0 < i \leq n+1$ we have the diagram



Hence, $s_n(\sigma)d_i^{n+1} = s_{n-1}(\sigma d_{i-1}^n)$ holds for $0 < i \leq n+1$. Clearly, $s_n(\sigma)d_0^{n+1} = \sigma$.

Proposition 7.10:

$s_\bullet : S_\bullet(X) \to S_{\bullet+1}(X)$ is a chain homotopy betwen Id and $\varepsilon_\bullet$.

Proof : For a 0-simplex $\sigma : \Delta^0 \to X$, we have

$$\partial(s\sigma) = (s\sigma)d_0 - (s\sigma)d_1 = \sigma - \sigma_0,$$

since $h(\sigma(0), 1) = x_0$. For a singular n-simplex $\sigma : \Delta^n \to X$ with $n \geq 1$, we compute

$$\begin{aligned}
\partial(s\sigma) &= \sum_{i=0}^{n+1} (-1)^i (s\sigma) d_i^{n+1} = (s\sigma)d_0^{n+1} - \sum_{i=1}^{n+1} (-1)^{i-1}(s\sigma)d_i^{n+1} \\
&= \sigma - \sum_{i=1}^{n+1} (-1)^{i-1} s(\sigma d_{i-1}^n) \\
&= \sigma - s\left(\sum_{i=0}^{n} (-1)^i \sigma d_i^n\right), \text{by replacing } i \leftrightarrow i-1, \text{and by linearity of } s \\
&= \sigma - s(\partial\sigma)
\end{aligned}$$

$$\Rightarrow \sigma - \underbrace{\varepsilon_n(\sigma)}_{0} = \partial(s\sigma) + s(\partial\sigma).$$

Thus, we have $\mathrm{Id} - \varepsilon = \partial s + s\partial$. In other words, $s_\bullet$ is a chain homotopy between Id and $\varepsilon_\bullet$ □

Theorem 7.11: (*Singular Homology of Contractible Space*)

Let X be a contracitble space. Then, $H_n(X) = 0$ for all $n \neq 0$.

Proof : Suppose $h : X \times [0,1] \to X$ is a homotopy that takes Id_X to the constant map c_{x_0} for some point $x_0 \in X$. Then, as above, construct the chain map $\varepsilon_\bullet : S_\bullet(X) \to S_\bullet(X)$. Since by Proposition 7.10 Id is chain homotopic to ε, an application of Proposition 7.8 it follows that they give the same map in homology. But homology is an additive functor. Hence, $\mathrm{Id} = H_n(\mathrm{Id}) = H_n(\varepsilon) = 0$ for $n \neq 0$. The only way identity map can be the zero map if the object is zero. Thus, $H_n(X) = 0$ for $n \neq 0$. □

Exercise 7.12: (0^{th}-*Homology and Path Component*)

Show that $H_0(X)$ is freely generated by the path components of X. In particular, $H_0(X) = \mathbb{Z}$ for a path connected space.