Course notes for

Algebraic Topology II (KSM4E02)

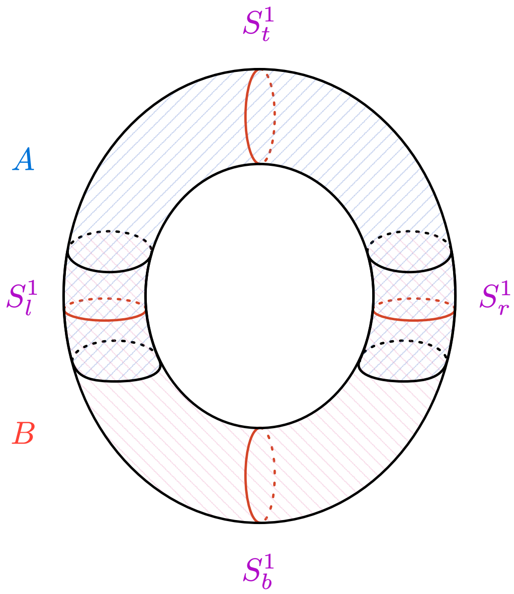
Instructor: Aritra Bhowmick

Day 6 : 6th February, 2026

`homology of` T^2 `– singular homology – singular homology of point – categorical digression : kernel, cokernel, abelian category – snake lemma`

6.1 Homology of Torus $T^2 = S^1 \times S^1$ (with $\mathbb{Z}$ coefficients)

Suppose $H_\star$ is a homology theory with $H_0(\star) = \mathbb{Z}$. Let us now compute the homology groups of the torus $T^2 = S^1 \times S^1$ using the Mayer-Vietoris sequence (Theorem 5.2). Consider the following decomposition $T^2 = A \cup B$.



Observe that $A \cong B \simeq S^1$, and $A \cap B \simeq S^1 \sqcup S^1$. Then, by Corollary 4.2, we have $H_1(S^1) = \mathbb{Z} = H_0(S^1)$, and all other homology of S^1 is zero. By Proposition 3.1, we have $H_\star(A \cap B) = H_\star(S^1) \oplus H_\star(S^1) = \mathbb{Z} \oplus \mathbb{Z}$ for $\star = 0, 1$, and 0 otherwise. It is easy to obeserve that $(A, A \cap B) \hookrightarrow (T^2, B)$ and $(B, A \cap B) \hookrightarrow (T^2, A)$ satisfy the hypothesis for excision, and thus, $(T^2;, A, B)$ is a proper triad.

Let us consider the diagrams

$$\begin{tikzcd} S^1_l \arrow[r, hook] \arrow[d, equal] & A \cap B \arrow[d, hook] & S^1_r \arrow[l, hook'] \arrow[d, equal] \\ S^1_t \arrow[r, hook] & A & S^1_t \arrow[l, hook'] \end{tikzcd} \qquad \begin{tikzcd} S^1_l \arrow[r, hook] \arrow[d, equal] & A \cap B \arrow[d, hook] & S^1_r \arrow[l, hook'] \arrow[d, equal] \\ S^1_b \arrow[r, hook] & B & S^1_b \arrow[l, hook'] \end{tikzcd}$$

which commute up to homtopy. Consequently, we can identify the map $\psi : H_\star(A \cap B) \to H_\star(A) \oplus H_\star(B)$ from the Mayer-Vietoris sequence as

$$\psi(x, y) = (x + y, -x - y).$$

Note that ψ is an injective map. Let us consider the exact sequence

$$\cdots \to \underbrace{H_2(A) \oplus H_2(B)}_{0} \to H_2(T^2) \to \underbrace{H_1(A \cap B)}_{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\psi} \underbrace{H_1(A) \oplus H_1(B)}_{\mathbb{Z}\oplus\mathbb{Z}} \to \dots.$$

As $\psi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is injective, we have

$$H_2(T^2) = \ker(\psi) = \{(x, y) \mid x = -y\} \cong \mathbb{Z}.$$

Next, we compute $H_1(T^2)$. Note that $\varphi : H_\star(A) \oplus H_\star(B) \to H_\star(T^2)$ is given by

$$\varphi(x, y) = x - y.$$

Now, using the *reduced* Mayer-Vietories sequence, we have

$$\cdots \to \underbrace{\tilde{H}_1(A) \oplus \tilde{H}_1(B)}_{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\varphi} \tilde{H}_1(T^2) \to \underbrace{\tilde{H}_0(A \cap B)}_{\mathbb{Z}} \xrightarrow{\psi} \underbrace{\tilde{H}_0(A) \oplus \tilde{H}_0(B)}_{0} \to \cdots$$

We have $\operatorname{im} \varphi = \mathbb{Z}$, and thus, injectivity of ψ gives the short exact sequence

$$0 \to \underbrace{\operatorname{im}(\varphi)}_{\mathbb{Z}} \to H_1(T^2) \to \mathbb{Z} \to 0.$$

This implies $H_1(T^2) = \mathbb{Z} \oplus$. Finally, again the reduced Mayer-Vietoris sequence gives the exact sequence,

$$\cdots \to \underbrace{\tilde{H}_0(A) \oplus \tilde{H}_0(B)}_{0} \to \tilde{H}_0(T^2) \to \underbrace{\tilde{H}_{-1}(A \cap B)}_{0} \to \cdots$$

Hence, $H_0(T^2) = \tilde{H}_0(T^2) \oplus H_0(\star) = 0 \oplus \mathbb{Z} = \mathbb{Z}$. It is easy to see, $H_k(T^2) = 0$ for $k \geq 3$ from the Mayer-Vietoris sequence. Hence, we have computed

$$H_k(T^2) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z} \oplus \mathbb{Z}, & k = 1 \\ \mathbb{Z}, & k = 2 \\ 0, & k > 2. \end{cases}$$

6.2 Singular Homology Theory

As a first step in defining the singular homology, let us begin with n-simplex.

Definition 6.1: (n*-simplex*)

The standard (or geometric) n*-simplex* is defined as

$$\Delta^n := \left\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum t_i = 1\right\}$$

Denoting $e_i := (0, \dots, 1, \dots, 0)$ to be the i^{th} standard unit vector in $\mathbb{R}^{n+1}$, we have Δ^n is the convex hull of $\{e_0, \dots, e_n\}$

In the language of category theory, the collection $\{\Delta_n\}$ is a *cosimplicial space*. Explicitly, recall from Example 1.3, the simplicial category $\mathbf{\Delta}$. We have a covariant functor $\Delta : \mathbf{\Delta} \to \mathbf{Top}$ given by $\Delta(n) = \Delta^n$. To describe the morphisms, consider a (weakly) order preserving map $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$. Then, we have

$$\Delta(\alpha) : \Delta^m \longrightarrow \Delta^n$$

$$\sum_{i=0}^{m} t_i e_i \mapsto \sum_{i=0}^{m} t_i e_{\alpha(i)}.$$

In $\mathbf{\Delta}$, there are some special maps. For $0 \le i \le n$, we have the *face maps* $\delta_i^n : [n-1] \to [n]$ given by

$$\begin{tikzcd}[row sep=large]
{[n-1]} \arrow[d, "\delta_i^n"'] & : & 0 \arrow[d] & 1 \arrow[d] & \cdots & i-1 \arrow[d] & & i \arrow[d] & i+1 \arrow[d] & \cdots & n-1 \arrow[d] \\
{[n]} & & 0 & 1 & & i-1 & i & i+1 & i+2 & & n
\end{tikzcd}$$

That is, δ_i^n is the map that *misses* i. On the other hand, for $0 \le i \le n$, we have the *degeneracy maps* $\sigma_i^n : [n+1] \to [n]$ given by

$$\begin{tikzcd}[row sep=large]
{[n+1]} \arrow[d, "\sigma_i^n"'] & : & 0 \arrow[d] & 1 \arrow[d] & \cdots & i \arrow[d] & i+1 \arrow[dl] & i+2 \arrow[dl] & \cdots & n+1 \arrow[dl] \\
{[n]} & & 0 & 1 & & i & i+1 & & n &
\end{tikzcd}$$

That is σ_i^n is the map that repeats i twice. It is a well-known fact that any morphism of $\mathbf{\Delta}$ can be written as finite composition of these δ_i^n and σ_j^m. These maps satisfy some identities, known as the *simplicial identities*:

$$\begin{aligned}
\delta_j^{n+1} \circ \delta_i^n &= \delta_i^{n+1} \circ \delta_{j-1}^n, && i < j, \\
\sigma_j^{n+1} \circ \sigma_i^n &= \sigma_i^{n+1} \circ \sigma_{j+1}^n, && i \le j, \\
\sigma_j^{n-1} \circ \delta_i^n &= \begin{cases} \delta_i^{n-1} \circ \sigma_{j-1}^n, & i < j, \\ \mathrm{Id}_{[n-1]}, & i = j, j+1, \\ \delta_{i-1}^{n-1} \circ \sigma_j^n, & i > j+1. \end{cases}
\end{aligned}$$

Denote the face and degeneracy maps

$$\begin{aligned} d_i^n := \Delta(\delta_i^n) : \Delta^{n-1} &\longrightarrow \Delta^n, \\ \sum t_j e_j &\longmapsto \sum t_j e_{\delta_i^n(j)}, \end{aligned} \qquad \begin{aligned} s_i^n := \Delta(\sigma_i^n) : \Delta^{n+1} &\longrightarrow \Delta^n \\ \sum t_j e_j &\longmapsto \sum t_j e_{\sigma_i^n(j)}. \end{aligned}$$

Definition 6.2: (*Singular Chain Complex*)

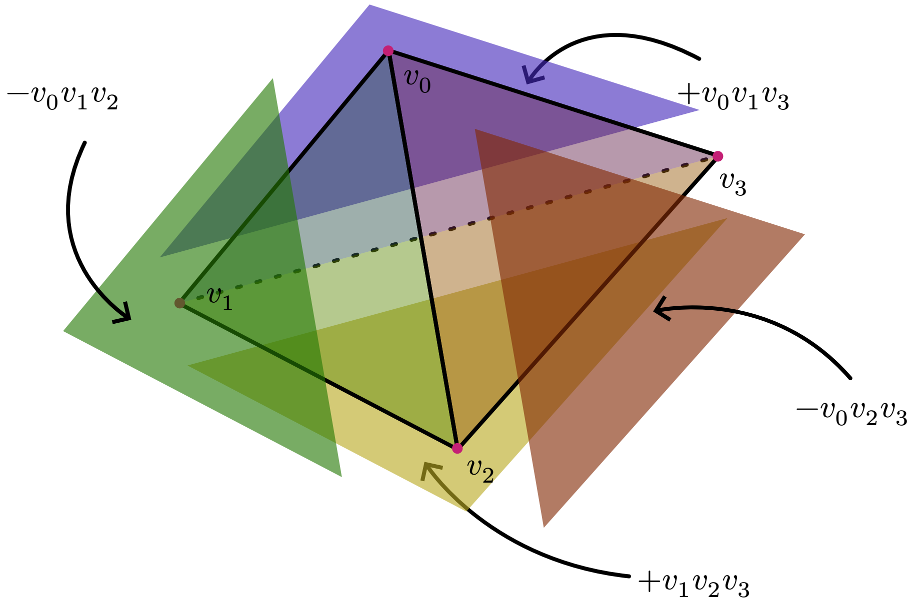
Given a space X, a *singular n-simplex* is a continuous map $\sigma : \Delta^n \to X$. The *singular chian complex* $S_\bullet(X)$ of X consists of the following data.

- The abelian group $S_n(X)$ freely generated by all singular n-simplicies. For $n < 0$, we have $S_n(X) = 0$.
- The boundary maps $\partial_n : S_n(X) \to S_{n-1}(X)$ defined on a generator $\sigma : \Delta^n \to X$ via

  $$\partial_n(\sigma) := \sum_{i=1}^{n} \sigma \circ d_i^n,$$

  and then extended $\mathbb{Z}$-linearly.

Any element of $S_n(X)$ is called a *singular n-chain*, an element of $\ker(\partial_n) \subset S_n(X)$ is called a *singular n-cycle*, an element of $\mathrm{im}(\partial_{n+1}) \subset S_n(X)$ is called a *singular n-boundary*.



Visual representaion of the boundary map : $\partial_2(\Delta^3)$ is the signed sum of the faces

Exercise 6.3: (*Singular Chain Complex is a Functor*)

Check that $S_\bullet : \mathrm{Top} \to \mathrm{Ch}$ is a functor, where Ch is the category of chain complexes and chain maps.

Hint : Given a map $f : X \to Y$, define $S_n(f)(\sigma) = f \circ \sigma$ for a singular n-simplex $\sigma : \Delta^n \to X$, and extend linearly. Check that $S_\bullet(f) : S_\bullet(X) \to S_\bullet(Y)$ is a chain map. Then, check that $S_\bullet(g \circ f) = S_\bullet(g) \circ S_\bullet(f)$ and $S_\bullet(\mathrm{Id}_X) = \mathrm{Id}_{S_\bullet(X)}$.

Proposition 6.4: (*Boundary in Singular Chain Complex*)

$\partial_n \circ \partial_{n+1} = 0$, and thus, $(S_\bullet(X), \partial)$ is indeed a chain complex.

Proof : We need to check on generators only. Consider $\sigma : \Delta^{n+1} \to X$. Then we compute

$$\begin{aligned}
\partial_n\partial_{n+1}(\sigma) &= \partial_n\left(\sum_{j=0}^{n+1}(-1)^j\sigma\circ d_j^{n+1}\right) = \sum_{j=0}^{n+1}(-1)^j\partial_n\left(\sigma\circ d_j^{n+1}\right) = \sum_{j=0}^{n+1}\sum_{i=0}^{n}(-1)^{i+j}\left(\sigma\circ d_j^{n+1}\circ d_i^n\right)\\
&= \sum_{i<j}(-1)^{i+j}\left(\sigma d_j^{n+1}d_i^n\right) + \sum_{j\le i}(-1)^{i+j}\left(\sigma d_j^{n+1}d_i^n\right)\\
&= \sum_{i<j}(-1)^{i+j}\left(\sigma d_i^{n+1}d_{j-1}^n\right) + \sum_{j\le i}(-1)^{i+j}\left(\sigma d_j^{n+1}d_i^n\right)\\
&= \sum_{i\le j'}(-1)^{i+j'+1}\left(\sigma d_i^{n+1}d_{j'}^n\right) + \sum_{j\le i}(-1)^{i+j}\left(\sigma d_j^{n+1}d_i^n\right)\\
&= -\sum_{i\le j}(-1)^{i+j}\left(\sigma d_i^{n+1}d_j^n\right) + \sum_{i\le j}(-1)^{i+j}\left(\sigma d_i^{n+1}d_j^n\right)\\
&= 0.
\end{aligned}$$

Hence, $\partial_n \circ \partial_{n+1} = 0$. Consequently, $(S_\bullet(X), \partial)$ is a chain complex. $\square$

Definition 6.5: (*Homology of a Chain Complex*)

Given any chain complex $C_\bullet = (C_n, \partial)$, the *$n^{\text{th}}$-homology group* is defined as the quotient

$$H_n(C_\bullet) = \frac{\ker(\partial_n : C_n \to C_{n-1})}{\operatorname{im}(\partial_{n+1} : C_{n+1} \to C_n)}.$$

Remark 6.6: (*Cochain Complex and Cohomology*)

We shall also consider the following case

$$C^\bullet : \cdots \to C^n \xrightarrow{\partial^n} C^{n+1} \to \cdots$$

such that $\partial^{n+1} \circ \partial^n = 0$. We say $C^\bullet$ is a *cochain complex*, and the *n^{th}-cohomology group* of $C^\bullet$ is defined as the quotient

$$H^n(C^\bullet) = \frac{\ker(\partial^n : C^n \to C^{n+1})}{\operatorname{im}(\partial^{n-1} : C^{n-1} \to C^n)}.$$

Observe that given a cochain complex $(C^\bullet, \partial^\bullet)$, we have a chain complex $(D_\bullet, \partial_\bullet)$ defined by $D_n := C^{-n}$, and $\partial_n = \partial^{-n}$.

Exercise 6.7: (*Homology is an Additive Functor*)

Given a chain map $f_\bullet : C_\bullet \to D_\bullet$, define

$$H_n(f_\bullet)([x]) = [f_n(x)], \quad [x] \in H_n(C).$$

Verify that this is well-defined, and $H_n : \mathrm{Ch} \to \mathrm{Ab}$ is a functor. Similarly, H^n is a functor as well from the category of cochain complexes to abelian groups. Moreover, verify that H_n is an *additive* functor, i.e, given $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$, we have $H_n(af_\bullet + bg_\bullet) = aH_n(f_\bullet) + bH_n(g_\bullet)$, for scalars a, b (i.e, $a, b \in R = \mathbb{Z}$)

Definition 6.8: (*Singular Homology*)

Given a space X, the *singular n-homology* is the n^{th}-homology group of the singular chain complex $S_\bullet(X)$, and it is dentoed as $H_n(X)$.

Remark 6.9: (*Homology is a Functor*)

It follows from Exercise 6.3 and Exercise 6.7 that $H_n : \mathrm{Top} \to \mathrm{Ab}$ is a functor as it is a composition of two functors.

Let us compute an example!

Proposition 6.10: (*Singular Homology of a Singleton*)

$H_n(\star) = \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0. \end{cases}$

<ins>**Proof**</ins> **:** Observe that for each n, there exists a unique map $\sigma_n : \Delta^n \to \star$, namely the constant map. Thus, we have $S_n(X) = \mathbb{Z}$ for $n \geq 0$. Let us figure out the boundary maps. For $n > 0$, we have

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i (\sigma_n \circ d_i^n) = \sum_{i=0}^{n} (-1)^i (\sigma_{n-1}) = \begin{cases} \sigma_{n-1}, & n \text{ is even} \\ 0, & n \text{ is odd.} \end{cases}$$

Thus, the singular n-complex $S_\bullet(\star)$ is given as

$$\cdots \to \underbrace{S_4}_{\mathbb{Z}} \xrightarrow{\mathrm{Id}} \underbrace{S_3}_{\mathbb{Z}} \xrightarrow{0} \underbrace{S_2}_{\mathbb{Z}} \xrightarrow{\mathrm{Id}} \underbrace{S_1}_{\mathbb{Z}} \xrightarrow{0} \underbrace{S_0}_{\mathbb{Z}} \to 0.$$

Immediately we get $H_n(\star) = \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0. \end{cases}$ □

The above shows that singular homology satisfies the dimension axiom from the Eilenberg-Steenrod axioms.

6.3 A Categorical Digression : Product, Coproduct, Kernel, Cokernel

In this section, we re-interpret some notions from algebra in category theoretic language. If it seems too abstract, you can skip it completely!

Definition 6.11: (*Product and Initial Object*)

Let $\mathcal{C}$ be a category, and $\{A_i\}_{\{i \in I\}}$ be a (possibly empty!) collection of objects of $\mathcal{C}$. The *product* is then defined via the following universal property.

> The product is an object $X \in \mathcal{C}$ and a collection of maps $\pi_i : X \to A_i$, such that given any other object $Y \in \mathcal{C}$ and any other collection of maps $f_i : Y \to A_i$, there exists a **unique** map $f : Y \to X$ such that $f_i = \pi_i \circ f$. We have the following diagram.

$$\begin{tikzcd} Y \arrow[d, dashed, "\exists! f"'] \arrow[dr, bend left, "f_i"] & \\ X \arrow[r, "\pi_i"'] & A_i \end{tikzcd}$$

An *initial object* in $\mathcal{C}$ is a product of an empty collection.

Since product is defined via an universal property, if it exists, it is unique up to unique isomorphism. We have an alternative, perhaps more useful definition of the initial object.

Exercise 6.12: (*Initial Object*)

Show that the initial object of a category $\mathcal{C}$ (if exists) can be equivalently defined as an object X such that given any object $Y \in \mathcal{C}$, there exists a unique morphism $X \to Y$.

Example 6.13: (*Examples of Product and Initial Objects*)

Here a few examples, that one should verify!

- Set : product is the Cartesian product of sets, and initial object is the emptyset.
- Group : product is the direct product, and initial object is the group with one element.
- Top : product is the Cartesian product of the underlying sets equipped with the product topology, and initial object is the emptyset.
- $\mathrm{Top}_\star$: product is the *smash product*, i.e, $\prod(X_i, x_i) = \frac{\prod X_i}{\bigvee X_i}$, and initial object is the singleton.

The dual notion to product is the coproduct.

Definition 6.14: (*Coproduct and Final Object*)

Let $\mathcal{C}$ be a category, and $\{A_i\}_{\{i\in I\}}$ be a (possibly empty!) collection of objects of $\mathcal{C}$. The *coproduct* is then defined via the following universal property.

> The coproduct is an object $X \in \mathcal{C}$ and a collection of maps $\iota_j : A_j \to X$, such that given any other object $Y \in \mathcal{C}$ and any other collection of maps $f_i : A_j \to Y$, there exists a **unique** map $f : X \to Y$ such that $f_i = f \circ \iota_j$. We have the following diagram.

$$\begin{tikzcd}
A_j \arrow[r, "\iota_j"] \arrow[d, "f_j"'] & X \arrow[dl, dashed, "\exists! f"] \\
Y &
\end{tikzcd}$$

A *final object* in $\mathcal{C}$ is a coproduct of an empty collection.

Again, coproduct (if exists) is unique up to unique isomorphism. Moreover, we have the following useful characterization of final objects.

Exercise 6.15: (*Final Object*)

Show that the final object of a category $\mathcal{C}$ (if exists) can be equivalently defined as an object X such that given any object $Y \in \mathcal{C}$, there exists a unique morphism $Y \to X$.

Example 6.16: (*Examples of Coproduct and Final Objects*)

- Set : coproduct is the disjoint union of sets, and final object is the singleton set.
- Ab : coproduct is the direct sum of Abelian groups, and final object is the group with one element.
- Group : coproduct is the *free product* of groups, and final object is the group with one element.
- Top : coproduct is the disjoint union of the underlying sets equipped with the disjoint union topology, and final object is the singleton.
- $\mathrm{Top}_\star$: coproduct is the wedge product of spaces, and final object is the singleton.

We now define the notion of (pre)additive categories, our motivation is the category Ab of Abelian groups.

Definition 6.17: (*Pre-additive Category*)

A category $\mathcal{C}$ is called an *pre-additive category* if for any $X, Y \in \mathcal{C}$ we have $\hom_{\mathcal{C}}(X, Y)$ is an Abelian group, and moreover, the composition

$$\hom_{\mathcal{C}}(X, Y) \times \hom_{\mathcal{C}}(Y, Z) \to \hom_{\mathcal{C}}(X, Z)$$

is bilinear. A functor $F : \mathcal{C} \to \mathcal{D}$ between pre-additive categories is called an *additive functor* if $F : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(F(X), F(Y))$ is a group homomorphism for all $X, Y \in \mathcal{C}$.

Proposition 6.18: (*Zero Object*)

In a pre-additive category $\mathcal{C}$, an object $\star$ is an initial object if and only if it is a final object, whence it is called a *zero object*

Proof : Suppose $\star \in \mathcal{C}$ is an initial object. Given any $X \in \mathcal{C}$, there exists a unique map $e : \star \to X$. Now, $\hom_{\mathcal{C}}(X, \star)$ is an Abelian group, and hence, we have the zero map $0_X : X \to \star$. If possible, suppose $f : X \to \star$ is another map. As $\mathcal{C}$ is a category, we have $\mathrm{Id}_\star : \star \to \star$, and as $\mathcal{C}$ is pre-additive, we have $0 : \star \to \star$. Then, by uniqueness, we have $\mathrm{Id}_\star = 0$. But then,

$$f = f \circ \mathrm{Id}_\star = f \circ 0 = 0_X,$$

where the last equality follows from the bilinearity of the composition. Thus, there exists a unique map $X \to \star$, which makes $\star$ into a final object. Similar argument works for the other direction as well. $\square$

In fact, one can generalize the above. In a pre-additive category, any finite product is a coproduct, and coversely, any finite coproduct is a product, whence they are called *biproduct*. Let $A, B \in \mathcal{C}$ be two objects in a pre-additive category. Suppose the product $A \oplus B \in \mathcal{C}$ exists. We have maps

$$\begin{tikzcd} A & A \oplus B \arrow[l, two heads, "\pi_A"'] \arrow[r, two heads, "\pi_B"] & B. \end{tikzcd}$$

From universal property of product, we get unique maps in the following diagrams.

$$\begin{tikzcd}
A \arrow[dr, dashed, "\iota_A"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "\mathrm{Id}_A"'] & & \\
& A \oplus B \arrow[r, "\pi_B"'] \arrow[d, "\pi_A"] & B \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[dr, dashed, "\iota_B"] \arrow[drr, bend left, "\mathrm{Id}_B"] \arrow[ddr, bend right, "0"'] & & \\
& A \oplus B \arrow[r, "\pi_B"'] \arrow[d, "\pi_A"] & B \\
& A &
\end{tikzcd}$$

Observe that,

$$\pi_A \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \mathrm{Id}_A \circ \pi_A + 0 \circ \pi_B = \pi_A + 0 = \pi_A,$$

and

$$\pi_B \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = 0 \circ \pi_A + \mathrm{Id}_B \circ \pi_B = 0 + \pi_B = \pi_B.$$

Then, from the universal property of the product, we have

$$\iota_A \circ \pi_A + \iota_B \circ \pi_B = \mathrm{Id}_{A \oplus B}.$$

Next, we check that

$$A \xhookrightarrow{\iota_A} A \oplus B \xhookleftarrow{\iota_B} B$$

is the coproduct. Indeed, for any $f : A \to C$ and $g : B \to C$, consider the map

$$h = f \circ \pi_A + g \circ \pi_B : A \oplus B \to C.$$

Then,

$$h \circ \iota_A = f \circ \pi_A \circ \iota_A + g \circ \pi_B \circ \iota_A = f \circ \mathrm{Id}_A + g \circ 0 = f + 0 = f,$$

and similarly,

$$h \circ \iota_B = f \circ \pi_A \circ \iota_B + g \circ \pi_B \circ \iota_B = f \circ 0 + g \circ \mathrm{Id}_B = 0 + g = g.$$

Now, consider $\theta : A \oplus B \to C$ be any map satisfying $\theta \circ \iota_A = f, \theta \circ \iota_B = g$. Then,

$$h = f \circ \pi_A + g \circ \pi_B = \theta \circ \iota_A \circ \pi_A + \theta \circ \iota_B \circ \pi_B = \theta \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \theta \circ \mathrm{Id}_{A \oplus B} = \theta.$$

Hence, $h : A \oplus B$ is the unique such map. This justifies that $A \xhookrightarrow{\iota_A} A \oplus B \xhookleftarrow{\iota_B} B$ is indeed a coproduct. Similarly, we can show that any finite coproduct is also a product.

Conversely, suppose we have objects A, B, C and maps $\iota_A : A \to C, \iota_B : B \to C, \pi_A : C \to A, \pi_B : C \to B$, satisfying

$$\pi_A \circ \iota_A = \mathrm{Id}_A, \pi_A \circ \iota_B = 0, \pi_B \circ \iota_A = 0, \pi_B \circ \iota_B = \mathrm{Id}_B, \iota_A \circ \pi_A + \iota_B \circ \pi_B = \mathrm{Id}_C.$$

Then, it follows that $A \xtwoheadleftarrow{\pi_A} C \xtwoheadrightarrow{\pi_B} B$ is a product, and $A \xhookrightarrow{\iota_A} C \xhookleftarrow{\iota_B} B$ is a coproduct.

Definition 6.19: (*Additive Category*)

A pre-additive category is called an *additive category* if it admits all finite products (hence all finite coproducts). In particular, the initial object exists (which is also the final object), and is called the *zero object* of the category. Given any two objects $X, Y \in \mathcal{C}$, we have the unique *zero map* $0 : X \to Y$ between them.

Proposition 6.20:

An additive functor (Definition 6.17) between additive categories preserve finite products. In particular, zero object is mapped to the zero object.

Proof : Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories. Firstly, let us show that $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$. Note that $\mathrm{Id}_{0_{\mathcal{C}}} = 0$. Since F is an additive functor, we have $\mathrm{Id}_{F(0_{\mathcal{C}})} = F\big(\mathrm{Id}_{0_{\mathcal{C}}}\big) = F(0) = 0$. That is, the identity map of $F(0_{\mathcal{C}})$ is the zero map. But then $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$.

Next, let $A, B \in \mathcal{C}$. We need to show that $F(A \oplus B) = F(A) \oplus F(B)$. Recall that $A \oplus B$ is both the product and coproduct. Thus, we have maps $\iota_A : A \to A \oplus B, \iota_B : B \to A \oplus B$ and $\pi_A : A \oplus B \to A, \pi_B : A \oplus B \to B$, which moreover satisfy

$$\pi_A \circ \iota_A = \mathrm{Id}_A, \pi_A \circ \iota_B = 0, \pi_B \circ \iota_A = 0, \pi_B \circ \iota_B = \mathrm{Id}_B, \iota_A \circ \pi_A + \iota_B \circ \pi_B = \mathrm{Id}_{A \oplus B}.$$

But then applying F we get the same relations for the object $F(A \oplus B)$. Hence, $F(A \oplus B)$ is the biproduct of $F(A)$ and $F(B)$. Inductively, it follows taht F takes any finite biproduct to a biproduct. $\square$

Example 6.21: (*Examples of Additive Category*)

Category of Abelian groups (and more generally, category of R-modules for a commutative ring R) is an additve category. The category Ch of chain complexes of Abelian groups (or even modules) is also an additive category.

Exercise 6.22:

Let R be a commutative ring with 1. Let $\mathscr{C}$ be the one-object category with hom-set R. Verify that $\mathscr{C}$ is a pre-additive category, which is not additive.

The goal is to now define (co)kernels, and relate them to monic and epic maps.

Definition 6.23: (*Kernel and Cokernel*)

Let $\mathscr{C}$ be an additive category, and $f : X \to Y$ is a map.

- The *kernel* of f is a map $\iota : \ker(f) \to X$ with $f \circ \iota = 0$, satisfying the universal property :

$$\begin{tikzcd} \ker(f) \arrow[r, "\iota"] & X \arrow[r, "f"] & Y \\ Z \arrow[u, dashed, "\exists! \tilde{g}"] \arrow[ur, "g"'] \arrow[urr, bend right, "0"'] & & \end{tikzcd}$$

given any map $g : Z \to X$ with $f \circ g = 0$, there exists a unique map $\tilde{g} : Z \to \ker(f)$ such that $\iota \circ \tilde{g} = g$.

- The *cokernel* of f is a map $\pi : X \to \mathrm{coker}(f)$ with $\pi \circ f = 0$, satisfying the universal property :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rrd, bend right, "0"'] & Y \arrow[r, "\pi"] \arrow[rd, "h"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists! \tilde{h}"] \\ & & W \end{tikzcd}$$

given any map $h : X \to W$ with $h \circ f = 0$, there exists a unique map $\tilde{h} : \mathrm{coker}(f) \to W$ such that $\tilde{h} \circ \pi = h$.

In the category of Abelian groups (or modules), the kernel of a map is indeed the 0-set. For a homomorphism $f : X \to Y$ of Abelian groups, we have $\mathrm{coker}(f) = \frac{B}{\mathrm{im}(f)}$.

Remark 6.24: (*Categorical Image and Coimage*)

Since we have defined (co)kernel as a map, we can now define the (co)kernel of them as well. This leads to the following definition : given a map $f : X \to Y$, the *image* is defined as $\ker(\mathrm{coker}(f))$ and the *coimage* is defined as $\mathrm{coker}(\ker(f))$. One can show that there is a natural map $\mathrm{coim}(f) \to \mathrm{im}(f)$, and the *first isomorphism theorem* states that this map is an isomorphism. In the category of Abelian groups, the categorical image matches with the set-theoretic image (which is already a subgroup), but in the category of groups one needs to take the *normal completion* of the set-theoretic image (which is only a subgroup).

Let us now define injective and surjective maps, and relate them kernel and cokernel.

Definition 6.25: (*Monic and Epic Maps*)

Let $f : X \to Y$ be a map in a category $\mathcal{C}$.

- f is called *monic* (or *monomorphism* or *injective*) if given any two maps $g_1, g_2 : Z \to X$ with $f \circ g_1 = f \circ g_2$, we have $g_1 = g_2$.
- f is called *epic* (or *epimorphism* or *surjective*) if given any two maps $h_1, h_2 : Y \to W$ with $h_1 \circ f = h_2 \circ f$, we have $h_1 = h_2$.

In the category Set, the above definition boils down to the usual definition of injective and surjective set maps. The categorical definition has the advantage that we do not need to the objects to be a set. Thus, for example, we can readily define monic/epic chain maps.

Caution 6.26: (*Monic, Epic and Iso*)

Suppose, $f : X \to Y$ admits a right inverse, i.e, there is a map $s : Y \to X$ such that $s \circ f = \mathrm{Id}_X$. Then, f is necessarily monic. But the converse may not be true! Consider the map $f : \mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2: clearly f is monic, but admits no right inverse. Similarly, any map admitting a left inverse is epic, but not conversely. By definition, an isomorphism admits both left and right inverse, and hence, is both monic and epic. In the category of commutative rings, the inclusion $\mathbb{Z} \to \mathbb{Q}$ is both monic and epic, but it is **not** an isomorphism!

Definition 6.27: (*Abelian Category*)

A category $\mathcal{C}$ is called an *Abelian category* if the following holds.

- $\mathcal{C}$ is an additive category (i.e, each $\hom$ is an Abelian groups, composition is bilinear, and all finite (co)products exists including the zero object).
- Any map in $\mathcal{C}$ has kernel and cokernel (this is sometimes called a *pre-Abelian category*).
- Every monic map is a kernel of some map, and every epic map is a cokernel of some map (this is called having *nomal monic* and *conormal epic* maps).

The category of modules over a commutative ring is an Abelian category, and so is the category of chain complexes of such modules! In the literature, following Grothendieck, additional axioms (which deals with arbitrary (co)products) are added to the definition of an Abelian category. In an Abelian category, a map which is both monic and epic is necessarily an iso.

Exercise 6.28: (*Monic and Epic maps in an Abelian Category*)

Let $f : X \to Y$ be a map in an Abelian category $\mathcal{C}$. Verify the following.

- f is monic if and only if $\ker(f) = 0$.
- $\iota : \ker(f) \to X$ is monic, and thus, $\ker(\ker(f)) = 0$.
- f is epic if and only if $\operatorname{coker}(f) = 0$.
- $\pi : Y \to \operatorname{coker}(f)$ is epic, and thus, $\operatorname{coker}(\operatorname{coker}(f)) = 0$.
- f is an iso $\Leftrightarrow$ f is both monic and epic $\Leftrightarrow$ $\ker(f) = 0 = \operatorname{coker}(f)$.

Note that by $\ker(f) = 0$ (and similarly, $\operatorname{coker}(f) = 0$), one should understand that the object $\ker(f)$ is the zero object, and the map $\ker(f) \to X$ is (necessarily) the 0 map.

6.4 Snake Lemma

After all the abstract nonsense, let us summarize the takeaway of the previous section! Suppose $f : X \to Y$ is a morphism of modules. Then, there exists an exact sequence

$$0 \to \ker(f) \hookrightarrow X \xrightarrow{f} Y \twoheadrightarrow \operatorname{coker}(f) \to 0.$$

Given a chain map $f_\bullet : C_\bullet \to D_\bullet$, kernel, cokernel, and image of $f_\bullet$ can be computed degree-wise, with naturally induced boundary maps. In particular, we have

$$0 \to P_\bullet \xrightarrow{f_\bullet} Q_\bullet \xrightarrow{g_\bullet} R_\bullet \to 0$$

is a short exact sequence of chain maps precisely when

$$0 \to P_n \xrightarrow{f_n} Q_n \xrightarrow{g_n} R_n \to 0$$

is a short exact sequence for each n. Note that any module A (and in particular 0), can be realized as a chain complex (*concentrated at degree* 0) by putting A at the 0^{th} place, and putting 0 everywhere else (and necessarily with 0 as boundary maps). Moreover, given any chain map $f_\bullet : C_\bullet \to D_\bullet$, we have an exact sequence

$$0 \to \ker(f)_\bullet \to C_\bullet \xrightarrow{f_\bullet} D_\bullet \to \operatorname{coker}(f)_\bullet \to 0$$

of chain complexes and maps.

We now state one of the most important results in homological algebra!

Lemma 6.29: (*Snake Lemma!*)

Consider the following commutative diagram

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' &
\end{tikzcd}$$

where the two rows are exact. Then, there exists an exact sequence

$$\ker(f) \to \ker(a) \to \ker(b) \to \ker(c) \xrightarrow{\partial} \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c) \to \operatorname{coker}(g'),$$

where ∂ is called the *boundary map*. Moreover, the sequence is natural.

<u>**Proof**</u> **:** We have the following diagram

$$\begin{tikzcd}
 & \ker(a) \arrow[r] \arrow[d] & \ker(b) \arrow[r] \arrow[d] & \ker(c) \arrow[d] \arrow[dddll, rounded corners, "\partial"] & \\
\ker(f) \arrow[r] \arrow[ur, bend left] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] \arrow[d] & C' \arrow[r] \arrow[d] & \operatorname{coker}(g') \\
 & \operatorname{coker}(a) \arrow[r] & \operatorname{coker}(b) \arrow[r] & \operatorname{coker}(c) \arrow[ur, bend right] &
\end{tikzcd}$$

The blue arrows can be induced naturally. Note that the colored arrows look like a snake! Let us define the map ∂. Say, $x \in \ker(c) \subset C$. Since g is surjective, we have some $y \in B$ such that $x = g(y)$. By commutativity, $g'b(y) = cg(y) = c(x) = 0 \Rightarrow b(y) \in \ker(g') = \operatorname{im}(f')$. Thus, there is some $z \in A'$ such that $f'(z) = b(y)$. Let us define $\partial(x) = [z] = z + \operatorname{im}(a)$. We need to show that ∂ is well-defined. Suppose, we have some $y' \in B$ such that $g(y') = c$. Then, $g'b(y') = cg(y') = 0 \Rightarrow b(y') \in \ker(g') = \operatorname{im}(f')$ and so, $b(y') = f'(z')$ for some $z' \in A'$. We have, $g(y-y') = g(y) - g(y') = x - x = 0 \Rightarrow y - y' \in \ker(g) = \operatorname{im}(f)$, and so $y - y' = f(w)$ for some $w \in A$. Now, $f'(z - z') = b(y - y') = bf(w) = f'a(w)$. As f' is injective, we have $z - z' = a(w)$. But then $[z] = [z']$ in $\operatorname{coker}(a)$. This proves that ∂ is well-defined. A similar argument shows that ∂ is a homomorphism. The exactness of the sequence is left as an exercise!

As for naturaility, we consider a commutative diagram of the form

$$\begin{tikzcd}
& & & A_2 \arrow[rr, "f"] \arrow[dd, "a_2"] & & B_2 \arrow[rr, "g"] \arrow[dd, "b_2"] & & C_2 \arrow[rr] \arrow[dd, "c_2"] & & 0 \\
& & A_1 \arrow[ur, red] \arrow[rr, "f"] \arrow[dd, "a_1"] & & B_1 \arrow[ur, red] \arrow[rr, "g"] \arrow[dd, "b_1"] & & C_1 \arrow[ur, red] \arrow[rr] \arrow[dd, "c_1"] & & 0 & \\
& 0 \arrow[rr] & & A_2' \arrow[rr, "f'"] & & B_2' \arrow[rr, "g'"] & & C_2' & & \\
0 \arrow[rr] & & A_1' \arrow[ur, red] \arrow[rr, "f'"'] & & B_1' \arrow[ur, red] \arrow[rr, "g'"'] & & C_1' \arrow[ur, red] & & &
\end{tikzcd}$$

where all the rows are exact. Then, the red arrwos determine a commutative diagram of the corresponding 8-term exact sequences. Again, the proof is a diagram chasing argument, and left as an exercise! □