Course notes for

# Algebraic Topology II (KSM4E02)

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## Day 5 : 27th January, 2026

`Mayer-Vietoris sequence – suspension isomorphism – homology of spheres – relative Mayer-Vietoris sequence – singular chains – singular homology`

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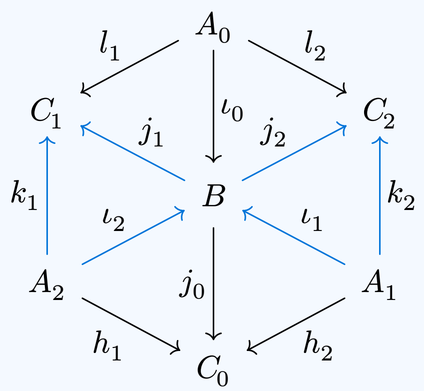
### 5.1 Mayer-Vietoris Sequence of a Proper Triad

Before describing the sequence, let us observe the following hexagonal lemma.

**Lemma 5.1:** (*Hexagonal Lemma*)

Suppose, we have a diagram of groups, where each triangle commutes.




Assume that $\ker(\iota_\alpha) = \mathrm{im}(j_\alpha)$ for $\alpha = 1, 2$, $j_0 \circ \iota_0 = 0$, and $k_1, k_2$ are isomorphisms. Then, the left and right sides of the hexagon differs by a side, i.e, $h_1 \circ k_1^{-1} \circ l_2 = -h_2 \circ k_2^{-1} \circ l_2$.

**Proof :** We can apply Lemma 4.10 to the blue part of the diagram. In particular, for any $a \in A_0$, we can uniquely write

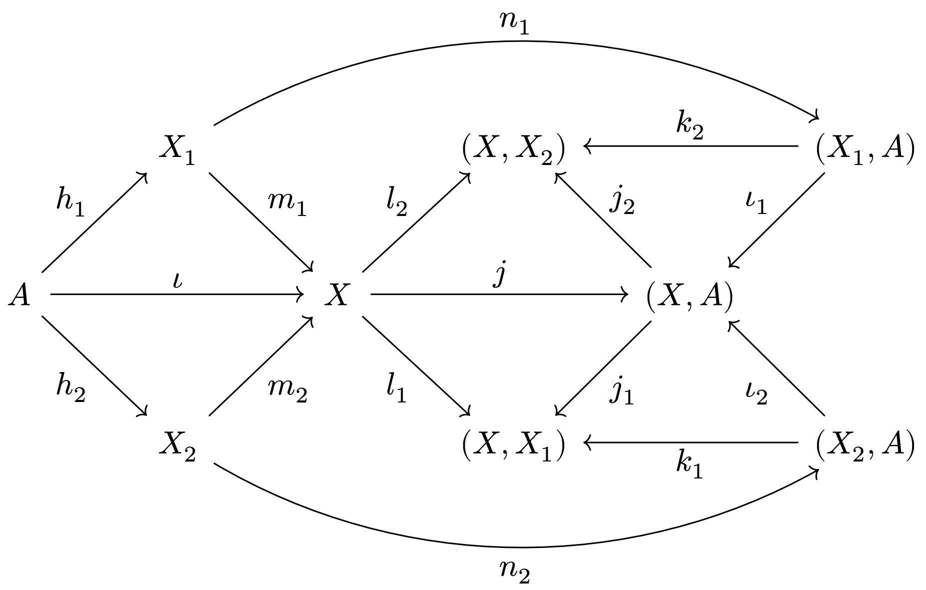
$$\iota_0(a) = \iota_1(k_2^{-1} j_2 \iota_0(a)) + \iota_2(k_1^{-1} j_1 \iota_0(a)).$$

Applying $j_0$, and using $j_0 \circ \iota_0 = 0$, we have

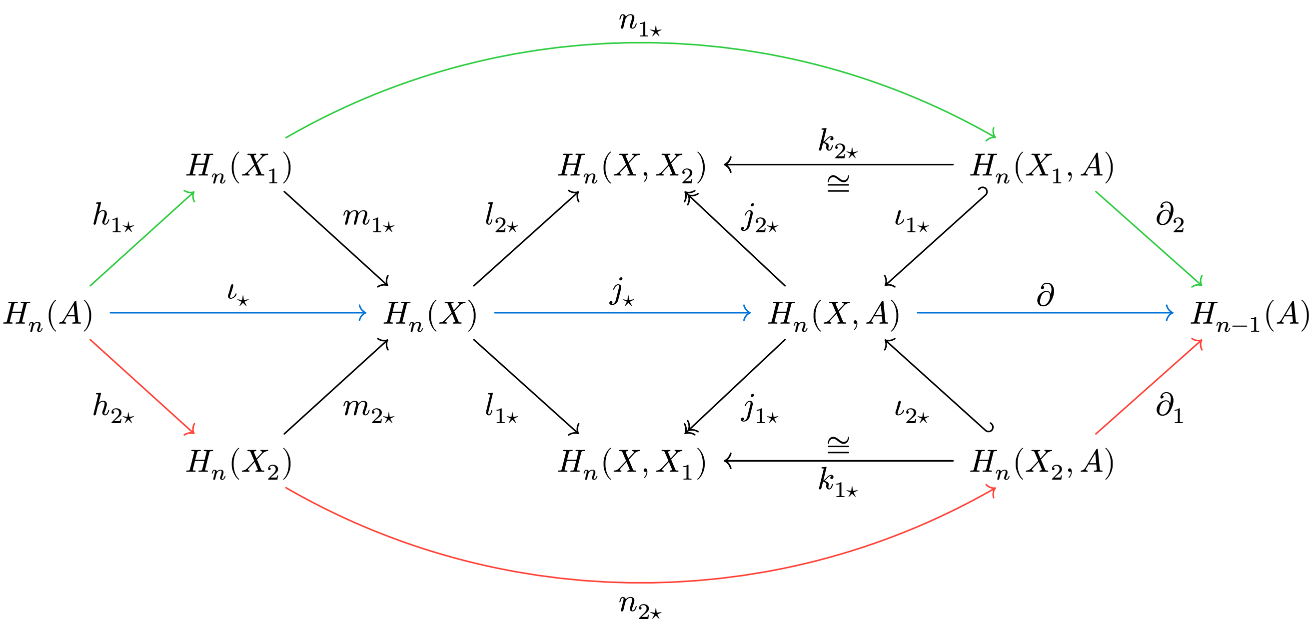
$$0 = j_0 \iota_0(a) = h_2 k_2^{-1} j_2 \iota_0(a) + h_1 k_1^{-1} j_1 \iota_0(a) = h_2 k_2^{-1} l_2(a) + h_1 k_1^{-1} l_1(a).$$

As $a \in A_0$ is arbitrary, we have the claim. □

Let us consider a proper triad $(X; X_1, X_2)$ with $X = X_1 \cup X_2$, and set $A = X_1 \cap X_2$. At the space level, we have the following commuting diagram.




Passing to homology, we have the following commuting diagram.




As the triad is proper, we have $k_{1\star}, k_{2\star}$ are isomorphisms, which gives the diagonal short exact sequences by Lemma 4.10. The commutativity involving the boundary maps follows from the naturality of the homology long exact sequence. The colored arrows are part of long exact sequence of $(X_1, A)$, $(X_2, A)$, and $(X, A)$. We define a sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{\psi} H_n(X_1) \oplus H_n(X_2) \xrightarrow{\varphi} H_n(X) \xrightarrow{\Delta} H_{n-1}(A) \longrightarrow \cdots$$

where the maps are as follows:

$$\begin{aligned} \psi(u) &= (h_{1\star}(u), -h_{2\star}(u)), && u \in H_n(A) \\ \varphi(v_1, v_2) &= m_{1\star}(v_1) + m_{2\star}(v_2), && v_1 \in H_n(X_1), v_2 \in H_n(X_2) \\ \Delta(w) &= -\partial_1 k_{1\star}^{-1} l_{1\star}(w) = \partial_2 k_{2\star}^{-1} l_{2\star}(w), && w \in H_n(X). \end{aligned}$$

The definition of $\Delta$ is a consequence of Lemma 5.1.

**Theorem 5.2:** (*Mayer-Vietoris Sequence for Proper Triad*)

Given a proper triad $(X; X_1, X_2)$ with $X = X_1 \cup X_2$ and $A = X_1 \cap X_2$, we have a long exact sequence,

$$\cdots \longrightarrow H_n(A) \xrightarrow{\psi} H_n(X_1) \oplus H_n(X_2) \xrightarrow{\varphi} H_n(X) \xrightarrow{\Delta} H_{n-1}(A) \longrightarrow \cdots$$

known as the *Mayer-Vietoris sequence*, which is natural with respect to morphisms of triads.

**Proof :** As usual, the proof requires explicitly checking the exactness at each point.

1. $\mathbf{ker}\,\varphi \supset \mathbf{im}\,\psi$ **:** For any $u \in H_n(A)$, we have

$$\varphi\psi(u) = \varphi(h_{1\star}(u), -h_{2\star}(u)) = m_{1\star}h_{1\star}(u) - m_{2\star}h_{2\star}(u) = \iota_\star(u) - \iota_\star(u) = 0.$$

   Thus, $\varphi \circ \psi = 0 \Rightarrow \ker\varphi \supset \operatorname{im}\psi$.

2. $\mathbf{ker}\,\varphi \subset \mathbf{im}\,\psi$ **:** Suppose $\varphi(v) = 0$ for some $v = (v_1, v_2) \in H_n(X_1) \oplus H_n(X_2)$. Then,

$$0 = j_\star\varphi(v) = j_\star m_{1\star}(v_1) + j_\star m_{2\star}(v_2) = \iota_{1\star}n_{1\star}(v_1) + \iota_{2\star}n_{2\star}(v_2)$$

   Now, by Lemma 4.10, we have $\iota_{1\star}, \iota_{2\star}$ are monic, and $\operatorname{im}(\iota_{1\star}) \cap \operatorname{im}(\iota_{2\star}) = 0$. Hence, $n_{1\star}(v_1) = 0 = n_{2\star}(v_2)$. By exactness of the long exact sequence of $(X_1, A)$ and $(X_2, A)$ respectively, there exists $u_1, u_2 \in H_n(A)$ such that $v_1 = h_{1\star}(u_1), v_2 = h_{2\star}(u_2)$. In other words,

$$0 = \varphi(v) = m_{1\star}h_{1\star}(u_1) + m_{2\star}h_{2\star}(u_2) = \iota_\star(u_1) + \iota_\star(u_2) = \iota_\star(u_1 + u_2).$$

   Now, $u_1 + u_2 \in \ker(\iota_\star) = \operatorname{im}\partial$. Thus, there is some $w \in H_{n+1}(X, A)$ such that $\partial(w) = u_1 + u_2$. Again by Lemma 4.10, there are (unique) $w_1 \in H_{n+1}(X_1, A), w_2 \in H_{n+1}(X_2, A)$ such that $w = \iota_{1\star}(w_1) + \iota_{2\star}(w_2)$. Then,

$$u_1 + u_2 = \partial(w) = \partial\iota_{1\star}(w_1) + \partial\iota_{2\star}(w_2) = \partial_2(w_1) + \partial_1(w_2).$$

   Set, $u = u_1 - \partial_2(w_1) = -(u_2 - \partial_1(w_2))$. We then have,

$$h_{1\star}(u) = h_{1\star}(u_1) - \cancel{h_{1\star}\partial_2(w_1)} = v_1, h_{2\star}(u) = -(h_{2\star}(u_2) - \cancel{h_{2\star}\partial_1(w_2)}) = -v_2.$$

   Hence, $\psi(u) = (h_{1\star}(u), -h_{2\star}(u)) = (v_1, v_2) = v$. This proves the claim.

3. $\mathbf{ker}\,\boldsymbol{\Delta} \supset \mathbf{im}\,\varphi$ **:** For $v = (v_1, v_2) \in H_n(X_1) \oplus H_2(X_2)$, we have

$$\Delta\varphi(v) = \Delta(m_{1\star}(v_1) + m_{2\star}(v_2)) = -\partial_1 k_{1\star}^{-1} l_{1\star} m_{1\star}(v_1) + \partial_2 k_{2\star}^{-1} l_{2\star} m_{2\star}(v_2) = 0.$$

   Thus, $\Delta \circ \varphi = 0 \Rightarrow \ker\Delta \supset \operatorname{im}\varphi$.

4. $\mathbf{ker}\,\boldsymbol{\Delta} \subset \mathbf{im}\,\varphi$ **:** Suppose for some $w \in H_n(X)$, we have $\Delta(w) = 0$. Thus, $\partial_1 k_{1\star}^{-1} l_{1\star}(w) = 0 = \partial_2 k_{2\star}^{-1} l_{2\star}(w)$. By exactness, for $\alpha = 1, 2$, there is $v_\alpha \in H_n(X_\alpha)$ such that $n_{\alpha\star}(v_\alpha) = k_{\alpha\star}^{-1} l_{\alpha\star}(w)$. By Lemma 4.10, we can (uniquely) write

$$\begin{aligned} j_\star(w) &= i_{1\star}k_{2\star}^{-1}j_{2\star}j_\star(w) + i_{2\star}k_{1\star}^{-1}j_{1\star}j_\star(w) \\ &= i_{1\star}k_{2\star}^{-1}l_{2\star}(w) + \iota_{2\star}k_{1\star}^{-1}l_{1\star}(w) \\ &= \iota_{1\star}n_{1\star}(v_1) + \iota_{2\star}n_{2\star}(v_2) \\ &= j_\star m_{1\star}(v_1) + j_\star m_{2\star}(v_2). \end{aligned}$$

   Thus, $w - m_{1\star}(v_1) - m_{2\star}(v_2) \in \ker(j_\star) = \operatorname{im}(\iota_\star)$. Hence, there exists some $u \in H_n(A)$ such that $\iota_\star(u) = w - m_{1\star}(v_1) - m_{2\star}(v_2)$. Set, $v_1' = v_1 + h_{1\star}(u), v_2' = v_2$. Then

$$\varphi(v_1', v_2') = m_{1\star}(v_1 + h_{1\star}(u)) + m_{2\star}(v_2)$$

$$= m_{1\star}(v_1) + (w - m_{1\star}(v_1) - m_{2\star}(v_2)) + m_{2\star}(v_2) = w.$$

This proves the claim.

5. $\mathbf{ker}\,\psi \supset \mathbf{im}\,\mathbf{\Delta}$ **:** For $w \in H_n(X)$, we have

$$\psi\Delta(w) = (h_{1\star}\partial_2 k_{2\star}^{-1})l_{2\star}(w), h_{2\star}\partial_1 k_{1\star}^{-1} l_{1\star}(w) = (0,0) = 0.$$

Thus, $\psi \circ \Delta = 0 \Rightarrow \ker\psi \supset \operatorname{im}\Delta$.

6. $\mathbf{ker}\,\psi \subset \mathbf{im}\,\mathbf{\Delta}$ **:** Suppose for some $u \in H_n(A)$ we have $\psi(u) = 0$. Thus, $h_{1\star}(u) = 0 = h_{2\star}(u)$. By exactness, for $\alpha = 1, 2$ there are $x_\alpha \in H_{n+1}(X_\alpha, A)$ such that $\partial_1(x_2) = u$ and $\partial_2(x_1) = -u$. Then,

$$\partial\iota_{1\star}(x_1) + \partial\iota_{2\star}(x_2) = \partial_2(x_1) + \partial_1(x_2) = -u + u = 0 \Rightarrow \iota_{1\star}(x_1) + \iota_{2\star}(x_2) \in \ker(\partial) = \operatorname{im}(j_\star).$$

Thus, there exists $w \in H_{n+1}(X)$ such that $j_\star(w) = -\iota_{1\star}(x_1) - \iota_{2\star}(x_2)$. Then,

$$\begin{aligned}\Delta(w) &= -\partial_1 k_{1\star}^{-1} l_{1\star}(w) = -\partial_1 k_{1\star}^{-1} j_{1\star} j_\star(w) \\ &= \partial_1 k_{1\star}^{-1} j_{1\star}(\iota_{1\star}(x_1) + \iota_{2\star}(x_2)) \\ &= 0 + \partial_1 k_{1\star}^{-1} j_{1\star}\iota_{2\star}(x_2) \\ &= \partial_1 k_{1\star}^{-1} k_{1\star}(x_2) = \partial_1(x_2) = u.\end{aligned}$$

This proves the claim.

Thus, the Mayer-Vietoris sequence is exact. One can *easily* prove the naturality of the sequence with respect to maps of triads (Check!). □

## 5.2 The Suspension Isomorphism

Given a space, recall the *(unreduced) cone* on $X$ is

$$CX = \frac{X \times [0,1]}{X \times \{0\}},$$

and the *(unreduced) suspension* of $X$ is

$$\Sigma X = \frac{CX}{X \times \{1\}}.$$

It is well known that $CX \simeq \star$, i.e, the cone is always contractible. Consequently we have the following.

**Proposition 5.3:** (*Cone on a Space Is Homologically Trivial*)

Given a space $X$, we have $CX$ is homologically trivial, i.e, $\tilde{H}_n(CX) = 0$ for all $n$.

Let us now prove a fundamental property of (ordinary) homology theory.

**Theorem 5.4:** (*Suspension Isomorphism*)

Given a space $X$, there exists a natural isomorphism $\tilde{H}_n(\Sigma X) \cong \tilde{H}_{n-1}(X)$.

**Proof :** Consider $A := q\left(X \times \left[0, \frac{3}{4}\right]\right), B := q\left(X \times \left[\frac{1}{4}, 1\right]\right) \subset \Sigma X$, where $q : X \times [0,1] \to \Sigma X$ is the quotient map. It is easy to see that $(\Sigma X; A, B)$ is a proper triad. Now,

$$\Sigma X = A \cup B, A \cong CX \simeq \star, B \cong CX \simeq \star, A \cap B \cong X \times \left[\frac{1}{4}, \frac{3}{4}\right] \simeq X.$$

From the reduced version of the Mayer-Vietoris sequence, we immediately see that $\Delta : \tilde{H}_n(\Sigma X) \to \tilde{H}_{n-1}(A \cap B)$ is an isomorphism. Since $A \cap B$ deformation ratracts onto $X$, we have the claim. $\square$

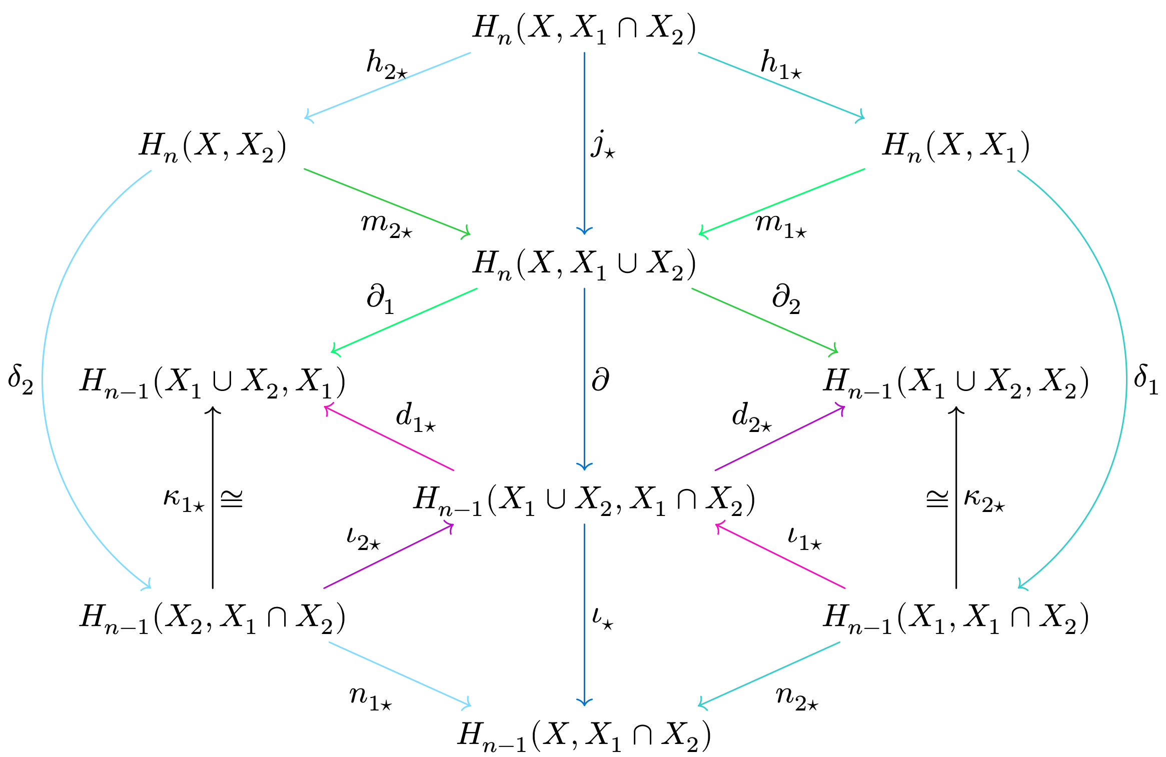
**Corollary 5.5:** (*Reduced Homology of Sphere*)

The reduced homology of the $n$-sphere is $\tilde{H}_k(S^n) = \begin{cases} 0, & k \neq n \\ H_0(\star), & k = n \end{cases}$.

**Proof :** We apply Theorem 5.4. Since $S^n = \Sigma S^{n-1}$, by induction, we only need compute the homology of $S^0$, which is just two points. By Proposition 3.1, we have $H_n(S^0) = H_n(\star) \oplus H_n(\star)$. Then, by the dimension axiom, it follows that $\tilde{H}_k(S^0) = \begin{cases} 0, & k \neq 0 \\ H_0(\star), & k = 0 \end{cases}$. $\square$

## 5.3 Relative Mayer-Vietoris Sequence

Let us now consider an arbitrary proper triad $(X; X_1, X_2)$. We have the following diagram.




The colored arrows are part of the corresponding long exact sequences of different triples. The isomorphims $\kappa_{1\star}, \kappa_{2\star}$ follows from the proper triad $(X_1 \cup X_2; X_1, X_2)$. In particular, we can apply Lemma 5.1. Let us define a sequence

$$\cdots \longrightarrow H_n(X, X_1 \cap X_2) \xrightarrow{\psi} H_n(X, X_1) \oplus H_n(X, X_2) \xrightarrow{\varphi} H_n(X, X_1 \cup X_2) \xrightarrow{\Delta} H_{n-1}(X, X_1 \cap X_2) \longrightarrow \cdots$$

where the maps are as follows:

$$\begin{aligned} \psi(u) &= (h_{1\star}(u), -h_{2\star}(u)), && u \in H_n(X, X_1 \cap X_2) \\ \varphi(v_1, v_2) &= m_{1\star}(v_1) + m_{2\star}(v_2), && v_1 \in H_n(X, X_1), v_2 \in H_n(X, X_2) \\ \Delta(w) &= -n_{1\star} k_{1\star}^{-1} \partial_1(w) = n_{2\star} k_{2\star}^{-1} \partial_2(w), && w \in H_n(X, X_1 \cup X_2). \end{aligned}$$

**Theorem 5.6:** (*Relative Mayer-Vietoris Sequence*)

Given a proper triad $(X; X_1, X_2)$, there exists a long exact sequence

$$\cdots \longrightarrow H_n(X, X_1 \cap X_2) \xrightarrow{\psi} H_n(X, X_1) \oplus H_n(X, X_2) \xrightarrow{\varphi} H_n(X, X_1 \cup X_2) \xrightarrow{\Delta} H_{n-1}(X, X_1 \cap X_2) \longrightarrow \cdots$$

known as the *relative Mayer-Vietoris sequence*, which is moreover natural with respect to morphism of triads.