Course notes for

Algebraic Topology II (KSM4E02)

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`reduced homology — triads — long exact sequence of proper triads`

4.1 Reduced Homology

Given a space X, we have the unique map $f : X \to \{\star\}$. If X is nonempty, fixing a point $x_0 \in X$, we have a continuous map $r : X \to P := \{x_0\}$. Then, P is a retract of X. Hence, by functoriality, we see $H_n(P)$ is a direct summand of $H_n(X)$ for all n. Let us write $H_n(X) = \tilde{H}_n(X) \oplus H_n(P)$. By the dimension axiom, for all $n \neq 0$, it follows that $\tilde{H}_n(X) = H_n(X)$.

Definition 4.1: (*Reduced Homology*)

Given a space X, its *reduced homology* groups are defined as $\tilde{H}_n(X) := \ker(H_n(f) : H_n(X) \to H_n(\star))$, for $f : X \to \{\star\}$ the unique constant map. If X is nonempty, we have $H_n(X) \cong \tilde{H}_n(X) \oplus H_n(\star)$.

Corollary 4.2: (*Reduced Homology of a Point*)

We have $\tilde{H}_n(X) = H_n(X)$ for all $n \neq 0$, and $\tilde{H}_n(\star) = 0$ for all n.

Lemma 4.3:

Let $f : X \to Y$ be a map. Then, we have an induced map $f_\star : \tilde{H}_0(X) \to \tilde{H}_0(Y)$.

<u>**Proof**</u> **:** We have a diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \tilde{H}_0(X) \arrow[r, hook] \arrow[d, dashed] & H_0(X) \arrow[r, two heads] \arrow[d, "H_0(f)"'] & H_0(\star) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \tilde{H}_0(Y) \arrow[r, hook] & H_0(Y) \arrow[r, two heads] & H_0(\star) \arrow[r] & 0
\end{tikzcd}$$

Note that for any $x \in \tilde{H}_0(X)$, we have $H_0(f)(x) \in \ker(H_0(Y) \to H_0(\star)) = \tilde{H}_0(Y)$. Hence, $H_0(f)$ restricts to a map $\tilde{H}_0(X) \to \tilde{H}_0(Y)$. $\square$

Theorem 4.4: (*Long Exact Sequence of Reduced Homology Groups*)

Given a pair (X, A), we have the long exact sequence

$$\cdots \to H_{n+1}(X, A) \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X, A) \to \cdots$$

<u>**Proof**</u> **:** We only need to consider the case $n = 0$, i.e, we need to prove the exactness of the part

$$\cdots \to H_1(X, A) \to \tilde{H}_0(A) \to \tilde{H}_0(X) \to H_0(X, A) \to \cdots$$

Let us consider the long exact sequence for the pair $(\star, \star)$. Naturality gives the following commutative diagram of long exact sequences.

$$\begin{tikzcd}
& & \tilde{H}_0(A) \arrow[r] & \tilde{H}_0(X) \arrow[dr, bend left] & & \\
\cdots \arrow[r] & H_1(X, A) \arrow[ur, bend left] \arrow[r] \arrow[d] & H_0(A) \arrow[r] \arrow[d] & H_0(X) \arrow[r] \arrow[d] & H_0(X, A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \underbrace{H_1(\star, \star)}_{0} \arrow[r] & H_0(\star) \arrow[r] & H_0(\star) \arrow[r] & \underbrace{H_0(\star, \star)}_{0} \arrow[r] & \cdots
\end{tikzcd}$$

Since $\tilde{H}_0(A)$ and $\tilde{H}_0(X)$ is defined as the kernel of the vertical maps, we get the induced blue arrows, and mroreover, the exactness for the reduced homology groups follows from the commutativity of the two rows (Check!). $\square$

Definition 4.5: (*Homologically Trivial Space*)

A space X is called *homologically trivial* if $H_n(X) = 0$ for all $n \neq 0$, and $\tilde{H}_0(X) = 0$ (or in other words, $\tilde{H}_n(X) = 0$ for all n). For $A \neq \emptyset$, the pair (X, A) is called homologically trivial if $H_n(X, A) = 0$ for all n.

Exercise 4.6:

Suppose $A \neq \emptyset$, and (X, A) is homologically trivial. Show that $\iota : A \hookrightarrow X$ induces isomorphism of both unreduced and reduced homology groups. Conversely, if $\iota_\star$ is an isomorphism (for either unreduced or the reduced homology groups), then show that (X, A) is homologically trivial.

4.2 Homology Exact Sequence of Triads

Definition 4.7: (*Proper Triad*)

A *triad* is a triple $(X; X_1, X_2)$, where $X_1, X_2 \subset X$ are subspaces. We say $(X; X_1, X_2)$ is a *proper triad* if the inclusion maps

$$\kappa_1 : (X_2, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_1), \quad \kappa_2 : (X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2)$$

induce isomorphisms of homology groups.

Note that $(X; X_1, X_2)$ and $(X; X_2, X_1)$ are distinct triads, but one of them is proper if and only if the other one is as well. A triad maybe proper with resepect to one homology theory, but fails to be proper with respect to another.

Exercise 4.8:

Suppose $X_1, X_2 \subset X$ are closed subsets, such that $X = X_1 \cup X_2$, and $\overline{X_1 \setminus (X_1 \cap X_2)} \cap \overline{X_2 \setminus (X_1 \cap X_2)} = \emptyset$. Then, the triad $(X; X_1, X_2)$ is a proper triad for any homology theory.

Hint : Check that the inclusion maps κ_1, κ_2 are excision maps.

Exercise 4.9: (*Triad and Triple*)

Suppose (X, A, B) is a triple, i.e, $B \subset A \subset X$. Then, show that $(X; A, B)$ is a proper triad.

Hint : Recall Lemma 3.6.

Lemma 4.10: (*Direct Sum Decomposition*)

Consider the commutative diagram of groups.

$$\begin{tikzcd}
C_1 & & C_2 \\
& B \arrow[ul, "j_1"'] \arrow[ur, "j_2"] & \\
A_2 \arrow[uu, "k_1"] \arrow[ur, "\iota_2"'] & & A_1 \arrow[uu, "k_2"'] \arrow[ul, "\iota_1"]
\end{tikzcd}$$

Suppose, $\ker(j_\alpha) = \operatorname{im}(\iota_\alpha)$ for $\alpha = 1, 2$. Then, the following are equivalent.

1. k_1, k_2 are isomorphisms
2. $0 \to A_i \to B \to C_i \to$ is exact for $i = 1, 2$. Moreover, $B = \operatorname{im}(\iota_1) \oplus \operatorname{im}(\iota_2)$ is a direct sum.

Proof : Suppose k_1, k_2 are isomorphisms. Commutativity implies that j_1, j_2 are epic, and ι_1, ι_2 are monic. Thus, $0 \to A_i \to B \to C_i \to$ is exact for $i = 1, 2$. Note that $\ker(j_1) \cap \ker(j_2) = \ker(j_1) \cap \operatorname{im}(\iota_2) = 0$, since otherwise k_1 will fail to be iso. Let us prove the direct sum decomposition. For any $b \in B$, consider

$$a_2 = (k_1)^{-1} j_1(b) \in A_2, \quad a_1 = (k_2)^{-1} j_2(b) \in A_1.$$

Set $b' = \iota_1(a_1) + \iota_2(a_2)$. Now, $j_1(b') = j_1(b)$ and $j_2(b') = j_2(b)$. Thus, $b - b' \in \ker(j_1) \cap \ker(j_2) = 0$, which means $b = b'$. Thus, every $b \in B$ can be written as a sum from $\operatorname{im}(\iota_1) + \operatorname{im}(\iota_2)$. On the other hand, $\operatorname{im}(\iota_1) \cap \operatorname{im}(\iota_2) = \ker(j_1) \cap \ker(j_2) = 0$. Hence, we have direct sum decomposition. $\square$

Exercise 4.11: (*Finite Additivity*)

Prove Proposition 3.1 for relative homology directly using Lemma 4.10.

Hint : Given (X_i, A_i), consider $(X = X_1 \sqcup X_2, A = A_1 \sqcup A_2)$. We have a diagram of spaces.

$$\begin{tikzcd}
 & (X, X_2 \sqcup A) & & (X, X_1 \sqcup A_2) & \\
 & & (X, A) \arrow[ul] \arrow[ur] & & \\
(X_1, A_1) \arrow[r] & (X_1 \sqcup A_2, A) \arrow[uu] \arrow[ur] & & (X_2 \sqcup A, A) \arrow[uu] \arrow[ul] & (X_2, A_2) \arrow[l]
\end{tikzcd}$$

Theorem 4.12:

A triad $(X; X_1, X_2)$ is proper if and only if $\iota_\alpha : (X_\alpha, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_1 \cap X_2)$ induces a monomorphism, and gives a direct sum decomposition of $H_\star(X_1 \cup X_2, X_1 \cap X_2)$.

Proof : At the space level, we have the following diagram.

$$\begin{tikzcd}
(X_1 \cup X_2, X_1) & & (X_1 \cup X_2, X_2) \\
 & (X_1 \cup X_2, X_1 \cap X_2) \arrow[ul, "j_1"'] \arrow[ur, "j_2"] & \\
(X_2, X_1 \cap X_2) \arrow[uu, "\kappa_1"] \arrow[ur, "\iota_2"'] & & (X_1, X_1 \cap X_2) \arrow[uu, "\kappa_2"'] \arrow[ul, "\iota_1"]
\end{tikzcd}$$

We have the triples $(X_1 \cup X_2, X_1, X_1 \cap X_2)$ and $(X_1 \cup X_2, X_2, X_1 \cap X_2)$. From Theorem 3.8, we we have $\ker{(j_\alpha)}_\star = \operatorname{im}{(\iota_\alpha)}_\star$ for $\alpha = 1, 2$. We conclude the proof from Lemma 4.10. $\square$

Definition 4.13: (*Boundary Operator of Triad*)

Given a proper triad $(X; X_1, X_2)$, the *boundary operator* is defined as the composition

$$\partial : H_n(X, X_1 \cup X_2) \xrightarrow{\partial} H_{n-1}(X_1 \cup X_2) \longrightarrow H_{n-1}(X_1 \cup X_2, X_2) \xrightarrow{(\kappa_2)_\star^{-1}} H_n(X_1, X_1 \cap X_2).$$

It should be noted that this boundary map is the boundary map of the triple $(X, X_1 \cup X_2, X_1)$, followed by (the inverse of) an excision isomorphism.

Theorem 4.14: (*Homology Long Exact Sequence of Proper Triad*)

Given a proper triad $(X; X_1, X_2)$ we have a long exact sequence of homology groups

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X, X_1 \cup X_2) \arrow[d, "\partial"'] & & \\
& H_n(X_1, X_1 \cap X_2) \arrow[r] & H_n(X, X_2) \arrow[r] & H_n(X, X_1 \cup X_2) \arrow[d, "\partial"'] & \\
& & & H_{n-1}(X_1, X_1 \cap X_2) \arrow[r] & \cdots
\end{tikzcd}$$

Moreove, the sequence is natural with respect to map of triads.

Proof : We have the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X, X_2) \arrow[r] & H_{n+1}(X, X_1 \cup X_2) \arrow[r, "\partial"] \arrow[dr, bend right, "\partial"'] & H_n(X_1 \cup X_2, X_2) \arrow[r] & H_n(X, X_2) \arrow[r] & \cdots \\
& & & H_n(X_1, X_1 \cap X_2) \arrow[u, "(\kappa_2)_\star", "\cong"'] \arrow[ur, bend right] & &
\end{tikzcd}$$

The top row is exact, being the homology long exact sequence of the triple $(X, X_1 \cup X_2, X_2)$. Hence, the blue sequence is also exact by commutativity of the triangles. Naturality can also be proved *easily*, as the top row is natural and so is the boundary map (Check!). □