

Exercise 3.2: (Finite Union of Pairs)

Let (X_j, A_j) be pairs of spaces for $1 \leq j \leq k$. Denote $(X, A) = (\sqcup X_j, \sqcup A_j)$. Show that $\iota_j : (X_j, A_j) \hookrightarrow (X, A)$ induces an isomorphism of $H_n(X_j, A_j)$ onto a direct summand of $H_n(X, A)$, and moreover, $H_n(X, A)$ is naturally isomorphic to the direct sum $\bigoplus_{j=1}^k H_n(X_j, A_j)$.

Hint : Use [Proposition 3.1](#) and the naturality of the long exact sequence ([Exercise 2.14](#)).

The finite additivity of homology theory cannot be improved to arbitrary sum, and there are examples of homology theories which does not split for arbitrary union of spaces. On the other hand, we shall see later that singular homology abides by this. Hence, Milnor added one extra axiom.

Definition 3.3: (Additive Homology Theory)

A homology theory is called an **additive homology theory** if given a collection of spaces $\{X_\alpha\}_{\alpha \in I}$, the inclusion maps $\iota_\alpha : X_\alpha \hookrightarrow X := \sqcup X_\alpha$ induces an isomorphism of $H_n(X_\alpha)$ onto a direct summand of $H_n(X)$, and moreover, $H_n(X)$ is naturally isomorphic to $\bigoplus H_n(X_\alpha)$.

3.2 Homology of Homotopy Equivalent Spaces

Recall, $f : X \rightarrow Y$ is a homotopy equivalence if there is a map $g : Y \rightarrow X$ such that

$$g \circ f \simeq \text{Id}_X, \quad f \circ g \simeq \text{Id}_Y.$$

One can similarly define homotopy equivalence $f : (X, A) \rightarrow (Y, B)$ for pairs, which essentially requires that that the homotopy restricts to a homotopy of the restricted map.

Proposition 3.4: (Homotopy Equivalence induces Homology Isomorphism)

Let $f : X \rightarrow Y$ be a homotopy equivalence. Then, $f_* : H_n(X) \rightarrow H_n(Y)$ is an isomorphism for all n . Similarly, a homotopy equivalence $(X, A) \rightarrow (Y, B)$ also induces isomorphism at the homology groups.

Proof : Let $g : Y \rightarrow X$ be a homotopy inverse. Now, from the functoriality and the homotopy invariance, we have

$$\text{Id} = H_n(\text{Id}_X) = H_n(g \circ f) = H_n(g) \circ H_n(f) = g_* \circ f_*,$$

and similarly, we have $f_* \circ g_* = \text{Id}$. Hence, $f_* : H_n(X) \rightarrow H_n(Y)$ is an isomorphism, with inverse g_* . Similar argument holds for $(X, A) \simeq (Y, B)$ as well. \square

Corollary 3.5: (Homeomorphism induces Homology Isomorphism)

Let $f : X \rightarrow Y$ be a homeomorphism of spaces. Then $f_* : H_n(X) \rightarrow H_n(Y)$ is an isomorphism. Similarly, homeomorphism $(X, A) \cong (Y, B)$ induces homology isomorphism as well.

3.3 Homology Long Exact Sequence of Triples

Let us consider a triple (X, A, B) of spaces where $B \subset A \subset X$. Then, we have inclusions

$$\iota : (A, B) \hookrightarrow (X, B), \quad j : (X, B) \hookrightarrow (X, A).$$

A map $f : (X, A, B) \rightarrow (Y, C, D)$ of triples is a continuous map $f : X \rightarrow Y$ such that $f|_A : A \rightarrow C$ and $f|_B : B \rightarrow D$ holds. A triple leads to a natural long exact sequence of homology groups. We shall need the following.

Lemma 3.6: (*Relative Homology of Space w.r.t. itself*)

For any space X , we have $H_n(X, X) = 0$.

Proof : By the excision isomorphism, it follows that $H_n(X, X) \cong H_n(\emptyset, \emptyset) = H_n(\emptyset)$. Now, for the pair (\emptyset, \emptyset) , we have long exact sequence $\cdots \rightarrow H_n(\emptyset) \xrightarrow{\iota_*} H_n(\emptyset) \xrightarrow{j_*} H_n(\emptyset) \rightarrow \cdots$. Since $\iota = \text{Id} = j$, we have $i_* = \text{Id} = j_*$. But by exactness, we have $j_* \circ \iota_* = 0$. Hence, we have $\iota_* = 0 = j_*$, and consequently, $H_n(\emptyset) = 0$. Thus, $H_n(X, X) = 0$ for any space X . \square

Exercise 3.7:

Give a proof of $H_n(X, X) = 0$ without using excision.

Hint : Use the long exact sequence of the pair (X, X) .

Theorem 3.8: (*Homology Long Exact Sequence of Triples*)

Let (X, A, B) be a triple with $B \subset A \subset X$. Then, there exists a natural long exact sequence

$$\cdots \rightarrow H_{n+1}(X, A) \xrightarrow{\partial} H_n(A, B) \xrightarrow{\iota_*} H_n(X, B) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B) \cdots,$$

which is natural with respect to maps of triples. The boundary map is given as the composition

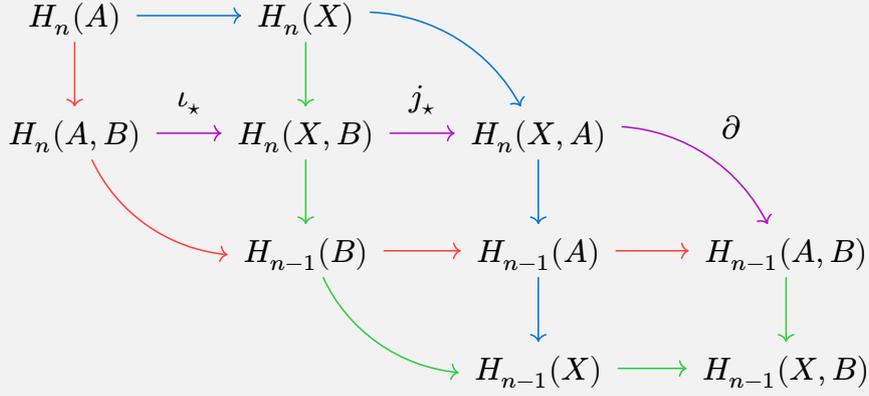
$$\partial : H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{j_*} H_{n-1}(A, B),$$

where the first boundary is from the long exact sequence of the pair (X, A) , and the second map is induced by the inclusion $A \hookrightarrow (A, B)$.

Proof : The proof boils down to checking exactness at each point, using the respective long exact sequences associated to the pairs (A, B) , (X, B) , and (X, A) . Let us color code the arrows as follows.

$$\begin{aligned} \cdots & \xrightarrow{\text{red}} H_{n+1}(A, B) \xrightarrow{\text{red}} H_n(A) \xrightarrow{\text{red}} H_n(B) \xrightarrow{\text{red}} H_n(A, B) \xrightarrow{\text{red}} H_{n-1}(A) \xrightarrow{\text{red}} \cdots \\ \cdots & \xrightarrow{\text{green}} H_{n+1}(X, B) \xrightarrow{\text{green}} H_n(X) \xrightarrow{\text{green}} H_n(B) \xrightarrow{\text{green}} H_n(X, B) \xrightarrow{\text{green}} H_{n-1}(X) \xrightarrow{\text{green}} \cdots \\ \cdots & \xrightarrow{\text{blue}} H_{n+1}(X, A) \xrightarrow{\text{blue}} H_n(X) \xrightarrow{\text{blue}} H_n(A) \xrightarrow{\text{blue}} H_n(X, A) \xrightarrow{\text{blue}} H_{n-1}(X) \xrightarrow{\text{blue}} \cdots \end{aligned}$$

Now, we have the following commutative diagram.



The horizontal arrows are induced by the inclusions $(A, B) \hookrightarrow (X, B) \hookrightarrow (X, A)$, and the commutativity of each square follows from the naturality of the long exact sequence of pairs. Let us now check the exactness at each position.

1. $\ker j_* \supset \text{im } \iota_*$: We need to show $j_* \circ \iota_* = 0$. We have a commuting diagram of spaces

$$\begin{array}{ccccc}
(A, B) & \xrightarrow{\iota} & (X, B) & \xrightarrow{j} & (X, A) \\
& & \searrow & & \nearrow \\
& & (A, A) & &
\end{array}$$

Since $H_n(A, A) = 0$ (Lemma 3.6), by functoriality, we have $j_* \circ \iota_* = 0$.

2. $\ker j_* \subset \text{im } \iota_*$: We look at the following diagram

$$\begin{array}{ccccc}
H_n(A) & \xrightarrow{\iota_*} & H_n(X) & & \\
\downarrow j_* & & \downarrow j_* & & \\
H_n(A, B) & \xrightarrow{\iota_*} & H_n(X, B) & \xrightarrow{j_*} & H_n(X, A) \\
& \searrow \partial & \downarrow \partial & & \downarrow \partial \\
& & H_{n-1}(B) & \xrightarrow{\iota_*} & H_{n-1}(A)
\end{array}$$

Suppose $j_*(x) = 0$ for some $x \in H_n(X, B)$. Then,

$$\iota_* \partial(x) = \partial j_* x = 0 \Rightarrow \partial(x) \in \ker(\iota_*) = \text{im}(\partial).$$

So, there exists some $y \in H_n(A, B)$ such that

$$\partial x = \partial y = \partial \iota_*(y) \Rightarrow \partial(x - \iota_*(y)) = 0 \Rightarrow x - \iota_*(y) \in \ker \partial = \text{im } j_*.$$

So, there exists some $z \in H_n(X)$ such that

$$j_*(z) = x - \iota_*(y) \Rightarrow j_*(z) = j_*(x - \iota_*(y)) = j_*(x) - j_* \iota_*(y) = 0 \Rightarrow z \in \ker j_* = \text{im } \iota_*.$$

So, there exists some $w \in H_n(A)$ such that $\iota_*(w) = z$. Define $y_1 = y + j_*(w) \in H_n(A, B)$. Then,

$$\iota_*(y_1) = \iota_*(y) + j_* \iota_*(w) = \iota_*(y) + j_*(z) = \iota_*(y) + x - \iota_*(y) = x.$$

This proves the claim.

3. $\ker \partial \supset \text{im } j_*$: We only need to show $\partial \circ j_* = 0$. We have the following diagram.

$$\begin{array}{ccc}
H_n(X, B) & \xrightarrow{j_*} & H_n(X, A) \\
\downarrow & & \downarrow \\
H_{n-1}(B) & \xrightarrow{\quad} & H_{n-1}(A) \\
& \searrow 0 & \downarrow \\
& & H_{n-1}(A, B)
\end{array}
\quad \partial$$

The 0 map is a consequence of the long exact sequence of the pair (A, B) . Then, chasing the diagram, it follows that $\partial \circ j_* = 0$.

4. $\ker \partial \subset \text{im } j_*$: We look at the following diagram.

$$\begin{array}{ccccc}
& w & & & \\
& H_n(X) & \xrightarrow{j_*} & & \\
& \downarrow j_* & & & \\
& z & & & x \\
& H_n(X, B) & \xrightarrow{j_*} & H_n(X, A) & \xrightarrow{\partial} \\
& \downarrow \partial & & \downarrow \partial & \searrow \\
& y & & & H_{n-1}(A, B) \\
& H_{n-1}(B) & \xrightarrow{\iota_*} & H_{n-1}(A) & \xrightarrow{j_*} \\
& \downarrow \iota_* & & \downarrow \iota_* & \\
& & & & H_{n-1}(X)
\end{array}$$

Suppose $\partial(x) = 0$ for some $x \in H_n(X, A)$. Now,

$$j_*(\partial(x)) = 0 \Rightarrow \partial(x) \in \ker(j_*) = \text{im}(\iota_*).$$

So, there exists some $y \in H_{n-1}(B)$ such that $\iota_*(y) = \partial(x)$. Since $B \hookrightarrow A \hookrightarrow X$, we get

$$\iota_*(y) = \iota_*(\iota_*(y)) = \iota_*(\partial(x)) = 0 \Rightarrow y \in \ker(\iota_*) = \text{im}(\partial).$$

So, there exists some $z \in H_n(X, B)$ such that $\partial(z) = y$. Now,

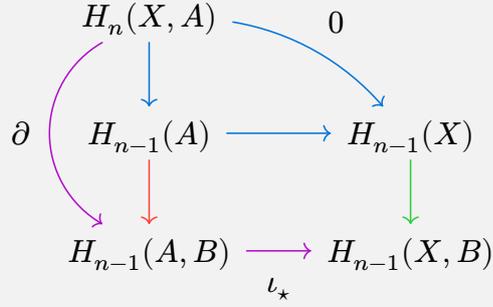
$$\partial(x - j_*(z)) = \partial(x) - \iota_*\partial(z) = \partial(x) - \iota_*(y) = 0 \Rightarrow x - j_*(z) \in \ker(\partial) = \text{im}(j_*).$$

So, there exists some $w \in H_n(X)$ such that $j_*(w) = x - j_*(z)$. Define, $z_1 = z + j_*(w) \in H_n(X, B)$. Then, we have

$$j_*(z_1) = j_*(z) + j_*(w) = j_*(z) + x - j_*(z) = x.$$

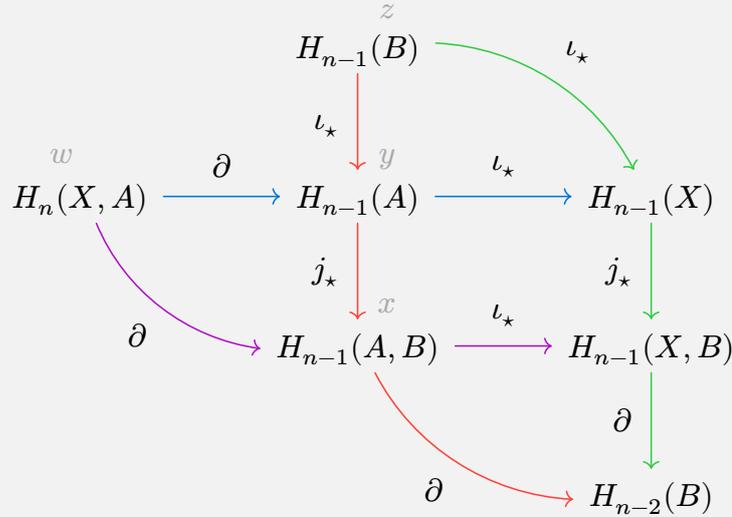
This proves the claim.

5. $\ker \iota_* \supset \text{im } \partial$: We only need to show $\iota_* \circ \partial = 0$. We have the following diagram.



The 0 map is a consequence of the long exact sequence of the pair (X, A) . Then, chasing the diagram, it follows that $\iota_\star \circ \partial = 0$.

6. $\ker \iota_\star \subset \text{im } \partial$: We look at the following diagram.



Suppose for some $x \in H_{n-1}(A, B)$ we have $\iota_\star(x) = 0$. Then,

$$\partial(x) = \partial(\iota_\star(x)) = 0 \Rightarrow x \in \ker(\partial) = \text{im}(j_\star).$$

So, there exists some $y \in H_{n-1}(A)$ such that $j_\star(y) = x$. Now,

$$j_\star(\iota_\star(y)) = \iota_\star(j_\star(y)) = \iota_\star(x) = 0 \Rightarrow \iota_\star(y) \in \ker(j_\star) = \text{im}(\iota_\star).$$

So, there exists some $z \in H_{n-1}(B)$ such that $\iota_\star(z) = \iota_\star(y)$. As $B \hookrightarrow A \hookrightarrow X$, we have

$$\iota_\star(y - \iota_\star(z)) = \iota_\star(y) - \iota_\star(z) = 0 \Rightarrow y - \iota_\star(z) \in \ker(\iota_\star) = \text{im}(\partial).$$

So, there exists some $w \in H_n(X, A)$ such that $\partial(w) = y - \iota_\star(z)$. We then have,

$$\partial(w) = j_\star(\partial(w)) = j_\star(y - \iota_\star(z)) = j_\star(y) - 0 = x.$$

This proves the claim.

Hence, we have proved that the sequence is exact at all points. Since all the maps involved are natural, one can *easily* show that the sequence is natural with respect to maps of triple as well (Check!). \square