Course notes for

Algebraic Topology II (KSM4E02)

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`additivity in homology theory – homology of homtopy equivalent spaces – homology long exact sequence of triple`

3.1 Additivity in Homology Theory

Proposition 3.1: (*Finite Additivity*)

Suppose $X = \sqcup_{j=1}^{k} X_j$ is a finite disjoint union of spaces. Then, the inclusion maps $\iota_j : X_j \hookrightarrow X$ induces isomorphism of $H_n(X)$ onto a direct summand of $H_n(X_j)$, and $H_n(X)$ is naturally isomorphic to $\oplus_{j=1}^{n} H_n(X_j)$ for each n.

Proof : We only conider the case $X \sqcup Y$, the general case follows by induction. Note that $\varphi_X : X \hookrightarrow (X \sqcup Y, Y)$ and $\varphi_Y : Y \hookrightarrow (X \sqcup Y, X)$ induces excision isomorphism in homology. Thus, we have the commuting diagrams

$$\begin{tikzcd} H_n(X) \arrow[r, hook, "(\iota_X)_\star"] \arrow[rr, bend right, "(\varphi_X)_\star"'] & H_n(X \sqcup Y) \arrow[r, two heads, "(j_X)_\star"] & H_n(X \sqcup Y, Y) \end{tikzcd} \qquad \begin{tikzcd} H_n(Y) \arrow[r, hook, "(\iota_Y)_\star"] \arrow[rr, bend right, "(\varphi_Y)_\star"'] & H_n(X \sqcup Y) \arrow[r, two heads, "(j_Y)_\star"] & H_n(X \sqcup Y, X) \end{tikzcd}$$

Then, from the long exact sequence for the pair $(X \sqcup Y, X)$, we get the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_n(X) \arrow[r] & H_n(X \sqcup Y) \arrow[r] & H_n(X \sqcup Y, X) \arrow[r] \arrow[l, bend right, "(\iota_X)_\star \circ (\varphi_X)_\star^{-1}"'] & 0. \end{tikzcd}$$

Thus, $(\iota_X)_\star$ maps $H_n(X)$ isomorphically onto a direct summand of $H_n(X \sqcup Y)$. Similarly, $(\iota_Y)_\star$ maps $H_n(Y)$ isomorphically onto a direct summand of $H_n(X \sqcup Y)$ as well. Finally, by the exicision isomorphism, we have $H_n(X \sqcup Y) \cong H_n(X) \oplus H_n(X \sqcup Y, X) \cong H_n(X) \oplus H_n(Y)$. $\square$

Exercise 3.2: (*Finite Union of Pairs*)

Let (X_j, A_j) be pairs of spaces for $1 \leq i \leq k$. Denote $(X, A) = (\sqcup X_i, \sqcup A_i)$. Show that $\iota_j : (X_j, A_j) \hookrightarrow (X, A)$ induces an isomorphism of $H_n(X_j, A_j)$ onto a direct summand of $H_n(X, A)$, and moreover, $H_n(X, A)$ is naturally isomorphic to the direct sum $\oplus_{j=1}^{k} H_n(X_j, A_j)$.

Hint : Use Proposition 3.1 and the naturality of the long exact sequence (Exercise 2.14).

The finite additivity of homology theory cannot be improved to arbitrary sum, and there are examples of homology theories which does not split for arbitrary union of spaces. On the other hand, we shall see later that singular homology abides by this. Hence, Milnor added one extra axiom.

Definition 3.3: (*Additive Homology Theory*)

A homology theory is called an *additive homology theory* if given a collection of spaces $\{X_\alpha\}_{\alpha \in I}$, the inclusion maps $\iota_\alpha : X_\alpha \hookrightarrow X := \sqcup X_\alpha$ induces an isomorphism of $H_n(X_\alpha)$ onto a direct summand of $H_n(X)$, and moreover, $H_n(X)$ is naturally isomorphic to $\oplus H_n(X_\alpha)$.

3.2 Homology of Homotopy Equivalent Spaces

Recall, $f : X \to Y$ is a homotopy equivalence if there is a map $g : Y \to X$ such that

$$g \circ f \simeq \mathrm{Id}_X, \quad f \circ g \simeq \mathrm{Id}_Y.$$

One can similarly define homotopy equivalence $f : (X, A) \to (Y, B)$ for pairs, which essentially requires that that the homotopy restricts to a homotopy of the restricted map.

Proposition 3.4: (*Homotopy Equivalence induces Homology Isomorphism*)

Let $f : X \to Y$ be a homotopy equivalence. Then, $f_\star : H_n(X) \to H_n(Y)$ is an isomorphism for all n. Similarly, a homotopy equivalence $(X, A) \to (Y, B)$ also induces isomorphism at the homology groups.

Proof : Let $g : Y \to X$ be a homotopy inverse. Now, from the functoriality and the homotopy invariance, we have

$$\mathrm{Id} = H_n(\mathrm{Id}_X) = H_n(g \circ f) = H_n(g) \circ H_n(f) = g_\star \circ f_\star,$$

and similarly, we have $f_\star \circ g_\star = \mathrm{Id}$. Hence, $f_\star : H_n(X) \to H_n(Y)$ is an isomorphism, with inverse $g_\star$. Similar argument holds for $(X, A) \simeq (Y, B)$ as well. □

Corollary 3.5: (*Homeomorphism induces Homology Isomorphism*)

Let $f : X \to Y$ be a homeomorphism of spaces. Then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism. Simlarly, homeomorphism $(X, A) \cong (Y, B)$ induces homology isomorphism as well.

3.3 Homology Long Exact Sequence of Triples

Let us consider a triple (X, A, B) of spaces where $B \subset A \subset X$. Then, we have inclusions

$$\iota : (A, B) \hookrightarrow (X, B), \quad j : (X, B) \hookrightarrow (X, A).$$

A map $f : (X, A, B) \to (Y, C, D)$ of triples is a continuous map $f : X \to Y$ such that $f|_A : A \to C$ and $f|_B : B \to D$ holds. A triple leads to a natural long exact sequence of homology groups. We shall need the following.

Lemma 3.6: (*Relative Homology of Space w.r.t. itself*)

For any space X, we have $H_n(X, X) = 0$.

Proof : By the exicision isomorphism, it follows that $H_n(X, X) \cong H_n(\emptyset, \emptyset) = H_n(\emptyset)$. Now, for the pair $(\emptyset, \emptyset)$, we have long exact sequence $\cdots \to H_n(\emptyset) \underset{\iota_\star}{\to} H_n(\emptyset) \underset{j_\star}{\to} H_n(\emptyset) \to \cdots$. Since $\iota = \mathrm{Id} = j$, we have $i_\star = \mathrm{Id} = j_\star$. But by exactness, we have $j_\star \circ \iota_\star = 0$. Hence, we have $\iota_* = 0 = j_\star$, and consequently, $H_n(\emptyset) = 0$. Thus, $H_n(X, X) = 0$ for any space X. □

Exercise 3.7:

Give a proof of $H_n(X, X) = 0$ without using excision.

Hint : Use the long exact sequence of the pair (X, X).

Theorem 3.8: (*Homology Long Exact Sequence of Triples*)

Let (X, A, B) be a triple with $B \subset A \subset X$. Then, there exists a natural long exact sequence

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A, B) \xrightarrow{\iota_\star} H_n(X, B) \xrightarrow{j_\star} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B)\cdots,$$

which is natural with respect to maps of triples. The boundary map is given as the composition

$$\partial : H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{j_\star} H_{n-1}(A, B),$$

where the first boundary is from the long exact sequence of the pair (X, A), and the second map is induced by the inclusion $A \hookrightarrow (A, B)$.

Proof : The proof boils down to checking exactness at each point, using the respective long exact sequences associated to the pairs (A, B), (X, B), and (X, A). Let us color code the arrows as follows.

$$\cdots \longrightarrow H_{n+1}(A, B) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(A, B) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_{n+1}(X, B) \longrightarrow H_n(X) \longrightarrow H_n(B) \longrightarrow H_n(X, B) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_{n+1}(X, A) \longrightarrow H_n(X) \longrightarrow H_n(A) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

Now, we have the following commutative diagram.

$$
\begin{tikzcd}
H_n(A) \arrow[r] \arrow[d] & H_n(X) \arrow[d] \arrow[dr, bend left] & & \\
H_n(A,B) \arrow[r, "\iota_\star"] \arrow[dr, bend right] & H_n(X,B) \arrow[r, "j_\star"] \arrow[d] & H_n(X,A) \arrow[d] \arrow[dr, bend left, "\partial"] & \\
& H_{n-1}(B) \arrow[r] \arrow[dr, bend right] & H_{n-1}(A) \arrow[r] \arrow[d] & H_{n-1}(A,B) \arrow[d] \\
& & H_{n-1}(X) \arrow[r] & H_{n-1}(X,B)
\end{tikzcd}
$$

The horizontal arrows are induced by the inclusions $(A,B) \hookrightarrow (X,B) \hookrightarrow (X,A)$, and the commutativity of each squeare follows from the naturality of the long exact sequence of pairs. Let us now check the exactness at each position.

1. $\mathbf{ker}\, \boldsymbol{j_\star} \supset \mathbf{im}\, \boldsymbol{\iota_\star}$: We need to show $j_\star \circ \iota_\star = 0$. We have a commuting diagram of spaces

$$
\begin{tikzcd}
(A,B) \arrow[r, "\iota"] \arrow[dr, bend right] & (X,B) \arrow[r, "j"] & (X,A) \\
& (A,A) \arrow[ur, bend right] &
\end{tikzcd}
$$

   Since $H_n(A,A) = 0$ (Lemma 3.6), by functoriality, we have $j_\star \circ \iota_\star = 0$.

2. $\mathbf{ker}\, \boldsymbol{j_\star} \subset \mathbf{im}\, \boldsymbol{\iota_\star}$: We look at the following diragram

$$
\begin{tikzcd}
\overset{w}{H_n(A)} \arrow[r, "\iota_\star"] \arrow[d, "j_\star"'] & \overset{z}{H_n(X)} \arrow[d, "j_\star"'] & \\
\overset{y}{H_n(A,B)} \arrow[r, "\iota_\star"] \arrow[dr, bend right, "\partial"'] & \overset{x}{H_n(X,B)} \arrow[r, "j_\star"] \arrow[d, "\partial"'] & H_n(X,A) \arrow[d, "\partial"] \\
& H_{n-1}(B) \arrow[r, "\iota_\star"'] & H_{n-1}(A)
\end{tikzcd}
$$

   Suppose $j_\star(x) = 0$ for some $x \in H_n(X,B)$. Then,

$$\iota_\star \partial(x) = \partial j_\star x = 0 \Rightarrow \partial(x) \in \ker(\iota_\star) = \mathrm{im}(\partial).$$

   So, there exists some $y \in H_n(A,B)$ such that

$$\partial x = \partial y = \partial \iota_\star(y) \Rightarrow \partial(x - \iota_\star(y)) = 0 \Rightarrow x - \iota_\star(y) \in \ker \partial = \mathrm{im}\, j_\star.$$

   So, there exists some $z \in H_n(X)$ such that

$$j_\star(z) = x - \iota_\star(y) \Rightarrow j_\star(z) = j_\star(x - \iota_\star(y)) = j_\star(x) - j_\star\iota_\star(x) = 0 \Rightarrow z \in \ker j_\star = \mathrm{im}\, \iota_\star.$$

   So, there exists some $w \in H_n(A)$ such that $\iota_\star(w) = z$. Define $y_1 = y + j_\star(w) \in H_n(A,B)$. Then,

$$\iota_\star(y_1) = \iota_\star(y) + j_\star\iota_\star(w) = \iota_\star(y) + j_\star(z) = \iota_\star(y) + x - \iota_\star(y) = x.$$

   This proves the claim.

3. $\mathbf{ker}\, \boldsymbol{\partial} \supset \mathbf{im}\, \boldsymbol{j_\star}$: We only need to show $\partial \circ j_\star = 0$. We have the following diagram.

$$
\begin{tikzcd}
H_n(X,B) \arrow[r, "j_\star"] \arrow[d] & H_n(X,A) \arrow[d] \arrow[dd, bend left, "\partial"] \\
H_{n-1}(B) \arrow[r] \arrow[dr, bend right, "0"'] & H_{n-1}(A) \arrow[d] \\
& H_{n-1}(A,B)
\end{tikzcd}
$$

The 0 map is a consequence of the long exact sequence of the pair (A,B). Then, chasing the diagram, it follows that $\partial \circ j_\star = 0$.

4. $\mathbf{ker}\,\boldsymbol{\partial} \subset \mathbf{im}\,\boldsymbol{j_\star}$: We look at the following diagram.

$$
\begin{tikzcd}
\overset{w}{H_n(X)} \arrow[d, "j_\star"'] \arrow[dr, bend left, "j_\star"] & & \\
\overset{z}{H_n(X,B)} \arrow[r, "j_\star"] \arrow[d, "\partial"'] & \overset{x}{H_n(X,A)} \arrow[d, "\partial"] \arrow[dr, bend left, "\partial"] & \\
\overset{y}{H_{n-1}(B)} \arrow[r, "\iota_\star"] \arrow[dr, bend right, "\iota_\star"'] & H_{n-1}(A) \arrow[r, "j_\star"] \arrow[d, "\iota_\star"] & H_{n-1}(A,B) \\
& H_{n-1}(X) &
\end{tikzcd}
$$

Suppose $\partial(x) = 0$ for some $x \in H_n(X,A)$. Now,

$$j_\star(\partial(x)) = 0 \Rightarrow \partial(x) \in \ker(j_\star) = \mathrm{im}(\iota_\star).$$

So, there exists some $y \in H_{n-1}(B)$ such that $\iota_\star(y) = \partial(x)$. Since $B \hookrightarrow A \hookrightarrow X$, we get

$$\iota_\star(y) = \iota_\star(\iota_\star(y)) = \iota_\star(\partial(x)) = 0 \Rightarrow y \in \ker(\iota_\star) = \mathrm{im}(\partial).$$

So, there exists some $z \in H_n(X,B)$ such that $\partial(z) = y$. Now,

$$\partial(x - j_\star(z)) = \partial(x) - \iota_\star\partial(z) = \partial(x) - \iota_\star(y) = 0 \Rightarrow x - j_\star(z) \in \ker(\partial) = \mathrm{im}(j_\star).$$

So, there exists some $w \in H_n(X)$ such that $j_\star(w) = x - j_\star(z)$. Define, $z_1 = z + j_\star(w) \in H_n(X,B)$. Then, we have

$$j_\star(z_1) = j_\star(z) + j_\star(w) = j_\star(z) + x - j_\star(z) = x.$$

This proves the claim.

5. $\mathbf{ker}\,\boldsymbol{\iota_\star} \supset \mathbf{im}\,\boldsymbol{\partial}$: We only need to show $\iota_\star \circ \partial = 0$. We have the following diagram.

$$\begin{tikzcd}
H_n(X,A) \arrow[d] \arrow[dr, bend left, "0"] \arrow[dd, bend right, "\partial"'] & \\
H_{n-1}(A) \arrow[r] \arrow[d] & H_{n-1}(X) \arrow[d] \\
H_{n-1}(A,B) \arrow[r, "\iota_\star"'] & H_{n-1}(X,B)
\end{tikzcd}$$

The 0 map is a consequence of the long exact sequence of the pair (X, A). Then, chasing the diagram, it follows that $\iota_\star \circ \partial = 0$.

6. $\boldsymbol{\ker \iota_\star \subset \operatorname{im} \partial}$: We look at the following diagram.

$$\begin{tikzcd}
& \overset{z}{H_{n-1}(B)} \arrow[d, "\iota_\star"'] \arrow[dr, bend left, "\iota_\star"] & \\
\overset{w}{H_n(X,A)} \arrow[r, "\partial"] \arrow[dr, bend right, "\partial"'] & \overset{y}{H_{n-1}(A)} \arrow[r, "\iota_\star"] \arrow[d, "j_\star"'] & H_{n-1}(X) \arrow[d, "j_\star"'] \\
& \overset{x}{H_{n-1}(A,B)} \arrow[r, "\iota_\star"] \arrow[dr, bend right, "\partial"'] & H_{n-1}(X,B) \arrow[d, "\partial"'] \\
& & H_{n-2}(B)
\end{tikzcd}$$

Suppose for some $x \in H_{n-1}(A, B)$ we have $\iota_\star(x) = 0$. Then,

$$\partial(x) = \partial(\iota_\star(x)) = 0 \Rightarrow x \in \ker(\partial) = \operatorname{im}(j_\star).$$

So, there exists some $y \in H_{n-1}(A)$ such that $j_\star(y) = x$. Now,

$$j_\star(\iota_\star(y)) = \iota_\star(j_\star(y)) = \iota_\star(x) = 0 \Rightarrow \iota_\star(y) \in \ker(j_\star) = \operatorname{im}(\iota_\star).$$

So, there exists some $z \in H_{n-1}(B)$ such that $\iota_\star(z) = \iota_\star(y)$. As $B \hookrightarrow A \hookrightarrow X$, we have

$$\iota_\star(y - \iota_\star(z)) = \iota_\star(y) - \iota_\star(z) = 0 \Rightarrow y - \iota_\star(z) \in \ker(\iota_\star) = \operatorname{im}(\partial).$$

So, there exists some $w \in H_n(X, A)$ such that $\partial(w) = y - \iota_\star(z)$. We then have,

$$\partial(w) = j_\star(\partial(w)) = j_\star(y - \iota_\star(z)) = j_\star(y) - 0 = x.$$

This proves the claim.

Hence, we have proved that the sequence is exact at all points. Since all the maps involved are natural, one can *easily* show that the sequence is natural with respect to maps of triple as well (Check!). $\square$