

# Algebraic Topology II (KSM4E02)

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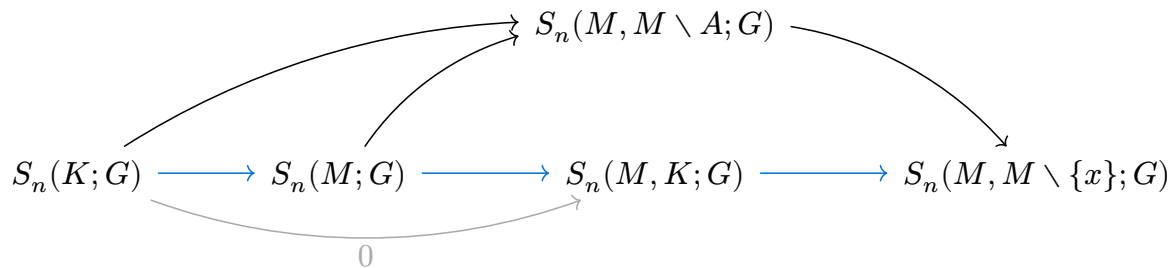
section of orientation sheaf – fundamental class – cap product – duality theorems – Poincaré duality

### 22.1 Fundamental Class

Suppose  $M$  is a manifold, and  $A \subset M$  is a closed subset. We consider homology with coefficients in a fixed Abelian group  $G$  (which could as well be a ring). Note that by UCT (Theorem 17.10), we have  $M_G = M_{\mathbb{Z}} \otimes G$ . For each  $\alpha \in H_n(M, M \setminus A; G)$ , we have the following map

$$\begin{aligned} \mathcal{J}^A(\alpha) : A &\rightarrow A; G \\ x &\mapsto r_x^A(\alpha). \end{aligned}$$

One can see that  $\mathcal{J}^A(\alpha)$  is a section of  $M_G$  over  $A$ . In fact, it has compact support in  $A$ . Indeed, choose a cycle  $c \in S_n(M; G)$  that represents  $\alpha$ . Then, there exists a compact set  $K \subset M$  such that image of singular simplices in  $c$  is contained in  $K$ , i.e,  $c \in S_n(K)$ . For any  $x \in A \setminus K$ , we then have the composition



which maps  $c$  to 0. Passing to homology, this shows  $\text{hat}(t) r_x^A(\alpha) = 0$ . Thus,  $\mathcal{J}^A(\alpha)$  has support in  $K \cap A$ , which is a compact set as  $A$  is closed. Thus,  $\mathcal{J}^A(\alpha) \in \Gamma_c(A; G)$ . We have the following.

**Theorem 22.1:** (Sections of Orientation Cover with Compact Support)

Let  $M$  be a Hausdorff, second countable  $n$ -dimensional manifold, and  $A \subset M$  be closed. Fix a group  $G$  (or an  $R$ -module) as coefficients for the homology. Then, the following holds

1.  $H_i(M, M \setminus A) = 0$  for  $i > n$ .
2.  $\mathcal{J}^A : H_n(M, M \setminus A) \rightarrow \Gamma_c(A; G)$  is an isomorphism.

**Proof :** The proof technique is like an *induction* over the complexity over all possible closed sets of  $M$ . For any closed set  $A \subset M$ , let us denote the statements

- $\mathcal{Z}(A) : H_i(M, M \setminus A) = 0$  for  $i > n$ .
- $\mathcal{J}(A) : \mathcal{J}^A : H_n(M, M \setminus A) \rightarrow \Gamma_c(A; G)$  is an isomorphism.

The proof now breaks down in to the following steps.

**Step 1.** We claim that if  $\mathcal{Z}(A), \mathcal{Z}(B)$  and  $\mathcal{Z}(A \cap B)$  holds, then so does  $\mathcal{Z}(A \cup B)$ . Similarly, if  $\mathcal{J}(A), \mathcal{J}(B)$  and  $\mathcal{J}(A \cap B)$  holds, then so does  $\mathcal{J}(A \cup B)$ .

We use the relative Mayer-Vietoris sequence ([Theorem 5.6](#)) for the *proper* triad  $(M \setminus (A \cap B); M \setminus A, M \setminus B)$  (Check!). We have the diagram

$$\begin{array}{ccc}
 H_{n+1}(M, M \setminus (A \cap B)) & \xrightarrow{\cong} & 0 \\
 \downarrow & & \downarrow \\
 H_n(M, M \setminus (A \cup B)) & \xrightarrow{\mathcal{J}^{A \cup B}} & \Gamma_c(A \cup B; G) \\
 \downarrow & & \downarrow \\
 H_n(M, M \setminus A) \oplus H_n(M, M \setminus B) & \xrightarrow[\cong]{\mathcal{J}^A \oplus \mathcal{J}^B} & \Gamma_c(A; G) \oplus \Gamma_c(B; G) \\
 \downarrow & & \downarrow \\
 H_n(M, M \setminus (A \cap B)) & \xrightarrow[\cong]{\mathcal{J}^{A \cap B}} & \Gamma_c(A \cap B; G)
 \end{array}$$

The left hand column is part of the relative Mayer-Vietoris sequence. The right hand side is also a long exact sequence induced by the (usual) Mayer-Vietoris sequence applied to each stalk (i.e, each fiber). Note that we can define the sheaves  $M_{G,k} = \sqcup_{x \in M} H_k(M, M \setminus \{x\}; G)$  for any  $k$ , and then,  $\Gamma_c(M_{G,n+1}) = 0$  as each stalk is 0. The commutativity follows from the naturality of the Mayer-Vietoris sequence. Then, by the 5-lemma, we have  $\mathcal{J}^{A \cup B}$  is an isomorphism, proving  $\mathcal{J}(A \cup B)$

For any  $i > n$ , we have the relative Mayer-Vietoris sequence

$$\cdots \rightarrow H_{i+1}(M, M \setminus (A \cap B)) \rightarrow H_i(M, M \setminus (A \cup B)) \rightarrow H_i(M, M \setminus A) \oplus H_i(M, M \setminus B) \rightarrow \cdots$$

$\qquad\qquad\qquad 0 \qquad\qquad\qquad 0 \qquad\qquad\qquad 0$

which proves that  $H_i(M, M \setminus (A \cup B)) = 0$ . This proves  $\mathcal{Z}(A \cup B)$ .

**Step 2.** Call a closed set *good* if  $A \subset U$  for some chart  $\varphi : U \rightarrow \mathbb{R}^n$  such that  $\varphi(A) \subset \mathbb{R}^n$  is convex. We claim that  $\mathcal{Z}(A)$  and  $\mathcal{J}(A)$  holds for any good compact sets. Note,  $M$  is Hausdorff and thus, compact sets are closed.

Choose a chart  $\varphi : U \rightarrow \mathbb{R}^n$  such that  $\varphi(A)$  is convex. Fix some  $x \in A$ , and assume that  $\varphi(x) = 0$ , and  $0 \in \varphi(A) \subset \mathring{D}^n$ , which can be done by some affine transform and then radially shrinking suitably. We consider the following diagram

$$\begin{array}{ccccccc}
 H_i(D^n, S^{n-1}) & \xrightarrow{\text{green}} & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi(A)) & \xleftarrow[\cong]{\varphi_*} & H_i(U, U \setminus A) & \xrightarrow{\text{blue}} & H_i(M, M \setminus A) \\
 \parallel & & \downarrow & & \downarrow & & \downarrow r_x^A \\
 H_i(D^n, S^{n-1}) & \xrightarrow{\text{green}} & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xleftarrow[\cong]{\varphi_*} & H_i(U, U \setminus \{x\}) & \xrightarrow{\text{blue}} & H_i(M, M \setminus \{x\})
 \end{array}$$

The commutativity is clear from the space level maps. The **green** arrows are homotopy equivalences, note that the standard radial homotopy pushes the *convex* set  $\varphi(A) \subset \mathring{D}^n$  to the boundary  $S^{n-1}$ . The **blue** arrows are excision isomorphisms, obtained from excising  $M \setminus U$ . Thus, it follows that  $r_x^A$  is an isomorphism.

Now, for  $i > n$ , we have  $H_i(M, M \setminus \{x\}) = H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = H_{i-1}(\mathbb{R}^n \setminus \{0\}) = 0$ , which shows  $H_i(M, M \setminus A) = 0$ , proving  $\mathcal{Z}(A)$ .

Next, note that a section of a covering space of a connected set (in this case  $A$ ) is determined by the value at a single point. Thus, the isomorphism  $r_x^A$  shows that  $\mathcal{J}^A$  is a bijection. This justifies  $\mathcal{J}(A)$ .

**Step 3.** Next, we show that both  $\mathcal{Z}(A)$  and  $\mathcal{J}(A)$  holds when  $A \subset U$ , where  $\varphi : U \rightarrow \mathbb{R}^n$  is a chart, and  $A = \bigcup_{i=1}^k K_i$  where  $K_i$  is a good compact set with respect to the chart map  $\varphi$ .

Recall that the intersection of compact sets of  $\mathbb{R}^n$  is again compact. By Step 2, we have  $\mathcal{Z}(K_i)$  holds for each  $i$ . Inductively assume that  $\mathcal{Z}(K_1 \cup \dots \cup K_{n-1})$  holds. Since  $(K_1 \cup \dots \cup K_{n-1}) \cap K_n$  is again a good compact set, we have  $\mathcal{Z}((K_1 \cup \dots \cup K_{n-1}) \cap K_n)$  holds. But then by Step 1, we have  $\mathcal{Z}(K_1 \cup \dots \cup K_n) = \mathcal{Z}(A)$  holds. Similar argument shows that  $\mathcal{J}(A)$  holds as well.

**Step 4.** Next, we claim that both  $\mathcal{Z}(A)$  and  $\mathcal{J}(A)$  holds when  $A \subset U$ , where  $\varphi : U \rightarrow \mathbb{R}^n$  is a chart, and  $A$  is compact.

Firstly, fix any open neighborhood  $A \subset W \subset U$ . Cover the image of  $\varphi(A)$  by the interior of finitely many balls, which are small enough to be contained in  $\varphi(W)$ . Then, the inverse images of the *closed* balls are good compact sets, and their union, say,  $V$  is of the same type as Step 3. Moreover,  $V$  is actually a neighborhood of  $A$  (i.e,  $A$  is contained in the interior of  $V$ )  $A \subset V \subset W$ , where  $W$  is arbitrary. In other words, we have arbitrarily small compact neighborhoods  $V$  of  $A$ .

The collection of all such  $A \subset V \subset U$  forms a directed system. Passing to the colimit, we have the diagram

$$\begin{array}{ccc} \operatorname{colim}_V H_i(M, M \setminus V) & \xrightarrow{\Phi} & H_i(M, M \setminus A) \\ \operatorname{colim}_V \mathcal{J}^V \downarrow & & \mathcal{J}^A \downarrow \\ \operatorname{colim}_V \Gamma_c(M_{G,i}|_V) & \xrightarrow{\Psi} & \Gamma_c(M_{G,i}|_A) \end{array}$$

The horizontal maps  $\Phi$  and  $\Psi$  are induced by the restriction to  $K$ , in particular, the  $\Phi$  is induced by the collection  $r_K^V$ . Commutativity follows from the universal property of the colimit.

- We check that  $\Phi$  is induced from an isomorphism at the chain level. Indeed, any chain has a compact support (i.e, it has a compact image). Thus, if there is a chain that is completely contained in  $M \setminus K$ , then there is a sufficiently small compact neighborhood  $A \subset V$  such that the chain is contained in  $M \setminus V$ . Then, any such chain lifts to the colimit  $\operatorname{colim}_V S_\bullet(M, M \setminus V)$ , proving the surjectivity. Injectivity, is easy to verify : an element in the colimit is represented by a chain in  $S_\bullet(M, M \setminus V)$  for some  $A \subset V$ , if this chain becomes 0 in  $S_\bullet(M, M \setminus A)$ , then it must have been 0 to begin with. Now, a general fact is that homology commutes with colimit, and hence we get  $\Phi$  is an isomorphism.
- The fact that  $\Psi$  is an isomorphism is a general fact about sheaf of sections of covering spaces. Essentially, surjectivity follows from the fact that any section on the compact set  $A$  can be extended to an arbitrarily small neighborhood of  $A$ . For injectivity, one can show that if two sections defined on some neighborhoods  $V_1, V_2$  of  $A$  matches on  $A$ , then they in fact matches on some smaller neighborhood  $A \subset V \subset V_1 \cap V_2$ .

For any  $i > n$ , we have  $\text{colim } H_i(M, M \setminus V) = 0$ , as  $\mathcal{Z}(V)$  holds. As  $\Phi$  is an isomorphism, we have  $H_i(M, M \setminus A) = 0$ , proving  $\mathcal{Z}(A)$ . Finally, let us take  $i = n$ , which immediately gives that  $\text{colim } \mathcal{J}^V$  is an isomorphism by  $\mathcal{J}(V)$ . Then,  $\mathcal{J}^A$  is isomorphism, proving  $\mathcal{J}(A)$ .

**Step 5.** We now claim that  $\mathcal{Z}(A)$  and  $\mathcal{J}(A)$  holds for *arbitrary* compact sets  $A \subset M$ .

Indeed, we can cover  $A$  by finitely many compact sets, each of which is contained some coordinate chart. Then,  $\mathcal{Z}(A)$  and  $\mathcal{J}(A)$  follows inductively from step 1.

**Step 6.** Next, suppose  $A = \cup_{i=1}^{\infty} K_i$ , where  $K_i$  is compact, and for each  $i \neq j$  there are disjoint open neighborhoods of  $K_i$  and  $K_j$ . Then, we show that  $\mathcal{Z}(A)$  and  $\mathcal{J}(A)$  holds.

It follows from the additivity of the homology theory, that  $\mathcal{J}^A = \sum \mathcal{K}^{K_i}$ , and similarly  $H_i(M, M \setminus A) = \oplus H_i(M, M \setminus K_i)$ . Since  $\mathcal{Z}(K_i)$  and  $\mathcal{J}(K_i)$  holds, we have  $\mathcal{J}^A$  is an isomorphism and  $H_i(M, M \setminus A) = 0$  for  $i > n$ . This shows  $\mathcal{Z}(A)$  and  $\mathcal{J}(A)$ .

**Step 7.** Finally, let  $A$  be an arbitrary closed set of  $M$ . Since  $M$  is locally compact,  $T_2$ , and second countable, we have compact sets  $K_i$  such that  $M = \cup_{i=1}^{\infty} K_i$  and  $K_i \subset \overset{\circ}{K}_{i+1}$ . Set  $A_i = A \cap (K_i \setminus \overset{\circ}{K}_{i-1})$ , with  $K_0 = \emptyset$ . Let

$$B = \bigcup_{i \text{ is even}} A_i, \quad C = \bigcup_{i \text{ is odd}} A_i.$$

By step 6,  $\mathcal{Z}, \mathcal{J}$  holds for  $B, C$  and  $B \cap C$ . By step 1, we conclude that  $\mathcal{Z}(A)$  and  $\mathcal{J}(A)$  holds as well.

This finishes the proof. □

As a consequence of [Theorem 22.1](#), we get the following.

**Theorem 22.2: (Orientation along Closed, Connected Set)**

Suppose  $A \subset M$  is a closed, connected subset of an  $n$ -fold  $M$ . Let  $R$  be a ring, and  $G$  be an  $R$ -module.

1. Suppose  $A$  is noncompact. then  $H_n(M, M \setminus A; G) = 0$ .
2. Suppose  $A$  is compact.
  - a. If  $M$  is  $R$ -orientable along  $A$ , then  $H_n(M, M \setminus A; G) = G$ . Moreover,  $r_x^A : H_n(M, M \setminus A; G) \rightarrow H_n(M, M \setminus \{x\}; G)$  is an isomorphism for each  $x \in M$ .
  - b. If  $R = \mathbb{Z}$ , and  $M$  is not orientable along  $A$ , then

$$H_n(M, M \setminus A; G) = {}_2G := \{g \in G \mid 2g = 0\}.$$

**Proof :** Suppose  $A$  is noncompact. Since  $A$  is connected, any section in  $s \in \Gamma(A; G)$  is completely determined by its value at a fixed point  $x \in A$ . If  $s(x) \neq 0$ , then  $s \neq 0$  on  $A$ . In particular, its support is  $A$ , which is noncompact. In other words, there are no nonzero section with compact support on  $A$ , i.e.,  $\Gamma_c(A; G) = 0$ . By [Theorem 22.1](#), we have  $H_n(M, M \setminus A; G) = 0$ .

Next, suppose  $A$  is compact. Then, any section on  $A$  has compact support, i.e.,  $H_n(M, M \setminus A; G) = \Gamma_c(A; G) = \Gamma(A; G)$ . Recall, a section is determined by its value at a point. We have a diagram

$$\begin{array}{ccc}
H_n(M, M \setminus A; G) & \xrightarrow[\cong]{\mathcal{J}^A} & \Gamma(A; G) \\
\downarrow r_x^A & & \downarrow b \\
H_n(M, M \setminus \{x\}; G) & \xrightarrow[\cong]{\mathcal{J}^x} & \Gamma(\{x\}; G)
\end{array}$$

Here,  $b$  is the restriction map. Since  $A$  is connected, any section is determined by its value at the point  $x$ , and in particular,  $b$  is injective.

- Suppose  $M$  is  $R$ -orientable along  $A$ . Then, it is  $G$ -orientable along  $A$ , in particular, there exists  $s \in \Gamma(A; G)$  such that  $b(s) = s(x)$  is a generator of  $H_n(M, M \setminus x; G) = G$ . Thus,  $b$  is surjective, and hence, an isomorphism. This implies,  $r_x^A$  is an isomorphism.
- Recall,  $M_G = \tilde{M} \times_{\mathbb{Z}_2} G$ , where  $\tilde{M} \subset M_{\mathbb{Z}}$  is the orientation double cover, and  $\mathbb{Z}_2$  acts on  $G$  by multiplication by  $-1$ . Thus, a section of  $\Gamma(A; G)$  corresponds to a map  $\lambda : \tilde{M} \rightarrow G$  such that  $\lambda \circ t = -\lambda$ , where  $t$  is the generator of  $\mathbb{Z}_2$ . Since  $M$  is not orientable along  $A$ , it follows that  $\tilde{M}|_A$  is connected, and thus,  $\lambda$  must be constant. Thus, image of  $b$  is precisely  ${}_2G$ . The commutativity shows that  $H_n(M, M \setminus A; G) = {}_2G$ .

This concludes the proof. □

For a compact, connected  $n$ -fold  $M$ , let us restate [Theorem 22.2](#).

**Theorem 22.3:** (*Oriention and Compact, Connected Manifold*)

Suppose  $M$  is a compact, connected  $n$ -fold. Then, precisely one of the following holds true.

1.  $M$  is orientable,  $H_n(M) = \mathbb{Z}$ , and for each  $x \in M$  the map  $r_x^A : H_n(M) \rightarrow H_n(M, M \setminus \{x\})$  is an isomorphism.
2.  $M$  is non-orientable and  $H_n(M) = 0$ .

**Proof :** Since  $M$  is compact and connected, we can take  $A = M$  in [Theorem 22.2](#). Note that  $H_n(M) = H_n(M, \emptyset) = H_n(M, M \setminus M)$ . When  $M$  is not orientable, we get  $H_n(M) = {}_2\mathbb{Z} = \{n \in \mathbb{Z} \mid 2n = 0\} = 0$ . This concludes the proof. □

In particular, over integers, orientability (resp. nonorientability) is equivalent to  $H_n(M) = \mathbb{Z}$  (resp.  $0$ ). A consequence of [Theorem 22.3](#) is the following important definition.

**Definition 22.4:** (*Fundamental Class*)

Suppose  $M$  is a compact, connected  $n$ -fold, which  $R$ -orientable for some ring  $R$ . Then, a *fundamental class* (or orientation class) for  $M$  is a generator  $[M]_R$  of  $H_n(M; R) = R$ .

Note that for each  $x \in M$  the map  $r_x^A : H_n(M; R) \rightarrow H_n(M, M \setminus \{x\}; R)$  maps a fundamental class  $[M]$  to a generator of  $H_n(M, M \setminus \{x\}; R) = R$ . When  $R = \mathbb{Z}$ , we simply denote a fundamental class as  $[M]$ .

Let us also point out another important consequence of [Theorem 22.2](#).

**Proposition 22.5:** (*Torsion in  $H_{n-1}(M)$* )

Suppose  $M$  is an  $n$ -fold, and  $A \subset M$  is a closed connected subset. Denote  $\mathcal{T} \subset H_{n-1}(M, M \setminus A; \mathbb{Z})$  as the torsion subgroup, i.e,

$$\mathcal{T} = \{a \in H_{n-1}(M, M \setminus A) \mid na = 0 \text{ for some } n \in \mathbb{Z}\}.$$

Then, we have the following dichotomy.

- $\mathcal{T} = \mathbb{Z}_2$  if  $A$  is compact and  $M$  is nonorientable along  $A$ .
- $\mathcal{T} = 0$  otherwise.

In particular, if  $M$  is a compact, connected, orientable  $n$ -fold, then  $H_{n-1}(M)$  is a free Abelian group.

**Proof :** Let  $q \in \mathbb{N}$  be a positive integer. If  $A$  is noncompact, then from the relative version of the UCT ([Theorem 17.10](#)), we have the short exact sequence

$$0 \rightarrow H_n(M, M \setminus A; \mathbb{Z}) \otimes \mathbb{Z}_q \rightarrow H_n(M, M \setminus A; \mathbb{Z}_q) \rightarrow \text{Tor}(H_{n-1}(M, M \setminus A; \mathbb{Z}), \mathbb{Z}_q) \rightarrow 0.$$

0 0

That the groups are 0 follows from [Theorem 22.2](#). But then the Tor term vanishes. In particular, the group  $H_{n-1}(M, M \setminus A; \mathbb{Z})$  has no  $q$ -torsion. As  $q$  is arbitrary, we get  $\mathcal{T} = 0$  for  $A$  noncompact. If  $M$  is orientable along a compact set  $A$ , again applying [Theorem 22.2](#) to the short exact sequence we have

$$0 \rightarrow H_n(M, M \setminus A; \mathbb{Z}) \otimes \mathbb{Z}_q \rightarrow H_n(M, M \setminus A; \mathbb{Z}_q) \rightarrow \text{Tor}(H_{n-1}(M, M \setminus A; \mathbb{Z}), \mathbb{Z}_q) \rightarrow 0.$$

$\mathbb{Z}$   $\mathbb{Z}_q$

That is,

$$0 \rightarrow \mathbb{Z}_q \rightarrow \mathbb{Z}_q \rightarrow \text{Tor}(H_{n-1}(M, M \setminus A; \mathbb{Z}), \mathbb{Z}_q) \rightarrow 0.$$

As the sequence splits, it follows that the Tor term vanishes. As  $q$  is arbitrary, we again get  $\mathcal{T} = 0$ .

Now, suppose  $A$  is compact, and  $M$  is nonorientable along  $A$ . For  $q$  odd, we have  ${}_2\mathbb{Z}_q = \{g \in \mathbb{Z}_q \mid 2g = 0\} = 0$ , as 2 is invertible in  $\mathbb{Z}_q$ . Hence, by [Theorem 22.2](#), we have  $H_n(M, M \setminus A; \mathbb{Z}_q) = 0$  for  $q$  odd. Thus, from the UCT sequence, we have  $\mathcal{T}$  has no odd torsion. Moreover, with  $G = \mathbb{Z}_4$ , we have  ${}_2\mathbb{Z}_4 = \{x \in \mathbb{Z}_4 \mid 2x = 0\} = \mathbb{Z}_2$ , and hence the sequence

$$0 \rightarrow H_n(M, M \setminus A; \mathbb{Z}) \otimes \mathbb{Z}_4 \rightarrow H_n(M, M \setminus A; \mathbb{Z}_4) \rightarrow \text{Tor}(H_{n-1}(M, M \setminus A; \mathbb{Z}), \mathbb{Z}_4) \rightarrow 0.$$

0  $\mathbb{Z}_2$

This shows that

$$\mathbb{Z}_2 = \text{Tor}(H_{n-1}(M, M \setminus A; \mathbb{Z}), \mathbb{Z}_4) = \text{Tor}(\mathcal{T}, \mathbb{Z}_4) = \{x \in \mathcal{T} \mid 4x = 0\}.$$

Since  $\mathcal{T}$  has no odd torsion, it follows that any element has finite order, which is a power of 2. But  ${}_4\mathcal{T}$  implies that there can only exist a unique element in  $\mathcal{T}$ , which then must have order 2. Thus,  $\mathcal{T} = \mathbb{Z}_2$ , as claimed.  $\square$

## 22.2 Cap Product

Modifying the cochain level cup product, we get the *cap product*. Given a ring  $R$ , define

$$\frown: S^k(X; R) \otimes S_{k+l}(X; R) \rightarrow S_l(X; R)$$

by the following formula. For  $k$ -cochain  $\varphi : S_k(X) \rightarrow R$ , and a singular simplex  $\sigma : \Delta^{k+l} \rightarrow X$ , define

$$\varphi \frown \sigma := \varphi(\sigma|_{[v_0, \dots, v_k]})\sigma|_{[v_k, \dots, v_{k+l}]},$$

and then extend  $R$ -linearly. In fact, we define  $\frown : S^k(X; R) \otimes S_n(X; R) \rightarrow S_{n-k}(X; R)$  by setting it 0 whenever  $n < k$ .

**Proposition 22.6: (Properties of Cap Product)**

Let  $\varphi \in S^p(X; R)$ ,  $\psi \in S^q(X; R)$  be cochains, and  $\sigma : \Delta^n \rightarrow X$  be a simplex. The cochain level cap product satisfies the following.

1.  $\partial(\varphi \frown \sigma) = (-1)^k(\varphi \frown \partial\sigma - \delta\varphi \frown \sigma)$ .
2.  $(\varphi \smile \psi) \frown \sigma = \varphi \frown (\psi \frown \sigma)$ .
3.  $(\varphi \smile \psi)(\sigma) = \psi(\varphi \frown \sigma)$ , for  $n = p + q$ .
4.  $1 \frown \sigma = \sigma$ , where  $1 : S_0(X) \rightarrow R$  is the constant map  $1_R$ .
5. For any  $f : Y \rightarrow X$ , and simplex  $\tau : \Delta^n \rightarrow Y$ , we have

$$f_{\#}(f^{\#}\varphi \frown \tau) = \varphi \frown f_{\#}\tau,$$

where  $f^{\#} : S^{\bullet}(X) \rightarrow S^{\bullet}(Y)$  and  $f_{\#} : S_{\bullet}(Y) \rightarrow S_{\bullet}(X)$  are the respective induced map.

Consequently, cap product induces the cohomological cap product

$$\frown : H^k(X; R) \otimes H_n(X; R) \rightarrow H_{n-k}(X; R)$$

which makes  $H_{\star}(X; R)$  into a graded  $H^{\star}(X; R)$ -module.

**Proof :** The proves are via explicit computation.

1. Let us assume  $n \geq k$ . In fact, for  $n = k$ , the LHS and RHS are zero. So we can take  $n \geq k + 1$ . We compute

$$\begin{aligned} \varphi \frown \partial\sigma &= \varphi \frown \left( \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \right) \\ &= \sum_{i=0}^n (-1)^i \varphi \frown \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \\ &= \sum_{i=0}^k (-1)^i \varphi(\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_{k+1}]})\sigma|_{[v_{k+1}, \dots, v_n]} + \sum_{i=k+1}^n (-1)^i \varphi(\sigma|_{[v_0, \dots, v_k]})\sigma|_{[v_{k+1}, \dots, \hat{v}_i, \dots, v_n]} \\ &= \varphi \left( \sum_{i=0}^{k+1} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_{k+1}]} \right) \sigma|_{[v_{k+1}, \dots, v_n]} - (-1)^{k+1} \varphi(\sigma|_{[v_0, \dots, v_k]})\sigma|_{[v_{k+1}, \dots, v_n]} \\ &\quad + (-1)^k \varphi(\sigma|_{[v_0, \dots, v_k]}) \left( \sum_{i=k+1}^n (-1)^{i-k} \sigma|_{[v_{k+1}, \dots, \hat{v}_i, \dots, v_n]} \right) \\ &= \varphi(\partial\sigma|_{[v_0, \dots, v_{k+1}]})\sigma|_{[v_{k+1}, \dots, v_n]} \\ &= \delta\varphi(\sigma|_{[v_0, \dots, v_{k+1}]})\sigma|_{[v_{k+1}, \dots, v_n]} + (-1)^k \varphi(\sigma|_{[v_0, \dots, v_k]}) \left( \sum_{i=0}^n (-1)^i \sigma|_{[v_k, \dots, \hat{v}_{k+i}, \dots, v_n]} \right) \\ &= \delta\varphi \frown \sigma + (-1)^k \varphi(\sigma|_{[v_0, \dots, v_k]}) (\partial\sigma|_{[v_k, \dots, v_n]}) \end{aligned}$$

$$\begin{aligned}
&= \delta\varphi \frown \sigma + (-1)^k \partial(\varphi(\sigma|_{[v_0, \dots, v_k]})\sigma|_{[v_k, \dots, v_n]}) \\
&= \delta\varphi \frown \sigma + (-1)^k \partial(\varphi \frown \sigma)
\end{aligned}$$

Thus, we have,  $\partial(\varphi \frown \sigma) = (-1)^k(\varphi \frown \partial\sigma - \delta\varphi \frown \sigma)$

2. Again, let us assume  $n \geq p + q$ . We compute

$$\begin{aligned}
(\varphi \smile \psi) \frown \sigma &= (\varphi \smile \psi)(\sigma|_{[v_0, \dots, v_{p+q}]})\sigma|_{[v_{p+q}, \dots, v_n]} \\
&= \varphi(\sigma|_{[v_0, \dots, v_p]})\psi(\sigma|_{[v_p, \dots, v_{p+q}]})\sigma|_{[v_{p+q}, \dots, v_n]} \\
&= \varphi(\sigma|_{[v_0, \dots, v_p]})(\psi \frown \sigma|_{[v_p, \dots, v_n]}) \\
&= (\varphi \frown (\psi \frown \sigma)).
\end{aligned}$$

3. We compute

$$\begin{aligned}
(\varphi \smile \psi)(\sigma) &= \varphi(\sigma|_{[v_0, \dots, v_p]}) \cdot \psi(\sigma|_{[v_p, \dots, v_n]}) \\
&= \psi(\varphi(\sigma|_{[v_0, \dots, v_p]}) \cdot \sigma|_{[v_p, \dots, v_n]}) \\
&= \psi(\varphi \frown \sigma)
\end{aligned}$$

1. We have

$$1 \frown \sigma = 1(\sigma|_{[v_0]})\sigma|_{[v_0, \dots, v_n]} = 1_R \cdot \sigma = \sigma.$$

2. For any  $f : Y \rightarrow X$ , we have

$$\begin{aligned}
f_{\#}(f^{\#}\varphi \frown \sigma) &= f_{\#}(f^{\#}\varphi(\sigma|_{[v_0, \dots, v_p]})\sigma|_{[v_p, \dots, v_n]}) \\
&= \varphi(f \circ \sigma|_{[v_0, \dots, v_p]})(f \circ \sigma|_{[v_p, \dots, v_n]}) \\
&= \varphi \frown f_{\#}\sigma.
\end{aligned}$$

This concludes the proof. □

### Exercise 22.7:

Define the cochain level cap product as

$$\varphi \frown \sigma = (-1)^{pq}\varphi(\sigma|_{[v_0, \dots, v_p]})\sigma|_{[v_{[v_p, \dots, v_{p+q}]}]}, \quad \varphi \in S^p(X; R), \sigma \in S_{p+q}(X; R).$$

Verify that  $\partial(\varphi \frown \sigma) = \delta\varphi \frown \sigma + (-1)^{|\varphi|}\varphi \frown \partial\sigma$ .

The cap product is actually the adjoint of the cup product. To see this, note that we have the *Kronecker pairing*

$$\langle \cdot, \cdot \rangle : H^k(X; R) \otimes H_k(X; R) \rightarrow R$$

induced by the map

$$\langle \cdot, \cdot \rangle : S^k(X; R) \otimes S_k(X; R) \rightarrow R$$

$$\varphi \otimes \sigma \mapsto \varphi(\sigma).$$

By definition,  $\delta\varphi(\sigma) = \varphi(\delta\sigma)$ , which means  $\langle \delta\varphi, \sigma \rangle = \langle \varphi, \delta\sigma \rangle$ , which justifies that the (co)chain level pairing gives rise to the Kronecker pairing (see also [Remark 17.4](#)). Then, from [Proposition 22.6](#), we have the following.

**Proposition 22.8:** (*Cap Product is Adjoint of Cup Product*)

For any  $\alpha \in H^p(X; R)$ ,  $\beta \in H^q(X; R)$ ,  $z \in H_{p+q}(X; R)$ , we have

$$\langle \alpha \smile \beta, z \rangle = \langle \beta, \alpha \frown z \rangle.$$

That is,  $\alpha \smile \_$  is adjoint to  $\alpha \frown \_$  with respect to the Kronecker pairing.

Let us also define the notion of *relative cap product*. Given a pair  $(X, A)$  with  $\iota : A \hookrightarrow X$  the inclusion, we have following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & S^k(X) \otimes S_n(A) & \xrightarrow{\text{Id} \otimes \iota_{\#}} & S^k(X) \otimes S_n(X) & \longrightarrow & S^k(X) \otimes S_n(X, A) \longrightarrow 0 \\
& & \downarrow \iota_{\#} \otimes \text{Id} & & \downarrow & & \downarrow \\
& & S^k(A) \otimes S_n(A) & & \smile & & \smile \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & S_{n-k}(A) & \xrightarrow{\iota_{\#}} & S_{n-k}(X) & \longrightarrow & S_{n-k}(X, A) \longrightarrow 0
\end{array}$$

Since we are tensoring a *split* exact sequence  $0 \rightarrow S_{\bullet}(A) \rightarrow S_{\bullet}(X) \rightarrow S_{\bullet}(X, A) \rightarrow 0$  by  $S^k(X)$ , it follows that both the rows are exact. The commutativity is a consequence of [Proposition 22.6](#). Hence, we have the induced relative cap product

$$\frown : S^k(X) \otimes S_n(X, A) \rightarrow S_{n-k}(X, A).$$

One can verify that these pass to cohomology, giving the relative cap product

$$\frown : H^k(X; R) \otimes H_n(X, A; R) \rightarrow H_{n-k}(X, A; R),$$

which makes  $H_{\star}(X, A; R)$  into a graded module over  $H^{\star}(X; R)$ .

On the other hand, we have the following situation. Given a ring  $R$ , we can define

$$\frown : S^k(X, A; R) \otimes S_n(X, A) \rightarrow S_{n-k}(X)$$

as follows. For relative  $k$ -cochain  $\varphi : S_k(X, A) \rightarrow R$  and an  $n$ -simplex  $\sigma : \Delta^n \rightarrow X$  not contained in  $A$ , we have

$$\varphi \frown \sigma = \varphi(\sigma|_{[v_0, \dots, v_k]} + S_k(A)) \cdot \sigma|_{[v_k, \dots, v_n]}.$$

One can check that this gives the relative cap product

$$\frown : H^k(X, A; R) \otimes H_n(X, A; R) \rightarrow H_{n-k}(X; R).$$

**Exercise 22.9:**

Verify that given a pair of maps  $f : (X, A) \rightarrow (Y, B)$ , we have  $f_*(f^*\alpha \frown z) = \alpha \frown f_*(z)$  for  $\alpha \in H^*(X, A), z \in H_*(X, A)$ . Also, check that  $\langle \alpha \smile \beta, z \rangle = \langle \alpha, \beta \frown z \rangle$  for  $\alpha \in H^p(X; R), \beta \in H^q(X; R), z \in H_{p+q}(X; R)$ .

More generally, for an excisive pair  $(A, B)$ , i.e, for  $S_\bullet(A) + S_\bullet(B) \hookrightarrow S_\bullet(A \cup B)$  is a chain homotopy equivalence, we have a relative cap product

$$\frown : H^k(X, A; R) \otimes H_n(X, A \cup B; R) \rightarrow H_{n-k}(X, B; R).$$

**22.3 Cohomology with Compact Support**

Given a space  $X$ , the collection of all compact subsets of  $X$  forms a directed system via inclusion. Given compact sets  $K \subset L \subset X$ , we have a map  $r_L^K : H^\bullet(X, X \setminus K) \rightarrow H^\bullet(X, X \setminus L)$  given by the inclusions. Passing to the colimit (or direct limit), we define the *cohomology with compact support*

$$H_c^\bullet(X) := \operatorname{colim}_{\substack{K \subset X \\ \text{compact}}} H^\bullet(X, X \setminus K).$$

If  $X$  is compact, it follows that  $H_c^\bullet(X) = H^\bullet(X)$ .

**Exercise 22.10:**

Denote  $S_c^n(X)$  to be the module generated by those  $n$ -cochains  $\varphi$  for which there exists a compact  $K \subset X$  such that  $\varphi$  vanishes for any singular simplex *not* completely contained in  $K$ . Verify that  $S_c^\bullet(X)$  is a cochain complex with the usual boundary map, and  $H^\bullet(S_c^\bullet(X)) = H_c^\bullet(X)$ .

Let us point out that in general, a map  $f : X \rightarrow Y$  does not induce a map in the cohomology with compact support. Indeed, for any compact set  $K \subset X$ , we have  $f(K) \subset Y$  is compact, but we don't necessarily get  $f(X \setminus K) \subset Y \setminus f(K)$ .

**Exercise 22.11: ( $H_c^\bullet$  is a Functor for Proper Maps)**

Recall, a map  $f : X \rightarrow Y$  is called *proper* if  $f^{-1}(L) \subset X$  is compact for any  $L \subset Y$  compact. Verify that given a proper map  $f : X \rightarrow Y$ , there exists a well-defined map  $H_c^\bullet(f) : H_c^\bullet(Y) \rightarrow H_c^\bullet(X)$ . In particular,  $H_c^\bullet$  is a contravariant functor when restricted to the class of proper maps.

**Hint :** Given a proper map, for any compact set  $L \subset Y$ , we have the composition

$$f^{*,L} : H^\bullet(Y, Y \setminus L) \rightarrow H^\bullet(X, X \setminus f^{-1}(L)) \rightarrow \operatorname{colim} H^\bullet(X, X \setminus K).$$

Verify that for  $L_1 \subset L_2 \subset Y$ , we have  $f^{*,L_2} \circ r_{L_2}^{L_1} = f^{*,L_1}$ , so that we can pass to colimit.

Let us note that for an *open* subset  $U \subset X$ , where  $X$  is  $T_2$ , we have a canonical map  $H_c^\bullet(U) \rightarrow H_c^\bullet(X)$ . Indeed, for any compact set  $K \subset U$ , note that  $(U, U \setminus K) \hookrightarrow (X, X \setminus K)$  induces an excision isomorphism  $H^\bullet(X, X \setminus K) \rightarrow H^\bullet(U, U \setminus K)$ . Naturality of excision isomorphism shows that the maps

$$H^\bullet(U, U \setminus K) \rightarrow H^\bullet(X, X \setminus K) \rightarrow H_c^\bullet(X)$$

are compatible with restrictions. Hence, passing to the colimit we get a map  $H_c^\bullet(U) \rightarrow H_c^\bullet(X)$ .

**Example 22.12:** (Compactly Supported Cohomology of  $\mathbb{R}^n$ )

Take  $U = \mathbb{R}^n \subset S^n$  as the complement of a point  $x \in S^n$ . For any  $K \subset U$  compact, there exists a contractible compact neighborhood  $x \in L \subset S^n$  such that  $K \cap L = \emptyset$ . Indeed, we can take  $L$  to be a small closed disc centered at  $x$  and disjoint from  $K$ . Since  $L$  is contractible, we have  $H^\bullet(S^n, S^n \setminus L) \rightarrow \tilde{H}^\bullet(S^n)$  is an isomorphism. Hence, passing to colimit, we get an isomorphism  $H_c^\bullet(U) = \tilde{H}^\bullet(S^n)$ . In particular,

$$H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & k = n \\ 0, & \text{otherwise.} \end{cases}$$

The above example shows that the compactly supported singular cohomology may not coincide with the usual one if the space is noncompact.

## 22.4 Poincaré Duality

Suppose  $M$  is an  $R$ -orientable  $n$ -fold. Fix a section  $\zeta \in \Gamma(M_R)$  representing the orientation. For any compact set  $K \subset M$ , we have the restriction  $\zeta|_K \in \Gamma(M_R|_K) = \Gamma_c(M_R|_K)$ . By [Theorem 22.1](#), we have the isomorphism  $\mathcal{J}^K : H_n(M, M \setminus K; R) \rightarrow \Gamma_c(M_R|_K)$ . Denote  $\zeta_K \in H^n(M, M \setminus K; R)$  as the element which maps to  $\zeta|_K$  under  $\mathcal{J}^K$ . Recall that  $\mathcal{J}^K(\zeta_K) = r_x^K(\zeta|_K) \in H_n(M, M \setminus \{x\}; R)$  for any  $x \in K$ . Using the *relative cap product*, we have a map

$$\begin{aligned} - \frown \zeta_K : H^q(M, M \setminus K; R) &\rightarrow H_{n-q}(M; R) \\ \varphi &\mapsto \varphi \frown \zeta_K. \end{aligned}$$

Note that here  $(M \setminus K, \emptyset)$  is obviously excisive. For  $K \subset L \subset M$  compact subsets, we have the inclusion  $\iota : (M, M \setminus L) \hookrightarrow (M, M \setminus K)$ . Note that  $r_L^K = \iota^*$ , and  $\zeta_K = \iota_* \zeta_L$ . Then, from [Exercise 22.9](#), we have

$$r_L^K \varphi \frown \zeta_L = \text{Id}_*(i^* \varphi \frown \zeta_L) = \varphi \frown i_* \zeta_L = \varphi \frown \zeta_K.$$

Thus, passing to colimit, we get a map

$$\mathcal{P} : H_c^q(M; R) \rightarrow H_{n-q}(M; R).$$

This map  $\mathcal{P}$  is known as the *Poincaré duality* map. In particular, if  $M$  is compact, we have

$$\mathcal{P} : H^q(M; R) \xrightarrow{\frown [M]_R} H_{n-q}(M; R),$$

where  $[M]_R$  is the fundamental class of  $M$ .

**Theorem 22.13:** (Poincaré Duality)

Suppose  $M$  is an  $R$ -orientable  $n$ -fold for some ring  $R$ , with a chosen orientation  $\zeta \in H_n(M; R)$ . Then, the Poincaré duality map  $\mathcal{P} : H_c^q(M; R) \rightarrow H_{n-q}(M; R)$  associated to  $\zeta$  is an isomorphism.

Let us point out an important consequence.

**Proposition 22.14:** (Non-degenerate Pairing)

Suppose  $M$  is an  $R$ -orientable compact, connected  $n$ -fold for some PID  $R$ . Assume that  $H_\bullet(M; R)$  is finitely generated. Then, there exists a non-degenerate pairing

$$\langle , \rangle : \frac{H^p(M; R)}{\text{Torsion}} \otimes \frac{H^{n-p}(M; R)}{\text{Torsion}} \rightarrow R$$

induced by the cup product. If  $R$  is a field, the pairing is  $H^p(M; R) \otimes H^{n-p}(M; R) \rightarrow R$

**Proof :** Fix an  $R$ -orientation  $\zeta \in H_n(M; R)$ . Then, using the Kronecker pairing, we have a pairing

$$\begin{aligned} \langle , \rangle : H^p(M; R) \otimes H^{n-q}(M; R) &\rightarrow R \\ \alpha \otimes \beta &\mapsto \langle \alpha \smile \beta, \zeta \rangle \end{aligned}$$

Say  $\alpha \in H^p(M; R)$  is a torsion element, i.e,  $r\alpha = 0$  for some  $0 \neq r \in R$ . Then, for any  $\beta \in H^{n-p}(M; R)$  we have

$$r(\alpha \smile \beta) = (r\alpha) \smile \beta = 0.$$

Since  $R$  is an integral domain, we must have  $\alpha \smile \beta = 0$ . This shows that the above pairing descends to a pairing modulo the torsion elements.

Let us now prove the nondegeneracy. Given any finitely generated  $R$ -module  $N$ , we have  $\text{Ext}_{R(N,R)}$  is torsion, i.e, every element is annihilated by some nonzero element of  $R$ . This follows from the structure theorem of finitely generated modules over a PID  $R$  and basic properties of  $\text{Ext}_R$ . From the UCT (Theorem 17.11), we have the split exact sequence

$$0 \rightarrow \text{Ext}(H_{k-1}(M; R), R) \rightarrow H^k(M; R) \rightarrow \text{hom}_R(H_k(M; R), R) \rightarrow 0,$$

where the  $\text{hom}$  is free. Thus,  $\text{Ext}(H_{k-1}(M; R), R)$  is precisely the torsion submodule of  $H^k(M; R)$ , and the free part is represented by the dual of  $H_k(M; R)$ . In particular, for any  $[\alpha] \in \frac{H^p(M; R)}{\text{Torsion}}$ , there is a unique  $a \in H_k(M; R)$  such that  $\alpha(a) = 1$ . By Poincaré duality, there exists some  $\beta \in H^{n-p}(M; R)$  such that  $\beta \smile \zeta = a$ . Now, we have

$$\langle \alpha, \beta \rangle = (-1)^{|\alpha||\beta|} \langle \beta \smile \alpha, \zeta \rangle = (-1)^{|\alpha||\beta|} \langle \alpha, \beta \smile \zeta \rangle = (-1)^{|\alpha||\beta|} \langle \alpha, a \rangle = (-1)^{|\alpha||\beta|} \neq 0.$$

Thus, the pairing is nondegenerate in the first variable. By the graded commutativity of the cup product, it then follows that the pairing is nondegenerate in both variables.

If  $R$  is a field, then the torsion submodule is 0, and so is the  $\text{Ext}(H_\bullet(M; R), R)$ . In particular,  $H^\bullet(M; R) = \text{hom}(H_\bullet(M; R), R)$  holds. Then, the nondegeneracy follows.  $\square$

The existence of this nondegenerate pairing leads to quick computations of some cohomology rings. As an example, let us give a proof of Theorem 21.16.

**Proposition 22.15: (Cohomology Rings of Projective Spaces)**

We have the cohomology rings

1.  $H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[X]/(X^{n+1})$  with  $|X| = 2$ .
2.  $H^*(\mathbb{H}P^n; \mathbb{Z}) = \mathbb{Z}[X]/(X^{n+1})$  with  $|X| = 4$ .
3.  $H^*(\mathbb{O}P^n; \mathbb{Z}) = \mathbb{Z}[X]/(X^{n+1})$  with  $|X| = 8$ , and  $n = 0, 1, 2$ .
4.  $H^*(\mathbb{R}P^n; \mathbb{Z}) = \mathbb{Z}_2[X]/(X^{n+1})$  with  $|X| = 1$ .

**Proof :** Using cellular (co)homology and UCT, we already have the isomorphism of graded modules in each of these cases.

Let us consider  $\mathbb{C}P^n$ . It is orientable; fix the fundamental class  $\zeta_n \in H_{2n}(\mathbb{C}P^n; \mathbb{Z})$ . Inductively, assume that  $H^*(\mathbb{C}P^{n-1}; \mathbb{Z}) = \mathbb{Z}[X]/(X^n)$  with  $|X| = 2$ . By cellular homology, we have  $\iota : \mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$  induces isomorphism in homology with degree  $< 2n$  (Example 12.5), Then, by UCT (Theorem 17.11), we have  $\iota^*$  is isomorphism in cohomology in degree  $< 2n$ .

Suppose  $\alpha \in H^2(\mathbb{C}P^n; \mathbb{Z})$  is a generator. Then  $\alpha' = \iota^* \alpha \in H^2(\mathbb{C}P^{n-1}; \mathbb{Z})$  is a generator. By induction,  $(\alpha')^{n-1}$  is a generator. On the other hand,  $(\alpha')^{n-1} = \iota^*(\alpha^n)$ . Since  $\iota^*$  is an isomorphism in degree  $< 2n$ , we must have  $\alpha^n$  is a generator of  $H^{2n-2}(\mathbb{C}P^n; \mathbb{Z})$ .

By Theorem 11.14, we have some  $\beta \in H^{2n-2}(\mathbb{C}P^n; \mathbb{Z})$  such that  $\langle \alpha, \beta \rangle = 1$ . We write  $\beta = m\alpha^n$  for some integer  $m$ . Then,

$$1 = \langle \alpha, \beta \rangle = \langle \alpha, m\alpha^n \rangle = m\langle \alpha, \alpha^n \rangle,$$

which forces  $m = \pm 1$ . Consequently, we have

$$1 = \pm \langle \alpha, \alpha^n \rangle = \pm \langle \alpha \smile \alpha^{n-1}, \zeta_n \rangle = \pm \alpha^n(\zeta_n).$$

Consequently,  $\alpha^n$  must be a generator of  $H^{2n}(\mathbb{C}P^n; \mathbb{Z})$ . This proves the claim for  $\mathbb{C}P^n$ .

Similar argument works for  $\mathbb{H}P^n$ , and for  $\mathbb{O}P^n$  (for  $n = 0, 1, 2$ ). For  $\mathbb{R}P^n$ , we work over the field  $\mathbb{Z}_2$ , where we again have the nondegenerate pairing since any manifold is  $\mathbb{Z}_2$ -orientable.  $\square$