Course notes for

Algebraic Topology II (KSM4E02)

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`Hopf invariant — relative cup product — cohomology ring of projective spaces — orientation`

21.1 Application of Cup Product : Hopf Invariant

Given a map $f : X \to Y$, recall the *mapping cone* is defined as

$$C_f = CX \cup_f Y,$$

where we attach the space Y to the base of the cone $CX = \frac{X \times [0,1]}{X \times \{0\}}$ by identifying $(x, 1) \sim f(x)$. It can be proved that if $f \simeq g : X \to Y$, then C_f is homotopy euquivalent to C_g.

Exercise 21.1:

If $h : f \simeq g : X \to Y$ is a homotopy, then show that

$$\begin{aligned} H : C_f &\to C_g \\ y &\mapsto y \\ [x,t] &\mapsto \begin{cases} h(x, 2t), & t \le \frac{1}{2} \\ h(x, 2t-1), & t \ge \frac{1}{2} \end{cases} \end{aligned}$$

is a homotopy equivalence, with homotopy inverse given by

$$\begin{aligned} G : C_g &\to C_f \\ y &\mapsto y \\ [x,t] &\mapsto \begin{cases} h(x, 1-2t), & t \le \frac{1}{2} \\ h(x, 2t-1), & t \ge \frac{1}{2}. \end{cases} \end{aligned}$$

Assume $n \ge 2$. Given a map $S^{2n-1} \to S^n$, consider the mapping cone C_f. Then, from cellular homology and UCT, it follows that

$$H^k(C_f) = H_k(C_f) = \begin{cases} \mathbb{Z}, & k = 0, n, 2n \\ 0, & \text{otherwise.} \end{cases}$$

Fix some generator $\alpha \in H^n(C_f), \gamma \in H^{2n}(C_f)$. Then, the cup product $\alpha \smile \alpha \in H^{2n}(C_f)$ is an integer multiple of γ. We write

$$\alpha \smile \alpha = h(f)\gamma.$$

The element $h(f) \in \mathbb{Z}$ is called the *Hopf invariant* of the map f, which is well-defined up to a sign (based on the choice of generators), and independent of the homotopy class of the map f. When n is odd, it is clear that $h(f) = 0$, since

$$\alpha \smile \alpha = (-1)^{n \cdot n} \alpha \smile \alpha = -\alpha \smile \alpha \Rightarrow 2\alpha \smile \alpha = 0 \Rightarrow \alpha \smile \alpha = 0.$$

Thus, let us consider maps $f : S^{4n-1} \to S^{2n}$ only.

Remark 21.2: (*Hopf Invariant One*)

One can explicitly produce maps $f : S^{4n-1} \to S^{2n}$ with $h(f) = 1$ for $n = 1, 2, 4$. Adams proved that these are the only possibilities for a map to have Hopf invariant 1 (up to a sign).

Let us now consider the space $S^{2n} \times S^{2n}$. It can be given a CW structure with one 0-cell, two $2n$-cells, and one $4n$-cell. We have the attaching map for the top $4n$-cell is given by $\rho : S^{4n-1} \to S^{2n} \vee S^{2n}$. Consider the composition

$$f : S^{4n-1} \xrightarrow{\rho} S^{2n} \vee S^{2n} \xrightarrow{\nabla} S^{2n},$$

where ∇ is the fold map.

Proposition 21.3:

The Hopf invariant is $h(f) = \pm 2$.

Proof : Consider the induced (quotient) map $q : S^{2n} \times S^{2n} = C_\rho \to X = C_f$. From cellular homology and Kunnenth formula, we have generators

$$u_1, u_2 \in H_1(S^{2n} \times S^{2n}), v \in H_2(S^{2n} \times S^{2n}),$$

and

$$\overline{u} \in H_1(S^{2n} \times S^{2n}), \overline{v} \in H_2(S^{2n} \times S^{2n}),$$

such that

$$q_\star(u_1) = \overline{u} = q_\star(u_2), \quad q_\star(v) = \overline{v}.$$

By UCT, it follows that the cohomology is dual to the homology, and by naturality $q^\star(\overline{u}^\star) = u_1^\star + u_2^\star, q^{\star(\overline{v}^\star)} = v^\star$. From Theorem 20.2, we get $u_1^\star \smile u_2^\star = \pm v^\star$, and $u_1^\star \smile u_1^\star = 0 = u_2^\star \smile u_2^\star$. Now,

$$\begin{aligned}
q^\star(\overline{u}^\star \smile \overline{u}^\star) &= q^\star(\overline{u}) \smile q^\star(\overline{u}) \\
&= (u_1^\star + u_2^\star) \smile (u_1^\star + u_2^\star) \\
&= \underbrace{u_1^\star \smile u_1^\star}_{0} + \underbrace{u_2^\star \smile u_2^\star}_{0} + u_1^\star \smile u_2^\star + u_2^\star \smile u_1^\star \\
&= u_1^\star \smile u_1^\star + (-1)^{2n \cdot 2n} u_1^\star \smile u_2^\star \\
&= 2u_1^\star \smile u_2^\star \\
&= \pm 2v^\star = \pm 2q^\star(\overline{v}^\star).
\end{aligned}$$

Hence, we have $h(f) = \pm 2$. □

As an immediate consequence, we have $\pi_{4n-1}(S^{2n}) \neq 0$. One can moreover prove that the Hopf invariant map

$$h : \pi_{4n-1}(S^{2n}) \to \mathbb{Z}$$
$$[f] \mapsto h(f)$$

is a group homomorphism, with some generators fixed. Consequently, we get that

$$\pi_{4n-1} S^{2n} = \mathbb{Z} \oplus \ker(h).$$

One can show that $\ker(h)$ is always a *finite* Abelian group, e.g, by Serre's technique. One can compute

$$\pi_3 S^2 = \mathbb{Z}, \quad \pi_7 S^4 = \mathbb{Z} \oplus \mathbb{Z}_{12}, \quad \pi_{15}(S^8) = \mathbb{Z} \oplus \mathbb{Z}_{120}.$$

21.2 Relative Cup Product

Suppose (X, A) is a pair. Recall, we have the short exact sequence

$$0 \to S_\bullet(A; R) \to S_\bullet(X; R) \to S_\bullet(X, A; R) \to 0$$

of chain complexes of singular simplices with coefficients in a ring R. Then, given an R-module M, the *relative cohomology* with coefficients in M is computed as

$$H^\bullet(X, A; M) := H^\bullet(\hom(S_\bullet(X, A), M))$$

A relative cochain $\varphi \in S^n(X, A; M)$ can be understood as a cochain on X which vanishes on all p-chains completely contained in A.

Now, suppose (X, A) and (X, B) are two pairs. Then, for $\varphi \in S^p(X, A)$ and $\psi \in S^q(X, B)$, consider the cochain $\varphi \smile \psi$. For any simplex $\sigma \in S_{p+q}(A) + S_{p+q}(B)$ (not a direct sum), i.e, a simplex that is contained in either A or B, we have

$$\varphi \smile \psi(\sigma) = \varphi\left(\sigma|_{[v_0, \ldots, v_p]}\right) \cdot \psi\left(\sigma|_{[v_p, \ldots, v_{p+q}]}\right) = 0,$$

since either the first term or the second term in the product vanishes! Thus, we have a cochain level cup product

$$\smile : S^p(X, A) \otimes S^q(X, B) \to \frac{S^{p+q}(X)}{S^{p+q}(A) + S^{p+q}(B)}.$$

Passing to cohomology, we get the cup product

$$\smile : H^p(X, A) \otimes H^q(X, B) \to H^{p+q}\left(\frac{S^\bullet(X)}{S^\bullet(A) + S^\bullet(B)}\right).$$

Exercise 21.4:

Verify that given a pair (X, A) we have relative cup products

$$\smile : H^p(X) \otimes H^q(X, A) \to H^{p+q}(X, A)$$
$$\smile : H^p(X, A) \otimes H^q(X) \to H^{p+q}(X, A)$$
$$\smile : H^p(X, A) \otimes H^q(X, A) \to H^{p+q}(X, A)$$

These makes $H^\star(X, A)$ as a graded algebra over the graded ring $H^\star(X)$.

Now, call the pair of subspaces (A, B) *excisive* if the inclusion map

$$S_\bullet(A) + S_\bullet(B) \hookrightarrow S_\bullet(A \cup B)$$

is a chain homotopy equivalence.

Example 21.5: (*Excisive Pairs*)

Here are a few useful examples of excisive pairs.

1. Suppose $A, B \subset X$ are open. Then, $\mathcal{U} = \{A, B\}$ is an open cover of $A \cup B$. By Lemma 9.4, we then have

$$S_\bullet^{\mathcal{U}}(X) = S_\bullet(A) + S_\bullet(B) \hookrightarrow S_\bullet(A \cup B)$$

   is a chain homotopy equivalence.

2. If X is a CW complex, and $A, B \subset X$ are subcomplexes we have $S_\bullet(A) + S_\bullet(B) \hookrightarrow S_\bullet(A \cup B)$ is a chain homotopy equivalence.

   To see this, note that A and B are deformation retracts of open sets in $A \cup B$. Then, by excision, we have that this map induces isomorphism in homology (i.e, a quasi-isomorphism). But then it must be a chain homotopy equivalence (fact : a quasi-equivalence between bounded below chain complex of free R-modules is a chain homotopy equivalence). One can also use the chain homotopy equivalences $S_\bullet(A) \to S_\bullet(\mathcal{O}(A)), S_\bullet(B) \to S_\bullet(\mathcal{O}(B))$, where $\mathcal{O}(A), \mathcal{O}(B) \underset{\text{open}}{\subset} A \cup B$ deforms onto A, B resepectively, and then use Lemma 9.4.

Given excisive pairs $A, B \subset X$, we have the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_\bullet(A) + S_\bullet(B) & \longrightarrow & S_\bullet(X) & \longrightarrow & \frac{S_\bullet(X)}{S_\bullet(A)+S_\bullet(B)} & \longrightarrow & 0 \\
 & & \downarrow \simeq & & \| & & \downarrow \simeq & & \\
0 & \longrightarrow & S_\bullet(A \cup B) & \longrightarrow & S_\bullet(X) & \longrightarrow & S_\bullet(X, A \cup B) & \longrightarrow & 0
\end{array}$$

Dualizing the induced chain homotopy equivalence, and then passing to the cohomology, we get natural isomorphism

$$H^\bullet(X, A \cup B) \xrightarrow{\cong} H^\bullet\left(\hom\left(\frac{S(X)}{S(A) + S(B)}, R\right)\right).$$

Definition 21.6: (*Relative Cup Product*)

Given a space X and two excisive subspaces $A, B \subset X$ (i.e, $S_\bullet(A) + S_\bullet(B) \hookrightarrow S_\bullet(A \cup B)$ is a chain homotopy equivalence), the *relative cup product* is defined as the composition

$$\smile : H^p(X, A) \otimes H^q(X, B) \to H^{p+q}\left(\frac{S^\bullet(X)}{S^\bullet(A) + S^\bullet(B)}\right) \xleftarrow{\cong} H^\bullet(X, A \cup B).$$

Exercise 21.7:

Suppose $A, B, C \subset X$ are open subsets. Verify the following.

1. The relative cup product is associative, i.e, the diagram commutes

$$\begin{array}{ccc} H^p(X,A)\otimes H^q(X,B)\otimes H^r(X,C) & \xrightarrow{\smile\,\otimes\,\mathrm{Id}} & H^{p+q}(X,A\cup B)\otimes H^r(X,C) \\ \Big\downarrow{\scriptstyle \mathrm{Id}\,\otimes\,\smile} & & \Big\downarrow{\scriptstyle \smile} \\ H^p(X,A)\otimes H^{q+r}(X,B\cup C) & \xrightarrow[\smile]{} & H^{p+q+r}(X,A\cup B\cup C) \end{array}$$

2. Relative cup product is graded commutative, i.e,

$$a\smile b=(-1)^{pq}b\smile a,\quad a\in H^p(X,A), b\in H^q(X,B).$$

3. There are unit elements $1_A\in H^0(X,A), 1_B\in H^0(X,B), 1_{A\cup B}\in H^0(X,A\cup B)$, and

$$1_A\smile 1_B=1_{A\cup B}.$$

Let us describe an interesting application of relative cup product.

Definition 21.8: (*Cup Length*)

Given a space X, the *cup-length* is defined as the largest integer n for which that there are n-many cohomology classes $a_i\in H^{p_i}(X)$, with $p_i\geq 1$ such that $a_1\smile\cdots\smile a_n\neq 0$.

Sated another way, if the cup-length of a space is n, then any k-fold cup product vanishes for $k<n$. Of course, $1\smile\ldots\smile 1=1\neq 0$, and thus we only consider elements in degree ≥ 1.

Example 21.9:

For the sphere S^n, the cup-length is 1, and for the torus $\mathbb{T}^2$ it is 2.

Next, let us define another related numerical invariant.

Definition 21.10: (*LS Category*)

The *LS category* (i.e, ***L**usternik–**S**chnirelmann category*) of a space X is the least integer n such that there are n-many open sets $U_i\subset X$ that covers X, i.e, $X=\cup_{i=1}^n U_i$, and U_i is contractible in X, i.e, $U_i\hookrightarrow X$ is null-homotopic.

Example 21.11:

One can see that S^n can be covered by the open sets $U=S^n\setminus\{N\}$ and $V=S^n\setminus\{S\}$, the complement of north and south poles. Clearly, $U,V\hookrightarrow S^n$ are null-homotopic. Thus, LS category of S^n is at most 2. Since S^n is not contractible, we can justify that the LS category is exact 2.

For the torus $\mathbb{T}^2$, from the standard pasting diagram, we can see that it can be covered by 3 open sets which are also contractible in $\mathbb{T}^2$. Thus, the LS category of $\mathbb{T}^2$ is at most 3. It is clearly not 1, since $\mathbb{T}^2$ is not contractible. Can we rule out that $\mathrm{LS\text{-}Cat}(\mathbb{T}^2)\neq 2$?!

We have the following proposition, that lets us get a lower bound on the LS category of a space.

Proposition 21.12: (*Cup Length and LS Category*)

Suppose $X = \cup_{i=1}^{n} U_i$, where $U_i \subset X$ is open and $U_i \hookrightarrow X$ is null-homotopic. Then, any n-fold cup product $a_1 \smile \dots \smile a_n = 0$, where $a_i \in H^{p_i}(X)$ with $p_i \geq 1$. In particular,

$$\text{LS-Cat }(X) \geq \text{Cup-Length }(X) + 1.$$

Proof : Since $\iota_j : U_j \hookrightarrow X$ is null-homotopic, from the long exact sequence of the pair (X, U_j)

$$\cdots H^{n-1}(U_j) \to H^n(X, U_j) \twoheadrightarrow H^n(X) \xrightarrow[0]{i_j^\star} H^n(U_j),$$

we have surjections $\eta_j^\star : H^n(X, U_j) \to H^n(X)$, where $\eta_j : X \hookrightarrow (X, U_j)$ are the inclusions. Let $a_i \in H^{p_i}(X)$ be cohomology classes with $p_i \geq 1$, and $\tilde{a}_i \in H^{p_i}(X, U_j)$ be the corresponding lifts, i.e, we have $\eta_j^\star(\tilde{a}_j) = a_j$. Since the relative cup product is associative, we have

$$\tilde{a}_1 \smile \dots \smile \tilde{a}_n \in H^{\sum p_i}(X, \cup_{i=1}^{n} U_i) = H^{\sum p_i}(X, X) = 0.$$

But then from the naturality of the cup product, we have

$$a_1 \smile \dots \smile a_k = \eta_1^\star(\tilde{a}_1) \smile \dots \smile \eta_n^\star(\tilde{a}_n) = \eta^\star(\tilde{a}_1 \smile \dots \smile \tilde{a}_n) = \eta^\star(0) = 0,$$

where $\eta : X \hookrightarrow (X, U_1 \cup \dots \cup U_n)$ is the inclusion. Thus, $\text{Cup-Length }(X) \lneq n$.

Now, suppose X has finite LS category (if it is inifinite, there is nothing to prove). Say, $\text{LS-Cat}(X) = n$. Hence, there are n-many open sets $U_j \subset X$ that covers X, such that $U_j \hookrightarrow X$ is null-homotopic. Then, we have

$$\text{Cup-Length}(X) + 1 \leq n = \text{LS-Cat }(X),$$

as required. □

As a corollary, we can now claim that $\text{LS-Cat}(\mathbb{T}^2) = 3$.

Exercise 21.13: (*LS Category of Higher Torus*)

Show that the LS category of the higher torus $\mathbb{T}^n = \underbrace{S^1 \times \dots \times S^1}_{n\text{-times}}$ is $\geq n + 1$. In fact, $\text{LS-Cat}(M) \leq \dim M + 1$, where M is a second countable, connected manifold, and thus, $\text{LS-Cat}(\mathbb{T}^n) = n + 1$.

Remark 21.14:

Note that in the literature, LS category is sometimes difined as follows : $\text{cat}(X)$ is the least integer n such that there are $(n+1)$-many open sets $U_i \subset X$ covering X such that $U_i \hookrightarrow X$ is null-homotopic. Thus, $\text{cat}(X) + 1 = \text{LS-Cat}(X)$. This defintion has the advantage that one can write

$$\text{cat}(X) \geq \text{Cup-Length}(X).$$

Moreover, one can prove $\text{cat}(X \times Y) \leq \text{cat }(X) \times \text{cat}(Y)$ holds. This led to the famous *Ganea conjecture* which states that $\text{cat}(X \times S^n) = \text{cat}(X) + 1$. Note $\text{cat}(S^n) = 1$. The conjecture was disproved by Norio Iwase.

21.3 Cohomology Ring of Projective Spaces

Let us denote $\mathbb{P}^n(d)$ to be the projective space of $\mathbb{K}^{n+1}$, where $\mathbb{K}$ is the vector space of dimension d over $\mathbb{R}$. We can only consider $d = 1, 2, 4$, which corresponds to

$$\mathbb{P}^n(1) = \mathbb{RP}^n, \quad \mathbb{P}^n(2) = \mathbb{CP}^n, \quad \mathbb{P}^n(4) = \mathbb{HP}^n.$$

Indeed, the definition of projectivization requires an *associative* algebra structure on $\mathbb{K}$. There is also the projective space $\mathbb{P}^n(4) = \mathbb{OP}^1$ corresponding to the octonionic numbers for $n = 0, 1, 2$, but they do not generalize to higher projective spaces.

Remark 21.15:

Stasheff defined a notion of A_n-space and proved that an A_n-space X admits a projective space $XP(i)$ of higher order associated to it for $0 \leq i \leq n$. They generalize the notion of usual projective spaces.

The cotonions $\mathbb{O}$ has a non-associative multiplication, making it into an H-space, or an A_2-space which is in fact not A_3. Thus, we can only have $\mathbb{OP}^n$ for $n = 0, 1, 2$.

The A_∞-spaces are the loop spaces (more precisely, they are weak homotopy equivalent to a loop space).

When the base field is $\mathbb{R}$, fix the ring $R = \mathbb{Z}/2\mathbb{Z}$, and otherwise fix $R = \mathbb{Z}$. Then, we can compute via UCT that

$$H^k(\mathbb{P}^n(d); R) = \begin{cases} R, & k = 0, d, ..., nd \\ 0, & \text{otherwise}. \end{cases}$$

As a graded module, we can write

$$H^\star(\mathbb{P}^n(d); R) = \frac{R[X]}{\langle X^{n+1} \rangle}, \quad |X| = d.$$

The right hand side is the truncated polynomial algebra.

Theorem 21.16: (*Cohomology Ring of Projective Space*)

The cohomology ring of the projective space $\mathbb{P}^n(d)$ with coefficient ring R is the truncated polynomial ring $\frac{R[X]}{\langle X^{n+1} \rangle}$ with $|X| = d$. Moreover, passing to the limit, we have

$$H^\star(\mathbb{P}^\infty(d)) = R[X], \quad |X| = d.$$

21.4 Orientation of a Manifold and Homology

Let us now focus on a *manifolds.*

Definition 21.17: (*Topological Manifold*)

A *manifold* is a space M which is *locally Euclidean,* i.e, given any $x \in X$ there exists an open neighborhood $x \in U \subset M$ such that U is homeomorphic to some open subset of the Euclidean space $\mathbb{R}^n$ for some n. The tuple (U, φ), where $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is the homeomorphism, is called a *coordinate chart*. Usually, we assume that a manifold is *second countable,* i.e, it admits a basis of countable open sets, and it is *Hausdorff.*

One can show (by invariance of domain) that for each connected component (which are same as path components), the integer n appearing in the definition remains constant. Thus, assuming M is connected, we can define $\dim M$ to be this integer, we say M is an n-fold (or n-dimensional manifold).

Example 21.18:

The real line is a manifold which is T_2 and second countable. The *real line with two origins* is a manifold which is second countable, but not T_2. The *long line* (or, *Sorgenfrey line*) is a manifold which is T_2 but not second countable. The sphere S^n, the projective spaces $\mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n$, their products etc are examples of compact manifolds, which are T_2 and second countable.

Remark 21.19: (*Manifold With Boundary*)

A space M is called a *manifold with boundary* if each $x \in M$ admits an open neighborhood which is homeomorphic to an open subset of the closed upper half plane

$$\mathbb{R}^n_{\geq 0} := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$

for some n. By invariance of domain, one can define $x \in M$ to be an *interior point* if x has a neighborhood homeoromphic to an open set in the open upper half plane

$$\mathbb{R}^n_{>0} := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

A point $x \in M$ which is not an interior point is called a *boundary point*. The collection of boundary point is defined to be ∂M. Every manifold M can be considered a manifold with boundary $\partial M = \emptyset$, but not the other way around.

Exercise 21.20:

Suppose M is an n-fold. Let $x \in M$, and $x \in V \subset M$ any open neighborhood. Show that there exists a open neighborhood $x \in U \subset V$, and a homeomorphism $\varphi : U \to \mathbb{R}^n$ such that $\varphi(x) = 0$.

Fix an n-fold M. Given subsets $K \subset L \subset M$, we have the restriction maps

$$r^L_K : H_\bullet(M, M \setminus L) \to H_\bullet(M, M \setminus K)$$

induced by the inclusions. When $K = \{x\}$ a singleton, we denote r^L_x as the restriction.

Lemma 21.21:

Given any open neighborhood $x \in W \subset M$, there exists a neighborhood $x \in U \subset W$ such that $r^U_y : H_\bullet(M, M \setminus U) \to H_\bullet(M, M \setminus y)$ is an isomorphism for all $y \in U$.

Proof : Get a chart (V, φ) such that $x \in V \subset W$ and $\varphi : V \to \mathbb{R}^n$ is a homeomorphism with $\varphi(x) = 0$ (Exercise 21.20). Set $U = \varphi^{\{-1\}} B^n$, where $B^n = \{v \in \mathbb{R}^n \mid \|v\| < 1\}$ is the open unit ball. For any $y \in U$, consider the diagram

$$\begin{array}{ccccc} H_\bullet(M, M \setminus U) & \longleftarrow & H_\bullet(V, V \setminus U) & \xrightarrow[\cong]{\varphi_\star} & H_\bullet(\mathbb{R}^n, \mathbb{R}^n \setminus B^n) \\ \downarrow r_y^U & & \downarrow & & \downarrow \\ H_\bullet(M, M \setminus \{y\}) & \longleftarrow & H_\bullet(V, V \setminus \{y\}) & \xrightarrow[\varphi_\star]{\cong} & H_\bullet(\mathbb{R}^n, \mathbb{R}^n \setminus \{\varphi(y)\}) \end{array}$$

The blue arrows are isomorphism as we are excising out $M \setminus V$. The green arrow is induced by a homotopy equivalence, and hence isomorphism. The commutativity then proves that r_y^U is an isomorphism for all $y \in U$. □

Next, fix a ring R, and consider the following set

$$M_R := \bigsqcup_{x \in M} H_n(M, M \setminus \{x\}; R),$$

along with the projection map $\pi : M_R \to M$. Observe that

$$H_n(M, M \setminus \{x\};) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong H_{n-1}(\mathbb{R}^n \setminus \{0\}) \cong R.$$

We assume $H_n(M, M \setminus \{x\}; R)$ has the discrete topology. We topologize M_R in the following way. For each $x \in M$, using Lemma 21.21, get a chart (U, φ) around x such that r_y^U is an isomorphism for each $y \in U$. Consider the map

$$\begin{aligned} \Phi_{x,U} : U \times H_n(M, M \setminus \{x\}; R) &\to \pi^{-1}(U) \\ (y, a) &\mapsto r_y^U (r_x^U)^{-1}(a). \end{aligned}$$

It is easy to see that $\Phi_{x,U}$ is a fiber-preserving bijection. Give M_R the coarsest topology that makes the collection of maps $\Phi_{x,U}$ continuous. Then, it follows that M_R is a covering space of M, and $\pi : M_R \to M$ is a covering map. This covering space is called the *orientation cover* of M with coefficients in R.

Exercise 21.22:

Check that $\Phi_{x,U}$ actually defines continuous *transition maps.* That is, show that the restricted map

$$\Phi_{y,V}^{-1} \circ \Phi_{x,U} : (U \cap V) \times H_n(M, M \setminus \{x\}; R) \to (U \cap V) \times H_n(M, M \setminus \{y\}; R)$$

is continuous.

Observe that for any U as in Lemma 21.21 and for $z \in H_n(M, M \setminus U)$, the map $y \mapsto r_y^U(z)$ is a local section of M_R. Let us denote the collection of all sections of M_R over some $A \subset M$ by $\Gamma(A; R)$. Since each fiber of M_R is an Abelian group, we can define a group structure in $\Gamma(A; R)$. Moreover, we can define the subgroup $\Gamma_c(A; R) \subset \Gamma(A; R)$ of sections with compact support, i.e, sections that are 0 outside a compact subset of A.

Remark 21.23:

One can more generally define $M_{R,k} := \bigsqcup_{x \in M} H_k(M, M \setminus \{x\}; R)$, and topologize it similarly. It follows that each $M_{G,k}$ is a covering space (with 0 as the fiber for $k \neq n$), and $M_{G,n} = M_G$ for an n-fold M.

Definition 21.24: (R*-Orientation*)

Given a ring R, and a subset $A \subset M$, an *R-orientation of M along A* is a section $s \in \Gamma(A; R)$ such that $s(a) \in H_n(M, M \setminus \{a\}; R) \cong R$ is a generator of R. For $A = M$, if such a section exists, we simply say it is an *R-orientation* of M, and M is then said to be *R-orientable*.

Exercise 21.25: ($\mathbb{Z}_2$*-Orientation*)

Justify that any manifold is $\mathbb{Z}_2$-orientable.

Exercise 21.26:

Justify that if M is R-orientable along some $A \subset M$, then it is R-orientable along any $B \subset A$ as well.

When $R = \mathbb{Z}$, we say an R-orientation in $\Gamma(M; \mathbb{Z})$ is an *orientation* of R. If such an orientation exists, we say M is *orientable*.

Remark 21.27: (*Orientation Sheaf*)

Technically, we are constructing the *sheaf* M_R whose stalk over each point $x \in M$ is the group (or ring) $H_n(M, M \setminus \{x\})$. This is also called the *orientaion sheaf* of M with coefficient R. An R-orientation is thus a global section of the sheaf M_R such that it takes values in generators.

Definition 21.28: (*Orientation Double Cover*)

Given a manifold M, the *orientation double cover* is defined as the subset $\widetilde{M} \subset M_{\mathbb{Z}}$, which consists of all the generators in all the fibers. As each fiber is isomorphic to $\mathbb{Z}$, which admits two generators, namely $\{\pm 1\}$, it follows that $\pi : \widetilde{M} \to M$ is a 2-sheeted cover of M.

We have the following useful result.

Proposition 21.29: (*Orientability and Orientation Double Cover*)

Given a manifold M, the following are equivalent.

1. M is orientable.
2. M is orientable along any compact set $K \subset M$.
3. The orientation covering $\widetilde{M}$ is trivial.
4. The covering $M_{\mathbb{Z}} \to M$ is trivial

Proof : $1 \Rightarrow 2$: If M is orientable, then it is orientable along any subset, and in particular, any compact subset.

$2 \Rightarrow 3$: Suppose M is orientable along any compact set. Since triviality can be verified over each connected component, we may assume M is connected. Since $\widetilde{M}$ is a double cover of M, it is trivial if and only if it is *not* connected. So, we have a path $\gamma : [0, 1] \to \widetilde{M}$ joining two distinct points in the fiber over some point $x \in M$. Composing with π, we get a compact, connected set $K = \pi(\gamma([0,1])) \subset M$. It follows that $\widetilde{M}|_K$ is nontrivial, as γ is a path joining two points in a fiber. But then M is not orientable along K, a contradiction.

$3 \Rightarrow 4$: Suppose $\sigma : M \to \widetilde{M}$ is a section. Then, we have

$$\begin{aligned} \rho : M \times \mathbb{Z} &\to M_{\mathbb{Z}} \\ (x,k) &\mapsto k\rho(x) \end{aligned}$$

is a trivialization of $\mathbb{Z}$. Indeed, it follows that $M_{\mathbb{Z}} = \widetilde{M} \times_{\mathbb{Z}_2} \mathbb{Z}$ is the Borel construction.

$4 \Rightarrow 1$: If $M_{\mathbb{Z}}$ is trivial, then we can choose a section with values in $\widetilde{M}$, i.e, we can choose an orientaion on M. □

Remark 21.30: (*Orientation Double Cover and Associated Bundle*)

If R is an arbitrary ring, we have the action of $\mathbb{Z}_2$ on R given by multiplication by -1. This leads to the following map

$$\begin{aligned} \widetilde{M} \times R &\to M_{\mathbb{Z}} \otimes R \\ (z,r) &\mapsto z \otimes r. \end{aligned}$$

Applying the UCT (Theorem 17.10) fiberwise, we get $M_{\mathbb{Z}} \otimes R \cong M_R$ (as the Tor vanishes). Hence, we have the induced isomorphism $\widetilde{M} \times_{\mathbb{Z}_2} R \to M_R$. Moreover, sections of $\Gamma(M;R)$ corresponds to $\mathbb{Z}_2$-equivariant maps $\lambda : \widetilde{M} \to R$.

Remark 21.31: (*Stiefel-Whitney Class*)

Since $\widetilde{M}$ is a 2-sheeted cover of M, we have a group homomorphism

$$\pi_1(M,x) \to \widetilde{M}_x = \mathbb{Z}_2.$$

As $\mathbb{Z}_2$ is Abelian, this map induces

$$\pi_1(M,x)^{\mathrm{ab}} \to \mathbb{Z}_2.$$

By the Hurewicz theorem, we then have a map

$$\omega : H_1(M;\mathbb{Z}) = \pi_1(M,x)^{\mathrm{ab}} \to \mathbb{Z}_2.$$

As $\mathbb{Z}_2$ is a field, by the UCT (Theorem 17.11), it follows that $H^1(M;\mathbb{Z}_2) = \hom(H_1(M;\mathbb{Z}),\mathbb{Z}_2)$. Thus, we have a cohomology class $\omega \in H^1(M;\mathbb{Z}_2)$. This class is called the *first Stiefel-Whitney class*, usually denoted as $\omega_1(M)$. From the definition, it follows that M is orientable if and only if $\omega_1(M) = 0$. For disconnected M, we work on each components separately.