

Algebraic Topology II (KSM4E02)

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cohomology ring of product – cohomology ring of wedge – cup product computation – sphere – torus – $\mathbb{R}P^2$ – klein bottle

20.1 Cohomology Ring of Product of Spaces

Recall, an *algebra* over a ring R is an R -module A with a map $\mu : A \otimes_R A \rightarrow A$. The algebra is *associative* if the diagram commutes

$$\begin{array}{ccc}
 A \otimes A \otimes A & \xrightarrow{\mu \otimes \text{Id}_A} & A \otimes A \\
 \text{Id}_A \otimes \mu \downarrow & \circlearrowleft & \downarrow \mu \\
 A \otimes A & \xrightarrow{\mu} & A
 \end{array}$$

The algebra is *unital* if there is an R -module map $\eta : R \rightarrow A$ such that the diagram commutes

$$\begin{array}{ccccc}
 R \otimes A & \xrightarrow{\eta \otimes \text{Id}_A} & A \otimes A & \xleftarrow{\text{Id}_A \otimes \eta} & A \otimes R \\
 & \searrow \cong & \downarrow \mu & \swarrow \cong & \\
 & & A & &
 \end{array}$$

An R -linear map $f : (A, \mu_A) \rightarrow (B, \mu_B)$ is an *algebra map* if the diagram commutes

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\
 \mu_A \downarrow & \circlearrowleft & \downarrow \mu_B \\
 A & \xrightarrow{f} & B
 \end{array}$$

If both A, B are unital, we moreover require that an algebra map preserves the unit, i.e, the diagram commutes

$$\begin{array}{ccc}
 & & A \\
 & \nearrow \eta_B & \downarrow f \\
 R & & B \\
 & \searrow \eta_A &
 \end{array}$$

When A is graded, we assume that the multiplication map $\mu : A \otimes A \rightarrow A$ and the unit map $\eta : R \rightarrow A$ has degree 0, and consequently, any algebra map of graded algebra must be a degree 0 map as well. The **tensor product** of two graded algebras is the graded module $A \otimes B$, equipped with the multiplication given by the composition

$$(A \otimes B) \otimes (A \otimes B) \xrightarrow{\text{Id}_A \otimes \tau \otimes \text{Id}_B} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B,$$

where $\tau : A \otimes B \rightarrow B \otimes A$ is the flip map (also known as the *braiding*). Explicitly,

$$\mu(a_1 \otimes b_1, a_2 \otimes b_2) = (-1)^{|b_1| \cdot |a_2|} \mu_A(a_1, a_2) \otimes \mu_B(b_1, b_2),$$

for homogeneous elements $a_1, a_2 \in A, b_1, b_2 \in B$. The flip map induces a sign according to the Koszul sign rule.

We have seen that given a space X and a ring R , the cohomology ring $H^*(X; R)$ is a graded commutative, unital, associative algebra over R .

Proposition 20.1: (Cross Product is an Algebra Map)

The cohomological cross product

$$\times : H^*(X; R) \otimes H^*(Y; R) \rightarrow H^*(X \times Y; R)$$

is a map of algebras over R .

Proof : Let $a_i \in H^{p_i}(X; R), b_i \in H^{q_i}(Y; R)$ be given for $i = 1, 2$. We have

$$\begin{aligned} (a_1 \times b_1) \smile (a_2 \times b_2) &= (\pi_X^* a_1 \smile \pi_Y^* b_1) \smile (\pi_X^* a_2 \smile \pi_Y^* b_2) \\ &= (-1)^{|\pi_Y^* b_1| |\pi_X^* a_2|} \pi_X^* a_1 \smile \pi_X^* a_2 \smile \pi_Y^* b_1 \smile \pi_Y^* b_2 \\ &= (-1)^{|b_1| |a_2|} \pi_X^* (a_1 \smile a_2) \smile \pi_Y^* (b_1 \smile b_2) \\ &= (-1)^{|b_1| |a_2|} (a_1 \smile a_2) \times (b_1 \smile b_2). \end{aligned}$$

Note that the cross product \times , and the cup product \smile has degree 0, and thus, there is no extra signs appearing while evaluating them! In other words, we have the commutativity of the diagram

$$\begin{array}{ccc} (H^*(X; R) \otimes H^*(Y; R)) \otimes (H^*(X; R) \otimes H^*(Y; R)) & \xrightarrow{\times \otimes \times} & H^*(X \times Y; R) \otimes H^*(X \times Y; R) \\ \text{Id} \otimes \tau \otimes \text{Id} \downarrow & & \downarrow \smile \\ H^*(X; R) \otimes H^*(X; R) \otimes H^*(Y; R) \otimes H^*(Y; R) & & \\ \smile \otimes \smile \downarrow & & \downarrow \\ H^*(X; R) \otimes H^*(Y; R) & \xrightarrow{\times} & H^*(X \times Y; R) \end{array}$$

Thus, we have that \times is a map of algebras. □

As a consequence, we get the following important result.

Theorem 20.2: (Künneth Algebra Isomorphism)

Let R be a PID, X, Y be spaces. Suppose that

- either $H_\bullet(X; R)$ or $H_\bullet(Y; R)$ are of finite type, and
- $\text{Tor}(H_i(X; R), H_j(X; R)) = 0$ for all i, j .

Then, the cohomological cross product $\times : H^\bullet(X; R) \otimes H^\bullet(Y; R) \rightarrow H^\bullet(X \times Y; R)$ is an algebra isomorphism of cohomology rings.

Proof : From the hypothesis, it follows that cross product [Theorem 18.12](#) is an isomorphism of R -modules, and by [Proposition 20.1](#) it is a map of R -algebras. Hence, the cross product is an algebra isomorphism. \square

20.2 Cohomology Ring of Wedge Sum of Spaces

Suppose (X, x_0) is a based space. Then, the inclusion map $\iota : \{\star\} \hookrightarrow (X, x_0)$ of the basepoint induces a map $\iota^\star : H^\bullet(X) \rightarrow H^\bullet(\star)$. The *reduced cohomology* is defined as $\tilde{H}^\bullet(X) = \ker(\iota^\star : H^\bullet(X) \rightarrow H^\bullet(\star))$. We immediately have

$$H^k(X) = \begin{cases} \tilde{H}^k(X) \oplus R, & k = 0 \\ \tilde{H}^k(X), & \text{otherwise.} \end{cases}$$

If $A \hookrightarrow X$ is a cofibration (e.g, if (X, A) is a CW pair), just like [Theorem 10.14](#), we have that the quotient map $q : X \rightarrow X/A$ induces an isomorphism $\tilde{H}^\bullet(X/A) \rightarrow H^\bullet(X, A)$.

Proposition 20.3: (Cohomology of Wedge Sum)

Suppose $\{(X_i, x_i)\}_{i \in I}$ be a given collection of based spaces, such that the inclusion $\{x_i\} \hookrightarrow X_i$ is a cofibration. Then, for any ring R -module M , there exists an isomorphism

$$\tilde{H}^\bullet\left(\bigvee_{i \in I} X_i; M\right) = \prod_{i \in I} \tilde{H}^\bullet(X_i; M),$$

induced componentwise by the maps $\iota_j : X_j \hookrightarrow \bigvee_{j \in I} X_j$

Proof : It is easy to see that the inclusion maps $\iota_j : X_j \hookrightarrow \bigsqcup_{i \in I} X_i$ induces chain level isomorphism

$$\sum S_\bullet(\iota_j) : \bigoplus_{j \in I} S_\bullet(X_j; R) \rightarrow S_\bullet\left(\bigsqcup_{j \in I} X_j; R\right).$$

Dualizing, we get the isomorphism of cochains

$$(S^\bullet(\iota_j))_{j \in I} : S^\bullet\left(\bigsqcup_{j \in I} X_j; M\right) \rightarrow \prod_{j \in I} S^\bullet(X_j; M),$$

since $\text{hom}(_, M)$ takes coproducts to products. Thus, we have isomorphism in cohomology

$$H^\bullet\left(\bigsqcup_{j \in I} X_j; M\right) \cong \prod_{j \in I} H^\bullet(X_j; M).$$

The same isomorphisms work for *relative* cohomology as well (Check!). Then, we have

$$\tilde{H}^\bullet\left(\bigvee_{i \in I} X_i; M\right) = H^\bullet\left(\bigsqcup_{i \in I} X_i, \bigsqcup_{i \in I} \{x_i\}; M\right) = \prod_{i \in I} H^\bullet(X_i, x_i; M) = \prod_{i \in I} \tilde{H}^\bullet(X_i; M).$$

One can see that componentwise the map is induced by $X_j \rightarrow (X_j, x_j) \rightarrow (\bigsqcup_{i \in I} X_i, \bigsqcup_{i \in I} x_i) \rightarrow \bigvee_{i \in I} X_i$, which is precisely $\iota_j : X_j \hookrightarrow \bigvee_{i \in I} X_i$. This proves the claim. \square

Our next goal is to understand the cohomology ring structure of a wedge sum.

Theorem 20.4: (Cohomology Ring of Wedge)

Let $(X, x_0), (Y, y_0)$ are based spaces, with the inclusions of the basepoints $\iota_X : \{\star\} \hookrightarrow X, \iota_Y : \{\star\} \hookrightarrow Y$ being cofibrations. Then, there exists an isomorphism of algebras

$$H^\star(X \vee Y) = \{(a, b) \in H^\star(X) \times H^\star(Y) \mid \iota_X^\star(a) = \iota_Y^\star(b)\} \subset H^\star(X) \times H^\star(Y),$$

where the cohomologies are computed with coefficients in some fixed ring.

Proof : Denote

$$\mathcal{R} = \{(a, b) \in H^\star(X) \times H^\star(Y) \mid \iota_X^\star(a) = \iota_Y^\star(b)\}.$$

Let us first observe that \mathcal{R} is actually a subalgebra of $H^\star(X) \times H^\star(Y)$, where we have componentwise product. Indeed, for $(a_1, b_1), (a_2, b_2) \in \mathcal{R}$, we have

$$\iota_X^\star(a_1 \smile a_2) = \iota_X^\star(a_1) \smile \iota_X^\star(a_2) = \iota_Y^\star(b_1) \smile \iota_Y^\star(b_2) = \iota_Y^\star(b_1 \smile b_2),$$

which implies $(a_1 \smile a_2, b_1 \smile b_2) \in \mathcal{R}$. Also, $\iota_X^\star(1_X) = 1_{\{\star\}} = \iota_Y^\star(1_Y)$, which implies that the unit $(1_X, 1_Y) \in \mathcal{R}$. Thus, \mathcal{R} is indeed a graded subring of $H^\star(X) \times H^\star(Y)$.

As for the isomorphism, note that $\mathcal{R}_k = H^k(X) \times H^k(Y) = H^k(X \vee Y)$ for $k \neq 0$, since ι_X, ι_Y induces zero map. For $k = 0$, we have

$$\mathcal{R}_0 = R \oplus \ker(\iota_X^\star) \times \ker(\iota_Y^\star) = R \oplus \ker(\iota_X^\star \vee \iota_Y^\star) = R \oplus \ker(\iota_{X \vee Y}^\star) = H^0(X \vee Y).$$

Thus, we have $H^\star(X \vee Y) = \mathcal{R}$, as graded modules. A direct computation shows that this an algebra isomorphism. Indeed, for any $z \in H^k(X \vee Y)$, with $k \neq 0$, we have $z = (\iota_1^\star z, \iota_2^\star z)$, where $\iota_1 : X \hookrightarrow X \vee Y, \iota_2 : Y \hookrightarrow X \vee Y$ are the inclusions. Thus, for $z \in H^k(X \vee Y), w \in H^l(X \vee Y)$, with $k, l \neq 0$, we have

$$z \smile w = (\iota_1^\star(z \smile w), \iota_2^\star(z \smile w)) = (\iota_1^\star z \smile \iota_1^\star w, \iota_2^\star z \smile \iota_2^\star w) = (\iota_1^\star z, \iota_2^\star z) \cdot (\iota_1^\star w, \iota_2^\star w).$$

As for $k = 0$ or $l = 0$, we only need to check the cup product for the unit $1_{X \vee Y}$. But $1_{X \vee Y} = (1_X, 1_Y)$ is clearly the unit in \mathcal{R} . Thus, we have an algebra isomorphism. \square

Exercise 20.5: (Cohomology Ring of Arbitrary Wedge)

Assume $\{(X_i, x_i)\}_{i \in I}$ is a collection based spaces, such that the basepoint inclusion map $\iota_j : \{\star\} \hookrightarrow X_j$ is a cofibration for all $j \in I$. Given a ring R , prove that there exists an algebra isomorphism

$$H^* \left(\bigvee_{i \in I} X_i; R \right) = \left\{ (a_i)_{i \in I} \in \prod_{i \in I} H^*(X_i; R) \mid \iota_j^*(a_j) = \iota_k^*(a_k) \forall j, k \right\}.$$

20.3 Computation of Cup Products

In this section, we shall compute the cup product in a few spaces. In fact, we shall identify the cohomology rings for these spaces. Let us recall a few notations. For a ring R , we have the usual polynomial ring $R[t]$. More generally, we shall write $R[t_1, \dots, t_k]$ with $|t_i| = n_i$ to mean that the indeterminates are at degree n_i respectively. Given polynomials f_1, \dots, f_k , we shall denote the ideal generated by them by $\langle f_1, \dots, f_k \rangle$.

20.3.1 Cup Product in Spheres Let us begin with the simplest example of the sphere.

Example 20.6: (Cup Product in Sphere)

Suppose $n > 0$. By the UCT (Theorem 17.1), we have $H^k(S^n; R) = \begin{cases} R, & k=0, n \\ 0, & \text{otherwise} \end{cases}$. Then, for degree reasons, all the cup products vanish, except for the unit! Indeed, for any generator $a \in H^n(S^n; R)$ we have $a^2 = a \smile a \in H^{2n}(S^n; R) = 0$, and so $a^2 = 0$. Thus, we have the cohomology ring $H^*(S^n; R) = \frac{R[t]}{\langle t^2 \rangle}$ with $|t| = n$. In fact, it is easy to see that the same remains true for $n = 0$.

Exercise 20.7: (Cohomology Ring of Product of Spheres)

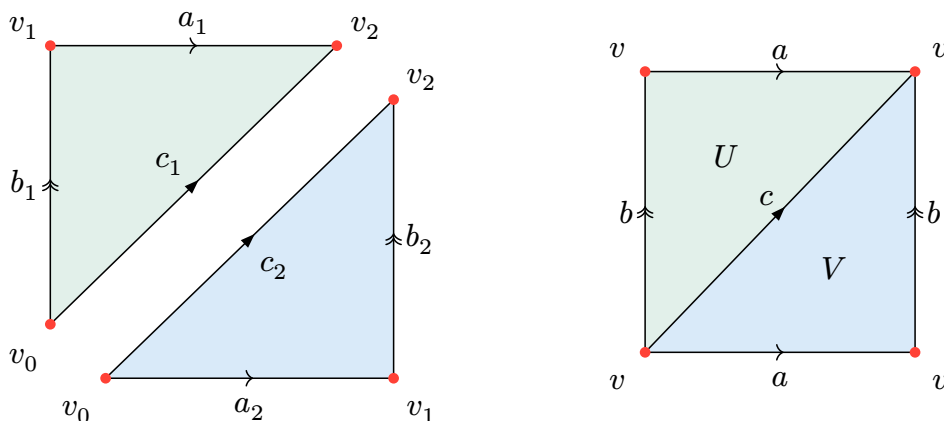
Verify that the cohomology ring of the space $X = \prod_{i=1}^k S^{n_i}$ for $n_i \geq 0$ over a PID R is isomorphic to the algebra

$$\frac{R[t_1, \dots, t_k]}{\langle t_1^2, \dots, t_k^2 \rangle}, \quad |t_1| = n_1, \dots, |t_k| = n_k.$$

Hint : We have an algebra isomorphism $R[X] \otimes R[Y] = R[X, Y]$, which preserves degrees of the indeterminates.

20.3.2 Cup Product in Torus Of course we can compute the cohomology ring of the torus $\mathbb{T}^2 = S^1 \times S^1$ by applying Theorem 20.2 and get $H^*(\mathbb{T}^2) = \frac{\mathbb{Z}[X, Y]}{\langle X^2, Y^2 \rangle}$, with $|X| = 1 = |Y|$. Instead, we compute it from the definition!

Let us consider the torus along with simplices given rise to cellular decomposition



The 1-simplices a, b generates the first homology $H_1(\mathbb{T}^2)$. Next, consider the two 2-simplices $U, V : \Delta^2 \rightarrow \mathbb{T}^2$, oriented such a way that

$$\partial U = a - c + b, \quad \partial V = b - c + a,$$

so that $\partial(V - U) = 0$. Thus, $[V - U]$ represents a homology class in $H_2(\mathbb{T}^2)$. Moreover, By cellular homology, it can be seen that $[V - U]$ is a generator of $H_2(\mathbb{T}^2)$.

By the UCT ([Theorem 17.1](#)), we have $H^\bullet(\mathbb{T}^2) = \text{hom}(H_\bullet(\mathbb{T}^2), \mathbb{Z})$, since the ext modules are zero. Then, we have generators $\alpha, \beta \in H^1(\mathbb{T}^2, \mathbb{Z})$ and $\gamma \in H^2(\mathbb{T}^2, \mathbb{Z})$ given as dual to $[a]$, $[b]$, and $[V - U]$ respectively. Let us find cocycle representations for α, β .

Suppose $\alpha = [\varphi]$ for some $\varphi : C_1^{\text{cell}}(\mathbb{T}^2) = \mathbb{Z}\langle a, b, c \rangle \rightarrow \mathbb{Z}$. Since $\alpha([a]) = 1, \alpha([b]) = 0$, we must have $\varphi(a) = 1, \varphi(b) = 0$. Since φ needs to be a cocycle, we require $\delta\varphi = 0$. This gives,

$$0 = \delta\varphi(U) = \varphi(\partial U) = \varphi(a - c + b) = \varphi(a) - \varphi(c) + \varphi(b) = 1 - \varphi(c) + 0 \Rightarrow \varphi(c) = 1.$$

Similarly, we have $\psi : C_1^{\text{cell}}(\mathbb{T}^2) \rightarrow \mathbb{Z}$ given by $\psi(a) = 0, \psi(b) = 1, \psi(c) = 1$, which is a cocycle representing β . Note that in order to make these into a *singular* cocycle, we need to consider all possible singular 1-simplices. One can do that as follows. Fix paths γ_x joining the vertex v to any point $x \in \mathbb{T}^2$. Then, for any $\sigma : \Delta^1 = [0, 1] \rightarrow \mathbb{T}^2$, we have the 1-cycle $\tilde{\sigma} = \gamma_{\sigma(0)} \star \sigma \star \bar{\gamma}_{\sigma(1)}$. Set $\varphi(\sigma) := \alpha([\tilde{\sigma}])$. Similarly, we can extend the definition of ψ to all of $S_1(\mathbb{T}^2)$.

Remark 20.8: (*Cellular Cohomology*)

We can instead work with *cellular cohomology*. That is, we consider the cellular chain complex C_\bullet^{cell} , apply the hom-functor $\text{hom}(_, M)$, and get the cohomology $H_{\text{cell}}^\bullet(X) = H^\bullet(\text{hom}(C^{\text{cell}}, M))$. Recall from [Theorem 13.18](#) that $H_\bullet^{\text{cell}}(X)$ is naturally isomorphic to $H_\bullet(X)$. One can apply the algebraic version of the UCT ([Theorem 17.1](#)) to the cellular chain complex $C_\bullet^{\text{cell}}(X)$, and then the naturality of the short exact sequence (and an application of five-lemma) will give us a natural isomorphism $H_{\text{cell}}^\bullet(X) \cong H^\bullet(X)$.

In fact, one can show that the singular chain complex $S_\bullet(X)$ is chain homotopy equivalent to the cellular chain complex $C_\bullet^{\text{cell}}(X)$, which can directly lead to the fact that $H_{\text{cell}}^\bullet(X) = H^\bullet(X)$.

Note that we are actually not defining how to compute cohomology for any arbitrary cellular cochain complex; the spaces that we consider gives rise to nice cellular structure whose cells are also simplices (sometimes these are called *delta complexes*).

Example 20.9: (Cup Product in the Torus \mathbb{T}^2)

The only interesting cup products appear involving the generators $\alpha, \beta \in H^1(\mathbb{T}^2, \mathbb{Z})$.

- By graded commutativity,

$$\alpha \smile \alpha = (-1)^{|\alpha| \cdot |\alpha|} \alpha \smile \alpha = -\alpha \smile \alpha \Rightarrow 2(\alpha \smile \alpha) = 0 \Rightarrow \alpha \smile \alpha = 0.$$

- By similar argument, we have $\beta \smile \beta = 0$.
- We claim $\alpha \smile \beta = \gamma$. This requires checking $(\alpha \smile \beta)([V - U]) = 1$. Indeed, we have

$$\begin{aligned} (\varphi \smile \psi)(U - V) &= \varphi(V|_{[v_0, v_1]}) \cdot \psi(V|_{[v_1, v_2]}) - \varphi(U|_{[v_0, v_1]}) \cdot \psi(U|_{[v_1, v_2]}) \\ &= \varphi(a) \cdot \psi(b) - \varphi(b) \cdot \psi(a) = 1. \end{aligned}$$

Thus, $\alpha \smile \beta = \gamma$ follows.

- By graded commutativity, $\beta \smile \alpha = (-1)^{1 \cdot 1} \alpha \smile \beta = -\alpha \smile \beta = -\gamma$.

Hence, we can identify the cohomology ring

$$H^*(\mathbb{T}^2) = \frac{\mathbb{Z}[X, Y]}{\langle X^2, Y^2 \rangle}, \quad |X| = 1 = |Y|,$$

with the understanding that it is a *graded* commutative algebra and thus, $XY = YX$. This also known as the *exterior algebra* $\Lambda_{\mathbb{Z}}(X, Y)$ over \mathbb{Z} , with two degree 1 generators.

Exercise 20.10: (Cohomology Ring of Torus with \mathbb{Z}_2 Coefficient)

Compute the comology ring of the the torus with $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ coefficients. Note: $2x = 0 \not\Rightarrow x = 0$ in \mathbb{Z}_2 !

20.3.3 Cup Product in Oriented Surfaces Recall, the *oriented* genus- g surface Σ_g is given as the the g -fold connected sum of 2-tori

$$\Sigma_g := \underbrace{\mathbb{T}^2 \# \dots \# \mathbb{T}^2}_{g\text{-times}}$$

Equivalently, Σ_g is constructed by considering $2g$ -fold wedge of circles $\bigvee_{i=1}^g S_{a_i}^1 \vee S_{b_i}^1$, and then attaching a 2-disc by the attaching map $S^1 \rightarrow \bigvee_{i=1}^{2g} S^1$ given by

$$[a_1, b_1] \cdots [a_g, b_g] = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}.$$

By Hurewicz theorem ([Theorem 10.1](#)), it follows that

$$H_1(\Sigma_g) = \pi_1(\Sigma_g)^{\text{ab}} = \bigoplus_{i=1}^g \mathbb{Z}\langle a_i \rangle \oplus \mathbb{Z}\langle b_i \rangle = \mathbb{Z}^{2g}.$$

By cellular homology, we also have $H_2(\Sigma_g) = \mathbb{Z}$. Then, by UCT ([Theorem 17.1](#)), it follows that $H^\bullet(\Sigma_g) = \text{hom}(H_\bullet(\Sigma_g), \mathbb{Z})$, and thus,

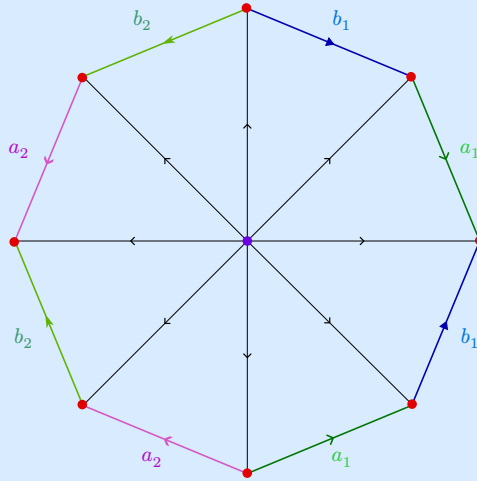
$$H^k(\Sigma_g) = \begin{cases} \mathbb{Z}, & k = 0, 2 \\ \mathbb{Z}^{2g}, & k = 1 \\ 0, & \text{otherwise.} \end{cases}.$$

Again, the only interesting cup products are $\smile: H^1(\Sigma_g) \otimes H^1(\Sigma_g) \rightarrow H^2(\Sigma_g)$. Denote, α_i, β_i as the duals of a_i, b_i respectively, which are generators of $H^1(\Sigma_g)$.

Exercise 20.11:

Compute the cup products $\alpha_i \smile \alpha_j, \alpha_i \smile \beta_j, \beta_i \smile \beta_j$ for all i, j by considering a suitable cellular decomposition of Σ_g .

Hint : Consider the following diagram for Σ_2



Since we have already computed the cohomology ring of $\Sigma_1 = \mathbb{T}^2$, let us try to use it as a base case of an induction.

Proposition 20.12: (Cohomology Ring of Σ_g)

The cohomology ring of the oriented genus- g surface is the graded commutative ring

$$H^*(\Sigma_g) = \frac{\mathbb{Z}[a_1, b_1, \dots, a_g, b_g]}{\langle a_i^2 \ \forall i, \ b_i^2 \ \forall i, \ a_i b_j \ \forall i \neq j \rangle}, \quad |a_i| = 1 = |b_j|.$$

Proof : Consider the quotient map $\Sigma_g \rightarrow X_i$ which collapses the two 1-cells labeled by a_i and b_i . It is then easy to see that X_i is homotopy equivalent to Σ_{g-1} . Composing with a homotopy equivalence, we then get a map $q_i: \Sigma_g \rightarrow \Sigma_{g-1}$. Moreover, this map is *surjective* on H_1 and isomorphism on H_2 , which can be verified by cellular homology. Hence, by the naturality of the UCT, we have maps $q_i^*: H^1(\Sigma_{g-1}) \hookrightarrow H^1(\Sigma_g)$, which maps the generators to the corresponding generators, skipping the i^{th} one. That is,

$$q_i^*(\alpha'_j) = \begin{cases} \alpha_j, & j < i \\ \alpha_{j+1}, & j \geq i \end{cases} \quad \text{and} \quad q_i^*(\beta'_j) = \begin{cases} \beta_j, & j < i \\ \beta_{j+1}, & j \geq i. \end{cases}$$

Moreover, $q_i^*(\gamma') = \gamma$, where γ' and γ are respectively the generators of $H^2(\Sigma_{g-1})$ and $H^2(\Sigma_g)$. Then, for $j_1 \neq i, j_2 \neq i$, we have

$$\alpha_{j_1} \smile \alpha_{j_2} = q_i^*(\alpha'_{j_1}) \smile q_i^*(\alpha'_{j_2}) = q_i^*(\alpha'_{j_1} \smile \alpha'_{j_2}) = q_i^*(0) = 0,$$

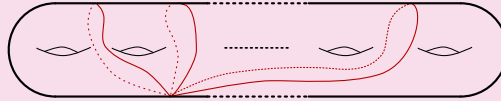
and similarly, $\beta_{j_1} \smile \beta_{j_2} = 0$. If $j_1 \neq j_2$, then $\alpha_{j_1} \smile \beta_{j_2} = 0$ as well. Finally, for any $j \neq i$ we have

$$\alpha_j \smile \beta_j = q_i^*(\alpha'_j \smile \beta'_j) = q_i^*(\gamma') = \gamma.$$

Since i is arbitrary, we have all the cup products. Inductively, we have the claim. \square

Exercise 20.13:

Consider Σ_g , the oriented genus- g surface



Observe that collapsing the red curves to a point, we have a quotient map $q : \Sigma_g \rightarrow \bigvee_{i=1}^g \mathbb{T}^2$ to g -fold wedge of the torus. Compute the cohomology ring of Σ_g using this map.

Hint : By computing the local degree ([Proposition 13.8](#)), it follows that $q_*(c) = c_1 + \dots + c_g$, where $H_2(\Sigma_g) = \mathbb{Z}\langle c \rangle$ and $H_2(\bigvee^g \mathbb{T}^2) = \mathbb{Z}\langle c_1, \dots, c_g \rangle$, with suitable choice of the 2-cells representing c_i . Then, by naturality of UCT, we get $q^*(\gamma_i) = \gamma$, where γ_i, γ are respectively dual of c_i, c .

20.3.4 Cup Product in \mathbb{RP}^2 Recall, the homology of the real projective space \mathbb{RP}^2 ([Example 13.13](#)) is

$$H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/2\mathbb{Z}, & k = 1 \\ 0, & \text{otherwise.} \end{cases}$$

An easy application of the UCT ([Theorem 17.11](#)) then gives

$$H^k(\mathbb{RP}^2; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/2\mathbb{Z}, & k = 2 \\ 0, & \text{otherwise.} \end{cases}$$

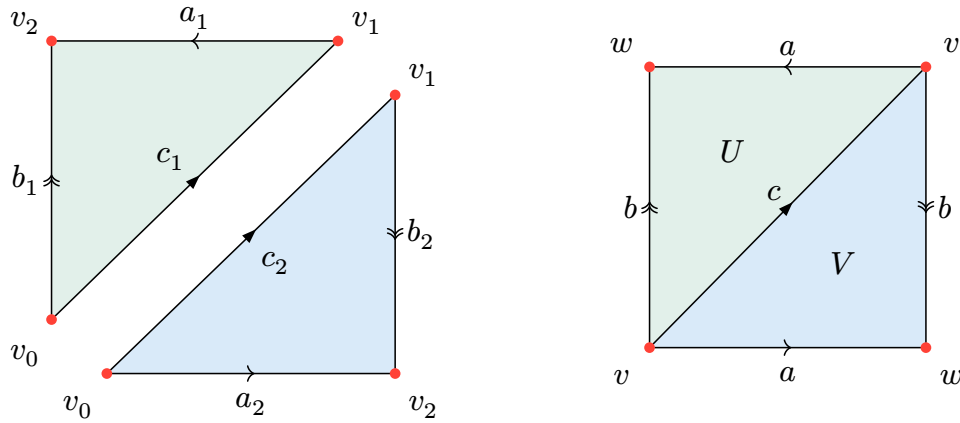
From degree reasons, it is clear that the cohomology ring structure of $H^*(\mathbb{RP}^2, \mathbb{Z})$ is not very interesting! Now, another application of UCT gives the cohomology groups with $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ coefficients

$$H^k(\mathbb{RP}^2; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & k = 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

There is a possibility of an interesting cup product

$$\smile : H^1(\mathbb{RP}^2, \mathbb{Z}_2) \otimes H^1(\mathbb{RP}^2, \mathbb{Z}_2) \rightarrow H^2(\mathbb{RP}^2, \mathbb{Z}_2).$$

Just like the computation of cup product in torus ([Example 20.9](#)), we consider the following cellular decomposition of \mathbb{RP}^2 .



Let us compute the cellular homology with coefficients in \mathbb{Z}_2 . We have the complex

$$0 \rightarrow C_2^{\text{cell}}(\mathbb{RP}^2; \mathbb{Z}_2) \rightarrow C_1^{\text{cell}}(\mathbb{RP}^2; \mathbb{Z}_2) \rightarrow C_0^{\text{cell}}(\mathbb{RP}^2; \mathbb{Z}_2) \rightarrow 0.$$

$\mathbb{Z}_2\langle U, V \rangle \qquad \mathbb{Z}_2\langle a, b, c \rangle \qquad \mathbb{Z}_2\langle v, w \rangle$

Note that $\partial b = w - v = \partial a$, and in particular, $H_0(\mathbb{RP}^2, \mathbb{Z}_2) = \mathbb{Z}_2\langle [v] = [w] \rangle = \mathbb{Z}_2$. Also, $\partial U = a - b + c$, $\partial V = b - a + c$. Over \mathbb{Z}_2 , we have $\partial(V - U) = 2(b - a) = 0$. So, $H_2(\mathbb{RP}^2; \mathbb{Z}_2) = \mathbb{Z}_2\langle [V - U] \rangle = \mathbb{Z}_2$ and $H_1(\mathbb{RP}^2; \mathbb{Z}_2) = \mathbb{Z}_2\langle [c] = [b - a] \rangle = \mathbb{Z}_2$.

Example 20.14: (Cohomology Ring of \mathbb{RP}^2 with \mathbb{Z}_2)

Since \mathbb{Z}_2 is a field, it follows from the UCT (Theorem 17.11) that $H^\bullet(\mathbb{RP}^2; \mathbb{Z}_2) = \text{hom}(H_\bullet(\mathbb{RP}^2; \mathbb{Z}_2), \mathbb{Z}_2)$, as the Ext terms vanish over a field. Hence, the nontrivial term $\alpha \in H^1(\mathbb{RP}^2; \mathbb{Z}_2) = \mathbb{Z}_2$ is represented by a 1-cocycle $\varphi : C_1^{\text{cell}}(\mathbb{RP}^2; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ such that $\varphi(b - a) = 1$. Since we must take value in \mathbb{Z}_2 , we can have $\varphi(b) = 1, \varphi(a) = 0$ (or, $\varphi(b) = 0, \varphi(a) = 1$). Let us assume the first case. Then, the cocycle condition gives

$$0 = \delta\varphi(U) = \varphi(\partial U) = \varphi(a - b + c) = \varphi(a) - \varphi(b) + \varphi(c) = 0 - 1 + \varphi(c) \Rightarrow \varphi(c) = 1.$$

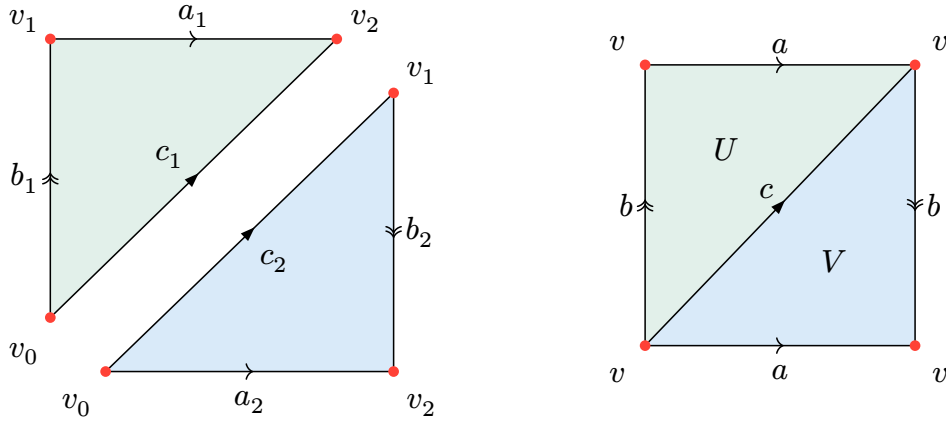
We compute

$$\begin{aligned} (\varphi \smile \varphi)(V - U) &= \varphi(V|_{[v_0, v_1]}) \cdot \varphi(V|_{[v_1, v_2]}) - \varphi(U|_{[v_0, v_1]}) \cdot \varphi(U|_{[v_1, v_2]}) \\ &= \varphi(c) \cdot \varphi(b) - \varphi(c) \cdot \varphi(a) = 1 \cdot 1 - 1 \cdot 0 = 1. \end{aligned}$$

Thus, $\alpha \smile \alpha$ is dual to the class $[V - U]$, and hence, is a generator of $H^2(\mathbb{RP}^2; \mathbb{Z}_2)$. Consequently, we have the cohomology ring

$$H^*(\mathbb{RP}^2, \mathbb{Z}_2) = \frac{\mathbb{Z}_2[X]}{\langle X^3 \rangle}, \quad |X| = 1.$$

20.3.5 Cup Product in Klein Bottle The Klein bottle K is given the following cellular decomposition



An argument similar to that of \mathbb{RP}^2 leads us to compute

$$H^k(K; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & k = 0, 2 \\ \mathbb{Z}_2^2, & k = 1 \\ 0, & \text{otherwise.} \end{cases}$$

The first cohomology is generated by duals of the generators $[a], [b]$ in the homology $H_1(K; \mathbb{Z}_2)$, say, α, β . Let $\alpha = [\varphi], \beta = [\psi]$ for some cellular 1-cocycles $\varphi, \psi : C_1^{\text{cell}}(K; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$. We have, $\varphi(a) = 1, \varphi(b) = 0, \psi(a) = 0, \psi(b) = 1$. Note that $\partial U = a - c + b, \partial V = b - a + c$, and thus $\partial(V - U) = (b - a + c) - (a - c + b) = 2(c - a) = 0$. In particular, $[V - U]$ generates $H_2(K; \mathbb{Z}_2)$, and its dual generates $H^2(K; \mathbb{Z}_2)$. The cocycle condition $\delta\varphi = 0$ gives

$$0 = \delta\varphi(U) = \varphi(\partial U) = \varphi(a - c + b) = \varphi(a) - \varphi(c) + \varphi(b) = 1 - \varphi(c) + 0 \Rightarrow \varphi(c) = 1,$$

and similarly, $\psi(c) = 1$. Then, we compute

$$\begin{aligned} (\varphi \smile \psi)(V - U) &= \varphi(V|_{[v_0, v_1]}) \cdot \psi(V|_{[v_1, v_2]}) - \varphi(U|_{[v_0, v_1]}) \cdot \psi(U|_{[v_1, v_2]}) \\ &= \varphi(c) \cdot \psi(b) - \varphi(b) \cdot \psi(a) = 1 \cdot 1 - 0 \cdot 0 = 1. \end{aligned}$$

In other words, $\alpha \smile \beta$ is the generator of $H^2(K; \mathbb{Z}_2)$. A similar computation shows $\beta \smile \beta$ is also a generator, but $\alpha \smile \alpha = 0$. Thus, we have the ring structure

$$H^*(K; \mathbb{Z}_2) = \frac{\mathbb{Z}_2[X, Y]}{\langle X^2, Y^2 + XY \rangle}, \quad |X| = 1 = |Y|.$$

By a change of basis $X \mapsto X + Y, Y \mapsto X$, we can get the isomorphism

$$H^*(K; \mathbb{Z}_2) = \frac{\mathbb{Z}_2[X, Y]}{\langle X^2 + Y^2, XY \rangle}, \quad |X| = 1 = |Y|.$$

20.3.6 Cup Product in Nonorientable Surfaces The nonorientable genus- g surface N_g is given as the g -fold connected sum of \mathbb{RP}^2 . It can be realized as a 2-dimensional CW complex whose 1-skeleton is the wedge of circles $\bigvee_{i=1}^g S_{a_i}^1$, and then a two cell is attached along the map $a_1^2 \cdots a_g^2$. By Hurewicz theorem (Theorem 10.1), it follows that

$$H_1(N_g) = \pi_1(N_g)^{\text{ab}} = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2.$$

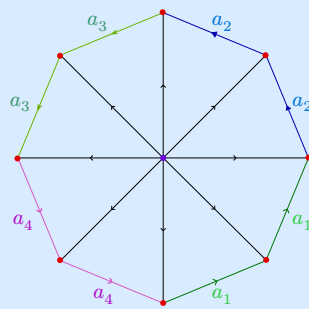
By cellular homology, $H_2(N_g; \mathbb{Z}) = 0$. Thus, cup products in \mathbb{Z} coefficients are not that interesting, and just like the case of $N_1 = \mathbb{RP}^2$ or $N_2 = K$ (Klein bottle), we focus on \mathbb{Z}_2 coefficient. By UCT, or by cellular homology with \mathbb{Z}_2 -coefficients, we have

$$H_k(N_g; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & k = 0, 2 \\ \mathbb{Z}_2^g, & k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 20.15:

Compute $H^\bullet(N_g; \mathbb{Z}_2)$ from UCT, and then compute the cup product by a suitable cellular decomposition as in [Exercise 20.11](#).

Hint : Consider the following diagram for N_4



We can also use an inductive argument, and show that

$$H^*(N_g; \mathbb{Z}_2) = \frac{\mathbb{Z}_2[a_1, \dots, a_g]}{\langle a_i a_j \quad \forall i \neq j, \quad a_i^2 + a_j^2 \quad \forall i \neq j \rangle}, \quad |a_i| = 1.$$