Course notes for

Algebraic Topology II (KSM4E02)

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`natural transformations – adjunction – Eilenberg-Steenrod axioms`

2.1 Natural Transformations

Definition 2.1: (*Natural transformation*)

Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta : F \Rightarrow G$ is a collection of morphisms $\eta_c : F(c) \to G(c)$ in $\mathcal{D}$, one for each $c \in \mathcal{C}$, such that given any morphism $f : X \to Y$ in $\mathcal{C}$, we have the following commutative diagram

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_Y"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

Exercise 2.2: (*Composition of Natural Transformations*)

Try to figure out two distinct ways compose natural transformations, namely, horizontally and vertically. The following diagram may help.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & \mathcal{D} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & \mathcal{E}
\end{tikzcd}$$

Try to determine the associativity of these compositions.

Definition 2.3: (*Natural Isomorphism*)

A natural transformation $\eta : F \Rightarrow G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is called a *natural isomorphism* if $\eta_c : F(c) \to G(c)$ is an isomorphism for all $c \in \mathcal{C}$.

Remark 2.4: (*Natural Transformation and Homotopy*)

You can imagine functors as maps between two spaces. Then, natural transformation can be interprated as homotopy between them! In fact this analogy can be made rigourous, and one can consider a level 3 morphism between two natural transformations, and so on and on to infinity.

Example 2.5:

Consider $\mathcal{C}$ to be the category of vector spaces over some field $\Bbbk$, with linear maps as morphisms. We have a functor $F : \mathcal{C} \to \mathcal{C}$ given by

$$F(V) = (V^*)^* = \hom_{\Bbbk}(\hom_{\Bbbk}(V, \Bbbk), \Bbbk).$$

Now, for each V, we have linear map given by evaluation

$$\begin{aligned} \eta_V : V &\to F(V) \\ \mathbf{v} &\mapsto (T \mapsto T(\mathbf{v})). \end{aligned}$$

Then, $\eta : \mathrm{Id} \Rightarrow F$ is a natural transformation (Check!).

2.2 Adjoint functors

Definition 2.6: (*Adjoint Functor*)

Let $\mathcal{C}, \mathcal{D}$ be two categories, and we have two fuctors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$. We say L is *left adjoint* to R, and R is *right adjoint* to L if there exists natural isomorphisms (i.e, set bijections)

$$\eta_{c,d} : \hom_{\mathcal{D}}(L(c), d) \to \hom_{\mathcal{C}}(c, R(d)), \quad c \in \mathcal{C}, d \in \mathcal{D}.$$

Here naturality means that given $c \in \mathcal{C}$ fixed, and any morphism $g : d_1 \to d_2$ in $\mathcal{D}$, we have a commutative diagram

$$\begin{tikzcd} \hom_{\mathcal{D}}(L(c), d_1) \arrow[r, "\eta_{c,d_1}"] \arrow[d, "\hom_{\mathcal{D}}(L(c), g)"'] & \hom_{\mathcal{C}}(c, R(d_1)) \arrow[d, "\hom_{\mathcal{C}}(c, R(g))"] \\ \hom_{\mathcal{D}}(L(c), d_2) \arrow[r, "\eta_{c,d_2}"'] & \hom_{\mathcal{C}}(c, R(d_2)) \end{tikzcd}$$

Similarly, given $d \in \mathcal{D}$ fixed, and any morphism $f : c_1 \to c_2$ in $\mathcal{C}$, we have a commutative diagram

$$\begin{tikzcd} \hom_{\mathcal{D}}(L(c_1), d) \arrow[r, "\eta_{c_1,d}"] & \hom_{\mathcal{C}}(c_1, R(d)) \\ \hom_{\mathcal{D}}(L(c_2), d) \arrow[r, "\eta_{c_2,d}"'] \arrow[u, "\hom_{\mathcal{D}}(L(f), d)"] & \hom_{\mathcal{C}}(c_2, R(d)) \arrow[u, "\hom_{\mathcal{C}}(f, R(d))"'] \end{tikzcd}$$

The pair of functors (L, R) is called an *adjunct pair*.

Exercise 2.7: (*Adjunction and Natural Isomorphisms*)

Let $L : \mathscr{C} \to \mathscr{D}, R : \mathscr{D} \to \mathscr{R}$ be a pair of functors. Show that (L, R) is an adjuct pair if and only if for each $c \in \mathscr{C}$ and each $d \in \mathscr{D}$ there are natural isomorphisms of functors

$$\hom_{\mathscr{D}}(L(c), \_) \Rightarrow \hom_{\mathscr{C}}(c, R(\_)), \quad \text{and} \quad \hom_{\mathscr{D}}(L(\_), d) \Rightarrow \hom_{\mathscr{C}}(\_, R(d)).$$

Exercise 2.8: (*Left and Right Adjoint of Forgetful Functor* $\mathrm{Top} \to \mathrm{Set}$)

Let $U : \mathrm{Top} \to \mathrm{Set}$ be the forgetful functor. Given a set X, we can either give it the discrete topology, or the indiscrete topology. This defines two functors $D, I : \mathrm{Set} \to \mathrm{Top}$. Show that D is left adjoint of U, and I is right adjoint of U.

Exercise 2.9: (*hom-tensor Adjunction*)

On the category Ab of Abelian groups, we have two functors for each $X \in \mathrm{Ab}$

$$\hom(X, \_) : \mathrm{Ab} \to \mathrm{Ab}, \quad \_ \otimes X : \mathrm{Ab} \to \mathrm{Ab}$$

Prove that the tensor is left adjoint to hom, i.e, for $X, Y, Z \in \mathrm{Ab}$ show that there exists natural isomorphism

$$\hom(X \otimes Y, Z) \to \hom(X, \hom(Y, Z)).$$

In fact, the isomorphism is an isomorphism of Abelian groups as well. Same statement holds true for R -modules as well.

2.3 Eilenberg-Steenrod Axioms of (co)Homology Theories

Consider the category $\mathrm{TopPair}$ whose objects are pairs of topological space (X, A) where $A \subset X$, and morphisms $f : (X, A) \to (Y, B)$ are continuous maps $f : X \to Y$, such that $f(A) \subset B$. For simplicity, we shall denote $X = (X, \emptyset)$. Given any pair (X, A), we have the following inclusion maps

$$\begin{tikzcd}
 & & (X, \emptyset) \arrow[dr, "j"] & & \\
(\emptyset, \emptyset) \arrow[r] & (A, \emptyset) \arrow[ur, "\iota"] \arrow[dr] & & (X, A) \arrow[r] & (X, X) \\
 & & (A, A) \arrow[ur] & &
\end{tikzcd}$$

For $I = [0, 1]$, we have $(X, A) \times I = (X \times I, A \times I)$. Two maps $f_0, f_1 : (X, A) \to (Y, B)$ are called homotopic if there is a map $h : (X \times A) \times I \to (Y, B)$ such that $h|_{(X,A)\times\{0\}} = f_0$ and $h|_{(X,A)\times\{1\}} = g$ holds. In particular, a homtopy of pairs restricts to a homotopy of $f_0|_A =$. Later on we shall put more restrictions on this category, one such possibility is to assume that (X, A) is a CW-pair, i.e, X is a CW complex, and A is a subcomplex.

Definition 2.10: (*Homology Theory*)

A *homology theory*, according to Eilenberg-Steenrod, is a collection, indexed by $\mathbb{Z}$, of

- functors $H_n : \mathrm{TopPair} \to \mathrm{Ab}$ (or, more generally, to category of R-mod), and
- natural transformations $\partial^n_{(X,A)} : H_n(X,A) \to H_{n-1}(A)$,

satisfying the following axioms. Without loss of generaligy, given $f : (X,A) \to (Y,B)$, we shall denote $f_* = H_n(f) : H_n(X,A) \to H_n(Y,B)$, provided the index n is understood from the context. Similarly, we shall denote $\partial = \partial^n_{(X,A)}$.

I. **Exactness Axiom** : Given any pair (X,A), there exists a long exact sequence of Abelian groups

$$\cdots \to H_{n+1}(X,A) \to H_n(A) \underset{\iota_\star}{\to} H_n(X) \underset{j_\star}{\to} H_n(X,A) \underset{\partial}{\to} H_{n-1}(A) \to \cdots$$

Here $\iota : A \hookrightarrow X$ and $j : X \hookrightarrow (X,A)$ are the inclusions.

II. **Homotopy Axiom** : Homotopic maps induce the same map in homology. That is, given two homotopic maps $f, g : (X,A) \to (Y,B)$, we have $f_\star = g_\star : H_n(X,A) \to H_n(Y,B)$.

III. **Excision Axiom** : Given $(X,A) \in \mathrm{TopPair}$ and $U \subset A$ satisfying $\overline{U} \subset \mathring{A}$, the inclusion map $\iota : (X \setminus U, A \setminus U) \hookrightarrow (X,A)$ induces isomoephism (called *excision isomorphism*) in all homology groups.

IV. **Dimension Axiom** : For the one-point space $P = \{\star\}$, we have $H_n(P) = 0$ for $n \neq 0$.

Remark 2.11: (*Extraodinary Homology Theory*)

A homology theory without the dimension axiom is called *extraodinary homology theory*. Topological K-theory is an example of extraodinary homology theory.

Remark 2.12: (*Excision*)

In their original work, Eilenberg-Steenrod assumed a weaker version of the excision axiom, namely they only considered *open* sets $U \subset X$ such that $\overline{U} \subset \mathring{A}$ holds. One of the most common homology theory (i.e, the singular homology theory) satisfies the stronger excision axiom where *any* subset $U \subset X$ with $\overline{U} \subset \mathring{A}$ can be excised out. We shall see that any (ordinary) homology theory defined on reasonable pairs (e.g. CW-paris) is isomorphic to the singular homology, and thus, satisfies the stronger excision axiom.

Exercise 2.13: (*Two Excision Axioms Can Be Equivalent*)

Consider $\mathscr{C}$ to be the category of pairs (X,A) of *compact* topological spaces, and continuous maps between them. Justify that the two excision axioms must be equivalent for this category.

Exercise 2.14:

Let $f : (X, A) \to (Y, B)$ be map. From the axiom, show that we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X,A) \arrow[r] \arrow[d] & H_n(A) \arrow[r] \arrow[d] & H_n(X) \arrow[r] \arrow[d] & H_n(X,A) \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n+1}(X,A) \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & H_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical arrows are induced by f. That is, the homology long exact sequence is natural.