Course notes for

Algebraic Topology II (KSM4E02)

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`cohomological cross product – cup product – cochain cup product`

19.1 The Cup Product

Let us revisit the Künneth theorem for cohomology. Suppose $C_\bullet, D_\bullet$ are some arbitrary chain complex of R-modules, and denote the dual cochain complexes $C^\bullet := \hom_R(C_\bullet, R), D^\bullet := \hom_R(D_\bullet, R)$. We have a map

$$\hom_R(C_\bullet, R) \otimes \hom_R(D_\bullet, R) \to \hom_R(C_\bullet \otimes D_\bullet, R)$$
$$\varphi \otimes \psi \mapsto \left(\sum_i c_i \otimes d_i \mapsto \sum_i \varphi(c_i) \cdot \psi(d_i) \right).$$

In general, this map is not an isomorphism. But if both $C_\bullet, D_\bullet$ are free, it is an isomorphism. This is precisely the first map in Theorem 18.4, called the *algebric cohomological cross product*

$$\times^{\mathrm{alg}} : H^p(C^\bullet) \otimes H^q(D^\bullet) \to H^{p+q}((C_\bullet \otimes D_\bullet)^\star).$$

Exercise 19.1: (*Dual of Tensor*)

Let M, N be R-modules, and $C_\bullet, D_\bullet$ be chain complexes of free R-modules. Then there exists a natural isomorphism

$$\hom_R(C_\bullet, M) \otimes_R \hom_R(D_\bullet, N) \to \hom_R(C_\bullet \otimes D_\bullet, M \otimes N).$$

This leads to a Künneth theorem in cohomology with coefficients.

Now, let X, Y be fixed spaces. Given a choice of EZ-map, say, $\mathrm{AW}_{X,Y} : S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$, dualizing we have a cochain map

$$\mathrm{AW}^\star_{X,Y} : (S_\bullet(X) \otimes S_\bullet(Y))^\star \to S^\bullet(X \times Y).$$

Definition 19.2: (*Cohomology Cross Product*)

Given two spaces X, Y, the *cohomology cross product* is defined as the composition

$$H^p(X) \otimes H^q(Y) \xrightarrow{\times^{\mathrm{alg}}} H^{p+q}((S_\bullet(X) \otimes S_\bullet(Y))^\star) \xrightarrow[\cong]{\mathrm{AW}^\star_{X,Y}} H^{p+q}(X \times Y),$$

where we consider cohomology with coefficients in a ring R.

Note that the definition is independent of the choice of the EZ-map since any such choice are chain homotopic and hence induces identical isomorphism at the cohomology level.

Definition 19.3: (*Cup Product*)

Given a space X, consider the *diagonal map* $\Delta : X \to X \times X$. The *cup product* on X is defined as the composition

$$\smile : H^p(X) \otimes H^q(X) \xrightarrow{\times} H^{p+q}(X \times X) \xrightarrow{\Delta^\star} H^{p+q}(X),$$

where we consider cohomology in a ring R.

We shall see computations of cup product in spaces, which shows cohomology as a vastly greater tool in distinguishing spaces.

Remark 19.4: (*Cup Product in Homology?*)

In homology, we have the cross product $H_p(X) \otimes H_q(X) \to H_{p+q}(X \times X)$. But the diagonal map induces $\Delta_\star : H_{p+q}(X) \to H_{p+q}(X \times X)$, which goes the *wrong way*. Thus, we cannot define a map $H_p(X) \otimes H_q(X) \to H_{p+q}(X)$.

One might ask: what about a coproduct $H_n(X) \to \sum H_p(X) \otimes H_q(X)$? Even this may fail! Recall that for a PID, the Künneth theorem in singular homology (Theorem 18.10) gives the *split* short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(X) \otimes H_j(X) \to H_n(X \times X) \to \bigoplus_{i+j=n-1} \operatorname{Tor}\left(H_i(X), H_j(X)\right) \to 0.$$

Since the sequence is split, we can define a composition

$$H_n(X) \xrightarrow{\Delta_\star} H_n(X \times X) \to \bigoplus_{i+j=n} H_i(X) \otimes H_j(X),$$

where the red arrow is a *choice,* and moreover, the choice is not natural as the splitting is not natural. Thus, in general we do not have coproduct in homology. Of course, if the Tor term vanishes (e.g, when working over a field) we do have a coproduct.

If (X, μ) is a H-space (i.e, X is a based space with a map $\mu : X \times X \to X$ such that $\mu|_{X \vee X} \simeq \nabla_X : X \vee X \to X$), we do get an induced map

$$H_p(X) \otimes H_q(X) \xrightarrow{\times} H_{p+q}(X \times X) \xrightarrow{\mu_\star} H_{p+q}(X).$$

This product is known as the *Pontryagin product*. This is important, e.g, when X is the loop space of another space; loop space is an H-space with the concatenation product.

We have the following useful relations between cup and cross product.

Proposition 19.5: (*Cup and Cross Product*)

Let $f : X' \to X, g : Y' \to Y$ be continuous map. Then, the following holds.

1. $(f \times g)^\star(\alpha \times \beta) = f^\star\alpha \times g^\star\beta$ for $\alpha \in H^\star(X), \beta \in H^\star(Y)$.
2. $\alpha \times \beta = \pi_X^\star\alpha \smile \pi_Y^\star\beta$ for $\alpha \in H^\star(X), \beta \in H^\star(Y)$.
3. $f^\star(\alpha \smile \beta) = f^\star\alpha \smile f^\star\beta$ for $\alpha, \beta \in H^\bullet(X)$.

Proof : The proofs essentially follow from definitions.

1. Since the short exact sequence in the Künneth theorem for cohomology (Theorem 18.12) is natural, it follows that the comological cross product is also natural. In particular, $(f \times g)^\star(\alpha \times \beta) = f^\star\alpha \times g^\star\beta$.
2. We have,

$$\begin{aligned}
\pi_X^\star\alpha \smile \pi_Y^\star\beta &= \Delta_{X\times Y}^\star(\pi_X^\star\alpha \times \pi_Y^\star\beta)\\
&= \Delta_{X\times Y}^\star(\pi_X \times \pi_Y)^\star(\alpha \times \beta)\\
&= ((\pi_X \times \pi_Y) \circ \Delta_{X\times Y})^\star(\alpha \times \beta)\\
&= \mathrm{Id}_{X\times Y}^\star(\alpha \times \beta)\\
&= \alpha \times \beta.
\end{aligned}$$

3. We have,

$$\begin{aligned}
f^\star(\alpha \smile \beta) &= f^\star\Delta^\star(\alpha \times \beta)\\
&= (\Delta \circ f)^\star(\alpha \times \beta)\\
&= ((f \times f) \circ \Delta)^\star(\alpha \times \beta)\\
&= \Delta^\star(f \times f)^\star(\alpha \times \beta)\\
&= \Delta^\star(f^\star\alpha \times f^\star\beta)\\
&= f^\star\alpha \smile f^\star\beta.
\end{aligned}$$

□

Proposition 19.6: (*Properties of the Cup Product*)

Given a space X, the following holds for cohomology with coefficients in a ring R.

1. **(Associativity)** : $a \smile (b \smile c) = (a \smile b) \smile c$ for any $a, b, c \in H^\star(X; R)$.
2. **(Graded Commutativity)** : $a \smile b = (-1)^{|a|\cdot|b|} b \smile a$ for $a \in H^{|a|}(X; R), b \in H^{|b|}(X; R)$.
3. **(Unitality)** : $1 \smile a = a = a \smile 1$ for any $a \in H^\star(X; R)$. Here $1 \in H^0(X; R)$ is the cohomology class induced by the map $\mathbf{1} : S^0(X) \to R$ taking every 0-simplex to $1_R \in R$.

Proof : The proofs are consequence of the properties of the EZ-maps (Theorem 18.6). □

As a consequence, we now have the following.

Definition 19.7: (*Cohomology Ring*)

Given a space X and a ring R, the *cohomology ring* of X is defined as the graded module $H^\star(X;R) = \bigoplus H^i(X;R)$, equipped with the cup product. It is a unital, associative, graded commutative algebra over R.

19.2 Cochain Cup Product

The importance of cup product warrants some explicit computations! Recall that at the level of cochains, using the Alexander-Whitney map, we have the cup product

$$\begin{aligned} \smile\colon S^p(X;R)\otimes S^q(X;R) &\to S^{p+q}(X;R) \\ \varphi\otimes\psi &\mapsto \Big(\sigma\mapsto \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big)\cdot\psi\big(\sigma|_{[v_p,\dots,v_{p+q}]}\big)\Big). \end{aligned}$$

We shall prove Proposition 19.6 via explicit computation involving this cochain level cup product.

Proposition 19.8: (*Cup Product is a Chain Map*)

We have a chain map $\smile\colon S^\bullet(X;R)\otimes S^\bullet(X;R)\to S^\bullet(X;R)$. Explicitly, for homogenous elements $\varphi,\psi\in S^\bullet(X;R)$, we have

$$\delta(\varphi\smile\psi) = \delta\varphi\smile\psi + (-1)^{|\varphi|}\varphi\smile\delta\psi.$$

As a consequence, we have the induced cup product

$$\smile\colon H^p(X;R)\otimes H^q(X;R)\to H^{p+q}(X;R).$$

Proof : Let $\sigma:\Delta^{p+q+1}\to X$ be a singular $(p+q+1)$-simplex, and $\varphi\in S^p(X;R), \psi\in S^q(X;R)$. We compute

$$\begin{aligned} \delta(\varphi\smile\psi)(\sigma) &= (\varphi\smile\psi)(\partial\sigma) \\ &= (\varphi\smile\psi)\left(\sum_{i=0}^{p+q+1}(-1)^i\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{p+q+1}]}\right) \\ &= \sum_{i=1}^{p+q+1}(-1)^i(\varphi\smile\psi)\big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{p+q+1}]}\big) \\ &= \sum_{i=0}^{p}(-1)^i\varphi\big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{p+1}]}\big)\cdot\psi\big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\big) \\ &\quad + \sum_{i=p+1}^{p+q+1}(-1)^i\varphi\big(\sigma|_{[v_0,\dots,v_p]}\big)\cdot\psi\big(\sigma|_{[v_p,\dots,\hat{v}_i,\dots,v_{p+q+1}]}\big) \\ &= \sum_{i=0}^{p}(-1)^i\varphi\big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{p+1}]}\big)\cdot\psi\big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\big) \\ &\quad + (-1)^{p+1}\varphi\big(\sigma|_{[v_0,\dots,v_p]}\big)\cdot\psi\big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\big) \\ &\quad + (-1)^{p}\varphi\big(\sigma|_{[v_0,\dots,v_p]}\big)\cdot\psi\big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\big) \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i=p+1}^{p+q+1} (-1)^i \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \psi\big(\sigma|_{[v_p,\dots,\hat{v}_i,\dots,v_{p+q+1}]}\big) \\
&= \left[\sum_{i=0}^{p+1} (-1)^i \varphi\big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{p+1}]}\big)\right] \cdot \psi\big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\big) \\
&\quad + (-1)^p \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \left[\sum_{i=p}^{p+q+1} (-1)^{i-p} \psi\big(\sigma|_{[v_p,\dots,\hat{v}_i,\dots,v_{p+q+1}]}\big)\right] \\
&= \varphi\left(\sum_{i=0}^{p+1} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{p+1}]}\right) \cdot \psi\big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\big) \\
&\quad + (-1)^p \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \psi\left(\sum_{i=0}^{q+1} (-1)^i \sigma|_{[v_p,\dots,\hat{v}_{p+i},\dots,v_{p+q+1}]}\right) \\
&= \varphi\Big(\partial\big(\sigma|_{[v_0,\dots,v_{p+1}]}\big)\Big) \cdot \psi\big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\big) \\
&\quad + (-1)^p \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \psi\big(\partial\sigma|_{[v_p,\dots,v_{p+q+1}]}\big) \\
&= \delta\varphi\big(\sigma|_{[v_0,\dots,v_{p+1}]}\big) \cdot \psi\big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\big) \\
&\quad + (-1)^p \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \delta\psi\big(\sigma|_{[v_p,\dots,v_{p+q+1}]}\big) \\
&= (\delta\varphi \smile \psi)(\sigma) + (-1)^p (\varphi \smile \delta\psi)(\sigma) \\
&= (\delta\varphi \smile \psi + (-1)^p \varphi \smile \delta\psi)(\sigma).
\end{aligned}
$$

Since σ was arbitrary, we have the claim.

As for the induced map, if both φ and ψ are cocycles, it follows that $\varphi \smile \psi$ is also a cocycle, and similarly, if they are coboundaries. Hence, we get the induced cup product at the cohomology. □

Next, we check associativity and unitality.

Proposition 19.9: (*Cochain Cup Product is Associative and Unital*)

The cochain level cup product is associative and unital. Consequently, so is the induced cohomological cup product.

Proof : Let $\varphi \in S^p(X;R), \psi \in S^q(X;R), \zeta \in S^r(X;R)$ are cochains, and consider $\sigma : \Delta^{p+q+r} \to X$ be a singular $(p+q+r)$-simplex. Then, we compute

$$
\begin{aligned}
((\varphi \smile \psi) \smile \zeta)(\sigma) &= (\varphi \smile \psi)\big(\sigma|_{[v_0,\dots,v_{p+q}]}\big) \cdot \zeta\big(\sigma|_{[v_{p+q},\dots,v_{p+q+r}]}\big) \\
&= \Big(\varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \psi\big(\sigma|_{[v_p,\dots,v_{p+q}]}\big)\Big) \cdot \zeta\big(\sigma|_{[v_{p+q},\dots,v_{p+q+r}]}\big) \\
&= \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \Big(\psi\big(\sigma|_{[v_p,\dots,v_{p+q}]}\big) \cdot \zeta\big(\sigma|_{[v_{p+q},\dots,v_{p+q+r}]}\big)\Big) \\
&= \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot (\psi \smile \zeta)\big(\sigma|_{[v_p,\dots,v_{p+q+r}]}\big) \\
&= (\varphi \smile (\psi \smile \zeta))(\sigma).
\end{aligned}
$$

Since σ is arbitrary, we have $(\varphi \smile) \smile \zeta = \varphi \smile (\psi \smile \zeta)$. This proves that the cochain level cup product is strictly associative, and so is the cohomological cup product.

Next, consider the cochain $1 : S_0(X) \to R$ which sends every 0-simplex to the unit $1_R \in R$. For a p-cochain $\varphi \in S^p(X;R)$ and a singular p-simplex $\sigma : \Delta^p \to X$, we compute

$$(\varphi \smile 1)(\sigma) = \varphi\left(\sigma|_{[v_0,\ldots,v_p]}\right) \cdot 1\left(\sigma|_{[v_p]}\right) = \varphi(\sigma) \cdot 1_R = \varphi(\sigma),$$
$$(1 \smile \varphi)(\sigma) = 1\left(\sigma|_{[v_0]}\right) \cdot \varphi\left(\sigma|_{[v_0,\ldots,v_p]}\right) = 1_R \cdot \varphi(\sigma) = \varphi(\sigma).$$

Thus, $1 \smile \varphi = \varphi = \varphi \smile 1$, which shows that 1 is the unit for the cochain level cup product. Hence, we get the induced unit for the cohomological cup product. □

The final claim is about commutativity.

Proposition 19.10: (*Cochain Cup Product is **Not** Commutative*)

Let X be a space and R be commutative ring. Then, the cochain level cup product on $S^\bullet(X;R)$ is not commutative, but the cohomological cup product on $H^\bullet(X;R)$ is graded commutative.

Proof : Let us define a map of degree -1, called the Steenrod cup-1 map,

$$\smile_1 : S^\bullet(X;R) \otimes S^\bullet(X;R) \to S^\bullet(X;R)$$

as follows. Say $\varphi \in S^p(X;R), \psi \in S^q(X;R)$ are cochains, and $\tau : \Delta^{p+q-1} \to X$ be a singular simplex. Define

$$(\varphi \smile_1 \psi)(\tau) = \sum_{0\le j<p} (-1)^{(p-j)(q+1)} \varphi\left(\tau|_{[v_0,\ldots,v_j,v_{j+q},\ldots v_{p+q-1}]}\right) \cdot \psi\left(\tau|_{[v_j,\ldots,v_{j+q}]}\right).$$

Next, say $\sigma : \Delta^{p+q} \to X$ is a singular $(p+q)$-simplex. We compute

$$\begin{aligned}
&\delta(\varphi \smile_1 \psi)(\sigma) \\
&= (\varphi \smile_1 \psi)(\partial\sigma) \\
&= \sum_{i=0}^{p+q} (-1)^i (\varphi \smile_1 \psi)\left(\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_{p+q}]}\right) \\
&= \sum_{0\le j<p} (-1)^{(p-j)(q+1)} \Bigg[\sum_{0\le i\le j} (-1)^i \varphi\left(\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_{j+1},v_{j+q+1},\ldots,v_{p+q}]}\right) \cdot \psi\left(\sigma|_{[v_{j+1},\ldots,v_{j+q+1}]}\right) \\
&\quad + \sum_{j<i\le j+q} (-1)^i \varphi\left(\sigma|_{[v_0,\ldots,v_j,v_{j+q+1},\ldots,v_{p+q}]}\right) \cdot \psi\left(\sigma|_{[v_j,\ldots,\hat{v}_i,\ldots,v_{j+q+1}]}\right) \\
&\quad + \sum_{j+q<i\le p+q} (-1)^i \varphi\left(\sigma|_{[v_0,\ldots,v_j,v_{j+q},\ldots,\hat{v}_i,\ldots,v_{p+q}]}\right) \cdot \psi\left(\sigma|_{[v_j,\ldots,v_{j+q}]}\right)\Bigg]
\end{aligned}$$

Next, we have

$$\begin{aligned}
&\delta\varphi \smile_1 \psi(\sigma) \\
&= \sum_{0\le j\le p} (-1)^{(p+1-j)(q+1)} \delta\varphi\left(\sigma|_{[v_0,\ldots,v_j,v_{j+q},\ldots,v_{p+q}]}\right) \cdot \psi\left(\sigma|_{[v_j,\ldots,v_{j+q}]}\right) \\
&= \sum_{0\le j\le p} (-1)^{(p+1-j)(q+1)} \left(\sum_{i=0}^{j} (-1)^i \varphi\left(\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_j,v_{j+q},\ldots,v_{p+q}]}\right)\right.
\end{aligned}$$

$$+\sum_{i=j}^{p}(-1)^{i+1}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q},\dots\hat{v}_{q+i},\dots,v_{p+q}]}\Big)\Bigg)\cdot\psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big)$$

$$=(-1)^{(p+1)(q+1)}\Bigg(\varphi\Big(\sigma|_{[v_q,\dots,v_{p+q}]}\Big)+\sum_{i=0}^{p}(-1)^{i+1}\varphi\Big(\sigma|_{[v_0,v_q,\dots,\hat{v}_{q+i},\dots,v_{p+q}]}\Big)\Bigg)\cdot\psi\Big(\sigma|_{[v_0,\dots,v_q]}\Big)$$

$$+\sum_{0\le j<p}(-1)^{(p-j)(q+1)}\sum_{i=0}^{j+1}(-1)^{i}\varphi\Big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{j+1},v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big)$$

$$+\sum_{0<j\le p}(-1)^{(p+1-j)(q+1)}\sum_{i=j}^{p}(-1)^{i+1}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q},\dots,\hat{v}_{q+i},\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big)$$

$$=(-1)^{(p+1)(q+1)}(\psi\smile\varphi)(\sigma)$$

$$+\sum_{0\le j<p}(-1)^{(p-j)(q+1)}\sum_{i=0}^{j}(-1)^{i}\varphi\Big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{j+1},v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big)$$

$$+\sum_{0\le j<p}(-1)^{(p-j)(q+1)}(-1)^{j+1}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big)$$

$$+\sum_{0\le j<p}(-1)^{(p-j)(q+1)}\sum_{j+q\le i\le p+q}(-1)^{i}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q},\dots,\hat{v}_i,\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big)$$

$$+(-1)^{p+q}\varphi\Big(\sigma|_{[v_0,\dots,v_p]}\Big)\cdot\psi\Big(\sigma|_{[v_p,\dots,v_{p+q}]}\Big)$$

$$=(-1)^{(p+1)(q+1)}(\psi\smile\varphi)(\sigma)+(-1)^{p+q}(\varphi\smile\psi)(\sigma)$$

$$+\sum_{0\le j<p}(-1)^{(p-j)(q+1)}\sum_{i=0}^{j}(-1)^{i}\varphi\Big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{j+1},v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big)$$

$$+\sum_{0\le j<p}(-1)^{(p-j)(q+1)}(-1)^{j+1}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big)$$

$$+\sum_{0\le j<p}(-1)^{(p-j)(q+1)}(-1)^{j+q}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big)$$

$$+\sum_{0\le j<p}(-1)^{(p-j)(q+1)}\sum_{j+q<i\le p+q}(-1)^{i}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q},\dots,\hat{v}_i,\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big)$$

$$=(-1)^{p+q}((\varphi\smile\psi)(\sigma)-(-1)^{pq}(\psi\smile\varphi)(\sigma))$$

$$+\sum_{0\le j<p}(-1)^{(p-j)(q+1)}\sum_{i=0}^{j}(-1)^{i}\varphi\Big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{j+1},v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big)$$

$$+\sum_{0\le j<p}(-1)^{(p-j)(q+1)}\sum_{j+q<i\le p+q}(-1)^{i}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q},\dots,\hat{v}_i,\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big)$$

$$+\sum_{0\le j<p}(-1)^{(p-j)(q+1)}(-1)^{j+1}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big)$$

$$+\sum_{0\le j<p}(-1)^{(p-j)(q+1)}(-1)^{j+q}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big)$$

Finally,

$$\varphi\smile_1\delta\psi(\sigma)$$

$$= \sum_{0\le j<p} (-1)^{(p-j)q}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot \delta\psi\Big(\sigma|_{[v_j,\dots,v_{j+q+1}]}\Big)$$

$$= \sum_{0\le j<p} (-1)^{(p-j)q}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot \left(\sum_{i=0}^{q+1}(-1)^i\psi\Big(\sigma|_{[v_j,\dots,\hat{v}_{j+i},\dots,v_{j+q+1}]}\Big)\right)$$

$$= \sum_{0\le j<p} (-1)^{(p-j)q}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot \left(\sum_{i=j}^{j+q+1}(-1)^{i-j}\psi\Big(\sigma|_{[v_j,\dots,\hat{v}_{i},\dots,v_{j+q+1}]}\Big)\right)$$

$$= (-1)^p\sum_{0\le j<p}(-1)^{(p-j)(q+1)}\sum_{j\le i\le j+q}(-1)^i\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_j,\dots,\hat{v}_i,\dots,v_{j+q+1}]}\Big)$$
$$+(-1)^p\sum_{0\le j<p}(-1)^{(p-j)(q+1)}(-1)^j\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big)$$
$$+(-1)^p\sum_{0\le j<p}(-1)^{(p-j)(q+1)}(-1)^{j+q+1}\varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big)\cdot\psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big)$$

Then, it follows that

$$\begin{aligned}\delta(\varphi\smile_1\psi)(\sigma)-(\delta\varphi\smile_1\psi)(\sigma)-(-1)^p(\varphi\smile_1\delta\psi)(\sigma)\\ =(-1)^{p+q+1}(\varphi\smile\psi)(\sigma)-(-1)^{pq}(\psi\smile\varphi)(\sigma).\end{aligned}$$

In other words,

$$\begin{aligned}\varphi\smile\psi-(-1)^{pq}\psi\smile\varphi\\ =(-1)^{p+q+1}(\delta(\varphi\smile_1\psi)-\delta\varphi\smile_1\psi-(-1)^p\varphi\smile_1\delta\psi).\end{aligned}$$

This implies that, passing to cohomology, the cup product is graded commutative! □

Remark 19.11: (*Cup-i Product*)

More generally, one can define the cup-i product as a map $\smile_i\colon S^p(X)\otimes S^q(X)\to S^{p+q-i}(X)$. They were originally defined by Steenrod in order to define the Steenrod square operations on cohomology with $\mathbb{Z}_2$-coefficincents. Generalizing even further, one gets to the notion of E_∞-algebra structure on the cochain complex $S^\bullet(X)$.