

# Algebraic Topology II (KSM4E02)

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Künneth theorem for chain complex – Eilenberg-Zilber theorem – Künneth theorem for spaces – homological cross product

### 18.1 Künneth Theorems in Homological Algebra

We now prove the Künneth theorems, which helps us compute the (co)homology of tensor product of chain complexes.

#### Theorem 18.1: (Künneth Theorem for Homology)

Let  $R$  be a PID, and  $C_\bullet$  be a chain complex of free  $R$ -modules. Then, for any chain complex  $D_\bullet$  of  $R$ -modules, we have a natural short exact sequence

$$0 \rightarrow \bigoplus_{i+j=n} H_i(C_\bullet) \otimes H_j(D_\bullet) \rightarrow H_n((C \otimes D)_\bullet) \rightarrow \bigoplus_{i+j=n-1} \text{Tor}(H_i(C_\bullet), H_j(D_\bullet)) \rightarrow 0.$$

Moreover, if  $D_\bullet$  consists of free  $R$ -modules, then the short exact sequence splits (not necessarily naturally).

**Proof :** Let us consider the graded modules  $Z(C_\bullet)$ ,  $B(C_\bullet)$  and  $H(C_\bullet)$  of respectively cycles, boundaries and homology of  $C_\bullet$  as chain complexes with *trivial* boundary maps. Since  $R$  is a PID, and  $C_\bullet$  is a complex of *free*  $R$ -modules, we have  $Z(C_\bullet)$  and  $B(C_\bullet)$  are also complexes of free  $R$ -modules. Now, we have the chain complex  $((Z(C) \otimes D)_\bullet, \partial_\bullet^{Z(C) \otimes D})$ , where the boundary is given as

$$\partial_n^{Z(C) \otimes D} = \sum_{i+j=n} \text{Id}_{Z_i(C_\bullet)} \otimes \partial_j^D.$$

Since  $Z_i(C_\bullet)$  is free, we have  $\text{Tor}(Z_i(C_\bullet), \_) = 0$ , and hence it follows that

$$\ker(\text{Id} \otimes \partial_j^D : Z_i(C_\bullet) \otimes D_j \rightarrow Z_i(C_\bullet) \otimes D_{j-1}) = Z_i(C_\bullet) \otimes Z_j(D_\bullet),$$

and

$$\text{im}(\text{Id} \otimes \partial_{j+1}^D : Z_i(C_\bullet) \otimes D_{j+1} \rightarrow Z_i(C_\bullet) \otimes D_j) = Z_i(C_\bullet) \otimes B_j(D_\bullet).$$

In particular, since (co)kernels for graded modules are computed degreewise, we have the equalities

$$(Z(C_\bullet) \otimes Z(D_\bullet))_n = \ker(\text{Id} \otimes \partial^D : (Z(C_\bullet) \otimes D_\bullet)_n \rightarrow (Z(C_\bullet) \otimes D_\bullet)_{n-1}),$$

and

$$(Z(C_\bullet) \otimes B(D_\bullet))_n = \text{im}(\text{Id} \otimes \partial^D : (Z(C_\bullet) \otimes D_\bullet)_{n+1} \rightarrow (Z(C_\bullet) \otimes D_\bullet)_n).$$

Moreover, tensoring the short exact sequence  $0 \rightarrow B_j(D_\bullet) \rightarrow Z_j(D_\bullet) \rightarrow H_j(D_\bullet) \rightarrow 0$  by  $Z_i(C_\bullet)$  gives a short exact sequence, and consequently, by the first isomorphism theorem,

$$\frac{Z_i(C_\bullet) \otimes Z_j(D_\bullet)}{Z_i(C_\bullet) \otimes B_j(D_\bullet)} \cong Z_i(C_\bullet) \otimes H_j(D_\bullet).$$

This implies that

$$H(Z(C_\bullet) \otimes D_\bullet) = Z(C_\bullet) \otimes H(D_\bullet),$$

as graded modules. By a similar argument, since  $B(C_\bullet)$  is a free (graded) module, we have

$$H(B(C_\bullet) \otimes D_\bullet) = B(C_\bullet) \otimes H(D_\bullet)$$

as graded modules. We now have a *degreewise free* resolution

$$0 \rightarrow B(C_\bullet) \rightarrow Z(C_\bullet) \rightarrow H(C_\bullet) \rightarrow 0.$$

Tensoring by  $H(D_\bullet)$ , and taking homology, we have the exact sequence

$$0 \rightarrow \text{Tor}(H(C_\bullet), H(D_\bullet)) \rightarrow B(C_\bullet) \otimes H(D_\bullet) \rightarrow Z(C_\bullet) \otimes H(D_\bullet) \rightarrow H(C_\bullet) \otimes H(D_\bullet) \rightarrow 0.$$

On the other hand, we have the *split* short exact sequence of chain complex of free  $R$ -modules

$$0 \rightarrow Z(C_\bullet) \rightarrow C_\bullet \rightarrow B(C_\bullet)[-1] \rightarrow 0.$$

Tensoring by  $D_\bullet$  gives the (split) short exact sequence

$$0 \rightarrow Z(C_\bullet) \otimes D_\bullet \rightarrow C_\bullet \otimes D_\bullet \rightarrow (B(C_\bullet) \otimes D_\bullet)[-1] \rightarrow 0.$$

Passing to the long exact sequence in homology, and identifying the (co)kernel of the maps  $\iota \otimes \text{Id} : B(C_\bullet) \otimes D_\bullet \rightarrow Z(C_\bullet) \otimes D_\bullet$ , we get the short exact sequence

$$0 \rightarrow H(C_\bullet) \otimes H(D_\bullet) \rightarrow H(C_\bullet \otimes D_\bullet) \rightarrow \text{Tor}(H(C_\bullet), H(D_\bullet))[-1] \rightarrow 0.$$

Since  $\text{Tor}$  is additive, we get the required short exact sequence.

Finally, assume  $D_\bullet$  is a chain complex of free  $R$ -modules. Then, we can *choose* retractions  $r_\bullet : C_\bullet \rightarrow Z_n(C_\bullet)$  and  $s_\bullet : D_\bullet \rightarrow Z_n(D_\bullet)$ . This induces a map  $(C_\bullet \otimes D_\bullet)_n \rightarrow H(C_\bullet) \otimes H(D_\bullet)$  given by  $c \otimes d \mapsto [r(c)] \otimes [s(d)]$ . One can verify that this map sends boundaries in  $C_\bullet \otimes D_\bullet$  to 0, and hence, induces a map  $\rho : H_n(C_\bullet \otimes D_\bullet) \rightarrow H(C_\bullet) \otimes H(D_\bullet)$ . Clearly,  $\rho$  is a retraction in the short exact sequence, which then splits. As  $r_\bullet, s_\bullet$  are choices, it follows that the splitting may not be natural.  $\square$

In the above theorem we have freely used homological algebraic machinery developed for  $R$ -modules to graded  $R$ -modules by degreewise consideration. This naive approach does not work when we consider chain complexes with nontrivial differentials.

### Remark 18.2: (Künneth Theorems for Spaces)

Can we directly use [Theorem 18.1](#) for spaces to compute homology of product?! Suppose  $X, Y$  are two spaces, and denote  $C_\bullet = S_\bullet(X), D_\bullet = S_\bullet(Y)$  to be the singular chain complexes (or the cellular ones, if we are working with CW complexes). Then, Künneth theorem is applicable for  $C_\bullet, D_\bullet$ , and we are able to compute  $H(C_\bullet \otimes D_\bullet) = H(S_\bullet(X) \otimes S_\bullet(Y))$ . On the other hand, the homology of the product  $X \times$

$Y$  is given as  $H(S_\bullet(X \times Y))$ . The challenge is then to relate  $S_\bullet(X) \otimes S_\bullet(Y)$  with  $S_\bullet(X \times Y)$ ! We shall see later that these two chain complexes are in fact quasi-isomorphic.

Similar to [Theorem 17.9](#), we have the following.

**Theorem 18.3:** (*Künneth Theorem for Cohomology*)

Let  $R$  be a PID, and  $C_\bullet, D_\bullet$  be cochain complexes of free  $R$ -modules. Assume that either  $H(C_\bullet)$  or  $H(D_\bullet)$  is of finite type. Then, for the dual complex  $C^\bullet = \text{hom}(C_\bullet, R), D^\bullet = \text{hom}(D_\bullet, R)$  there exists a natural short exact sequence

$$0 \rightarrow \bigoplus_{i+j=n} H^i(C^\bullet) \otimes H^j(D^\bullet) \rightarrow H^n(C^\bullet \otimes D^\bullet) \rightarrow \bigoplus_{i+j=n+1} \text{Tor}(H^i(C^\bullet), H^j(D^\bullet)) \rightarrow 0,$$

which split.

## 18.2 Künneth Theorem for Spaces

Fix a PID  $R$ , and denote by  $\text{Ch}_{\geq 0}$  the category of chain complexes of  $R$ -modules, with 0 in the strictly negative degrees. We then have two functors

$$\begin{aligned} \mathcal{F} : \text{Top} \times \text{Top} &\rightarrow \text{Ch}_{\geq 0} & \mathcal{G} : \text{Top} \times \text{Top} &\rightarrow \text{Ch}_{\geq 0} \\ (X, Y) &\rightarrow S_\bullet(X) \otimes S_\bullet(Y), & (X, Y) &\rightarrow S_\bullet(X \times Y). \end{aligned}$$

Here,  $S_\bullet(\_)$  is the singular chain complex functor with  $R$ -coefficients. It is easy to see that

$$S_0(X) \otimes S_0(Y) = R\langle X \rangle \otimes R\langle Y \rangle \cong R\langle X \times Y \rangle = S_0(X \times Y).$$

Indeed, 0-simplicies are nothing but points, and hence, sending the 0-simplices  $\sigma_x \in S_0(X), \sigma_y \in S_0(Y)$  to the simplex  $\sigma_{x,y} \in S_0(X \times Y)$  defines a natural isomorphism, with inverse given by  $\sigma_{x,y} \mapsto \sigma_x \otimes \sigma_y$ . But this strategy does not generalize immediately.

**Definition 18.4:** (*Eilenberg-Zilber Maps*)

A pair of natural transformations

$$\mathfrak{P} : S_\bullet(\_) \otimes S_\bullet(\_) \Rightarrow S_\bullet(\_ \times \_) \quad \text{and} \quad \mathfrak{Q} : S_\bullet(\_ \times \_) \Rightarrow S_\bullet(\_) \otimes S_\bullet(\_)$$

are called *Eilenberg-Zilber maps* (or *EZ-maps*) if they extend the canonical isomorphism at the 0<sup>th</sup> level.

By an application of the *acyclic model theorem*, one gets the following.

**Theorem 18.5:** (*Eilenberg-Zilber Theorem*)

EZ-maps exist. Moreover, the following holds.

- For any pair  $(\mathfrak{P}, \mathfrak{Q})$  of EZ-maps, we have  $\mathfrak{P} \circ \mathfrak{Q}$  and  $\mathfrak{Q} \circ \mathfrak{P}$  are naturally chain homotopic to identity. In particular,  $\mathfrak{P}_{X,Y}$  and  $\mathfrak{Q}_{X,Y}$  are natural chain equivalences.
- Given two pairs  $(\mathfrak{P}, \mathfrak{Q})$  and  $(\mathfrak{P}', \mathfrak{Q}')$  of EZ-maps, we have  $\mathfrak{P}, \mathfrak{P}'$  (and similarly,  $\mathfrak{Q}, \mathfrak{Q}'$ ) are naturally chain homotopic.
- An EZ map  $\mathfrak{P}$  is

- ▶ *associative* up to natural homotopy, i.e, the following diagram commutes up to natural chain homotopy:

$$\begin{array}{ccc}
 S_{\bullet}(X) \otimes S_{\bullet}(Y) \otimes S_{\bullet}(Z) & \xrightarrow{\mathfrak{P}_{X,Y} \otimes \text{Id}} & S_{\bullet}(X \times Y) \otimes S_{\bullet}(Z) \\
 \text{Id} \otimes \mathfrak{P}_{Y,Z} \downarrow & & \downarrow \mathfrak{P}_{X \times Y, Z} \\
 S_{\bullet}(X) \otimes S_{\bullet}(Y \times Z) & \xrightarrow{\mathfrak{P}_{X, Y \times Z}} & S_{\bullet}(X \times Y \times Z)
 \end{array}$$

- ▶ *commutative* up to natural homotopy, i.e, the following diagram commutes up to natural chain homotopy:

$$\begin{array}{ccc}
 S_{\bullet}(X) \otimes S_{\bullet}(Y) & \xrightarrow{\tau_{X,Y}} & S_{\bullet}(Y) \otimes S_{\bullet}(X) \\
 \mathfrak{P}_{X,Y} \downarrow & & \downarrow \mathfrak{P}_{Y,X} \\
 S_{\bullet}(X \times Y) & \xrightarrow{S_{\bullet}(t_{X,Y})} & S_{\bullet}(Y \times X)
 \end{array}$$

- An EZ map  $\Omega$  is

- ▶ *coassociative* up to natural homotopy, i.e, the following diagram commutes up to natural chain homotopy:

$$\begin{array}{ccc}
 S_{\bullet}(X \times Y \times Z) & \xrightarrow{\Omega_{X \times Y, Z} \otimes \text{Id}} & S_{\bullet}(X \times Y) \otimes S_{\bullet}(Z) \\
 \Omega_{X, Y \times Z} \downarrow & & \downarrow \Omega_{X, Y} \otimes \text{Id} \\
 S_{\bullet}(X) \otimes S_{\bullet}(Y \times Z) & \xrightarrow{\text{Id} \otimes \Omega_{Y, Z}} & S_{\bullet}(X) \otimes S_{\bullet}(Y) \otimes S_{\bullet}(Z)
 \end{array}$$

- ▶ *cocommutative* up to natural homotopy, i.e, the following diagram commutes up to natural chain homotopy:

$$\begin{array}{ccc}
 S_{\bullet}(X \times Y) & \xrightarrow{S_{\bullet}(t_{X,Y})} & S_{\bullet}(Y \times X) \\
 \Omega_{X,Y} \downarrow & & \downarrow \Omega_{Y,X} \\
 S_{\bullet}(X) \otimes S_{\bullet}(Y) & \xrightarrow{\tau_{X,Y}} & S_{\bullet}(Y) \otimes S_{\bullet}(X)
 \end{array}$$

Here,  $t_{X,Y} : X \times Y \rightarrow Y \times X$  is natural flip map, and  $\tau_{X,Y} : S_{\bullet}(X) \otimes S_{\bullet}(Y) \rightarrow S_{\bullet}(Y) \otimes S_{\bullet}(X)$  is given by  $\tau_{X,Y}(\alpha \otimes \beta) = (-1)^{|\alpha||\beta|} \beta \otimes \alpha$ .

Although [Theorem 18.5](#) is existential in nature, there are explicit constructions of EZ-maps, which were found out later by Eilenberg and MacLane.

**Definition 18.6:** (Alexander-Whitney Map)

Given spaces  $X, Y$ , the *Alexander-Whitney maps*

$$AW_{X,Y} : S_{\bullet}(X \times Y) \rightarrow S_{\bullet}(X) \otimes S_{\bullet}(Y)$$

are defined on an  $n$ -simplex  $\sigma : \Delta^n \rightarrow X \times Y$  by the formula

$$AW(\sigma) = \sum_{p+q=n} \pi_X \circ \sigma \circ \lambda_p^n \otimes \pi_Y \circ \sigma \circ \rho_q^n,$$

where  $\lambda_p^n : \Delta^p \rightarrow \Delta^n$  and  $\rho_q^n : \Delta^q \rightarrow \Delta^n$  are defined linearly as  $\lambda_p^n(e_i) = e_i$  for  $0 \leq i \leq p$ , and  $\rho_q^n(e_j) = e_{n-q+j}$  for  $0 \leq j \leq q$ . Intuitively,  $\lambda_p^n$  embeds  $\Delta^p$  as the *front*  $p$ -face of  $\Delta^n$ , and  $\rho_q^n$  embeds  $\Delta^q$  as the *back*  $q$ -face of  $\Delta^n$ .

One can verify that  $AW$  is a natural chain map, extending the natural isomorphism at the  $0^{\text{th}}$  level, i.e,  $AW$  is an EZ map. Moreover,  $AW$  is *strictly* coassociative.

**Definition 18.7:** (Eilenberg-MacLane Map)

Given spaces  $X, Y$  the (explicit) *Eilenberg-Zilber maps* (or *Eilenberg-MacLane formula*)

$$EM_{X,Y} : S_{\bullet}(X) \otimes S_{\bullet}(Y) \rightarrow S_{\bullet}(X \times Y)$$

are defined via *shuffle products*. Explicitly, treat  $\Delta^n \subset \mathbb{R}^n$  as the subset  $\Delta^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \leq x_1 \leq \dots \leq x_n \leq 1\}$ . Recall, a  $(p, q)$ -shuffle  $\sigma$  is a bijection  $\sigma : \{1, \dots, p+q\} \rightarrow \{1, \dots, p+q\}$  such that  $\sigma(1) \leq \dots \leq \sigma(p)$  and  $\sigma(p+1) \leq \dots \leq \sigma(p+q)$ ; thus  $\text{Sh}(p, q) \subset S_{p+q}$  is a subset of permutations and in particular the signum function  $\text{sgn}(\sigma)$  makes sense. Given a  $(p, q)$ -shuffle  $\sigma$ , consider the linear embedding  $\ell_{\sigma} : \Delta^{p+q} \rightarrow \Delta^p \times \Delta^q$  given by

$$\ell_{\sigma}(x_1, \dots, x_p, x_{p+1}, \dots, x_{p+q}) = \left( (x_{\sigma(1)}, \dots, x_{\sigma(p)}), (x_{\sigma(p+1)}, \dots, x_{\sigma(p+q)}) \right).$$

Then, for  $\alpha : \Delta^p \rightarrow X, \beta : \Delta^q \rightarrow Y$  one can define

$$EM(\alpha \otimes \beta) = \sum_{\sigma \in \text{Sh}(p,q)} \text{sgn}(\sigma)(\alpha \times \beta) \circ \ell_{\sigma}.$$

Again, it can be verified that  $EM$  is a natural chain map, extending the natural isomorphism at the  $0^{\text{th}}$  level, making it an EZ-map! Moreover,  $EM$  is strictly coassociative. Note that we changed the definition of simplices, but they are naturally homeomorphic to [Definition 6.1](#), which lets us translate the definition of  $EM$  back to the usual singular chain complexes where the formula becomes even more complicated. There are other combinatorial ways to define such maps, e.g, by identifying a simplex with paths in an integer grids.

**Example 18.8:** (EZ-map from AW)

The composition

$$\begin{array}{ccc}
S_{\bullet}(X \times Y) & \xrightarrow{\text{EZ}_{X,Y}} & S_{\bullet}(X) \otimes S_{\bullet}(Y) \\
\downarrow S_{\bullet}(\Delta_{X \times Y}) & & \uparrow S_{\bullet}(\pi_X) \otimes S_{\bullet}(\pi_Y) \\
S_{\bullet}(X \times Y \times X \times Y) & \xrightarrow{\text{AW}_{X \times Y, X \times Y}} & S_{\bullet}(X \times Y) \otimes S_{\bullet}(X \times Y)
\end{array}$$

defines a strictly associative EZ-map as well. In fact, for any EZ-map  $S_{\bullet}(- \times -) \Rightarrow S_{\bullet}(-) \otimes S_{\bullet}(-)$  the above composition gives a EZ-map, which may not be strictly coassociative.

As a consequence of [Theorem 18.5](#), we can now get the Künneth theorem for spaces.

**Theorem 18.9:** (*Künneth Theorem for Singular Homology*)

Let  $R$  be a PID, and  $X, Y$  be spaces. Then, there exists a natural short exact sequence

$$0 \rightarrow \bigoplus_{i+j=n} H_i(X; R) \otimes H_j(Y; R) \rightarrow H_n(X \times Y; R) \rightarrow \bigoplus_{i+j=n-1} \text{Tor}(H_i(X; R), H_j(Y; R)) \rightarrow 0,$$

which is split, but not naturally.

**Proof :** Proof is immediate from the Künneth theorem in homological algebra ([Theorem 18.1](#)) and the Eilenberg-Zilber theorem ([Theorem 18.5](#)). Indeed, naturality of the short exact sequence lets us replace  $S_{\bullet}(X) \otimes S_{\bullet}(Y)$  by the naturally chain equivalent complex  $S_{\bullet}(X \times Y)$ . □

The first map in the Künneth theorem gives a product structure on singular homology.

**Definition 18.10:** (*Homology Cross Product*)

Given two spaces  $X, Y$ , the *homology cross product* for homology with coefficients in some ring is defined as the composition

$$\times : H_p(X) \otimes H_q(Y) \rightarrow H_{p+q}(S_{\bullet}(X) \otimes S_{\bullet}(Y)) \xrightarrow{\cong} H_{p+q}(X \times Y),$$

where the natural isomorphism is via an EZ-map.

Explicitly, given  $\sigma : \Delta^p \rightarrow X, \tau : \Delta^q \rightarrow Y$ , and a choice of an EZ-map  $\Omega$ , we have the element  $\Omega_{X,Y}(\sigma \otimes \tau) \in S_{p+q}(X \times Y)$ , which induces the cross product in homology.

In order to get the Künneth theorem in singular cohomology, we need to take some extra assumptions.

**Theorem 18.11:** (*Künneth Theorem for Singular Cohomology*)

Let  $R$  be a PID, and  $X, Y$  be spaces. Suppose that either  $H_{\bullet}(X; R)$  or  $H_{\bullet}(Y; R)$  are of finite type. Then, there exists a natural short exact sequence

$$0 \rightarrow \bigoplus_{i+j=n} H^i(X; R) \otimes H^j(Y; R) \rightarrow H^n(X \times Y; R) \rightarrow \bigoplus_{i+j=n+1} \text{Tor}(H^i(X; R), H^j(Y; R)) \rightarrow 0,$$

which splits, but not naturally.

**Proof :** The proof follows from the algebraic Künneth theorem for cohomology ([Theorem 18.3](#)), and the Eilenber-Zilber theorem ([Theorem 18.5](#)). □

**Remark 18.12:** (*Relative Künneth Theorems*)

There are appropriate versions of Künneth theorems for *relative* singular (co)homology as well.