Course notes for

Algebraic Topology II (KSM4E02)

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`universal coefficient theorem for cohomology – singular (co)homology with coefficients – universal coefficient theorem for singular (co)homology – graded modules – tensor of chain complexes – Koszul sign rule`

17.1 Universal Coefficient Theorems (Cont.)

Similar to Theorem 16.18, we now prove a universal coefficient theorem for cohomology, which lets us compute the cohomology with coefficients from the homology.

Theorem 17.1: (*UCT for Cohomology*)

Let R be a PID, $C_\bullet$ be a chain complex of free R-modules, and M be an arbitray R-module. Then, there exists a short exact sequence

$$0 \to \mathrm{Ext}\ (H_{n-1}(C_\bullet), M) \to H^n(\hom_R(C_\bullet, M)) \to \hom_R(H_n(C_\bullet), M) \to 0,$$

which is natural in both $C_\bullet$ and the coefficient module M. Moreover, the short exact sequence splits, the splitting is natural in the coefficient module, but may not be natural with respect to the chain complex.

Proof : Let us again denote the modules

$$Z_n := \ker\left(C_n \xrightarrow{\partial} C_{n-1}\right), \qquad B_n := \mathrm{im}\left(C_{n+1} \xrightarrow{\partial} C_n\right), \qquad H_n := H_n(C_\bullet) = B_n/Z_n.$$

We have the short exact sequence

$$0 \to Z_n \to C_n \to B_{n-1} \to 0,$$

which is split exact as R is a PID. Applying $\hom_R(\_, M)$, we have a split exact sequence

$$0 \to \hom_R(B_{n-1}, M) \to \hom_R(C_n, M) \to \hom_R(Z_n, M) \to 0.$$

Considering $\hom_R(Z_\bullet, M)$ and $\hom_R(B_\bullet, M)$ as cochain complexes with 0-codifferentials, we have a short exact sequence of cochain complexes

$$0 \to \hom_R(B_\bullet, M)[-1] \to \hom_R(C_\bullet, M) \to \hom_R(Z_\bullet, M) \to 0.$$

Applying Theorem 7.1, we have a long exact sequence in cohomology

$$\cdots \to \hom_R(B_{n-1}, M) \to H^n(\hom_R(C_\bullet, M)) \to \hom_R(Z_n, M) \to \hom_R(B_n, M) \to H^{n+1}(\hom_R(C_\bullet, M)) \to \cdots.$$

The boundary map can be easily identified with $\iota^* : \hom_R(Z_n, M) \to \hom_R(B_n, M)$, where $\iota : Z_n \hookrightarrow B_n$ is the inclusion map. This leads to the short exact sequences

$$0 \to \operatorname{coker}(i^\star) \to H^n(\hom_R(C_\bullet, M)) \to \ker(\iota^\star) \to 0.$$

Now, we have another short exact sequence

$$0 \to B_n \xrightarrow{\iota} Z_n \to H_n \to 0,$$

which can be treates a free resolution of H_n. This gives,

$$\operatorname{Ext}(H_n, M) \cong \operatorname{coker}(\iota^\star), \qquad \ker(\iota^\star) = \hom_R(H_n, M).$$

Hence, we have the short exact sequence

$$0 \to \operatorname{Ext}(H_{n-1}, M) \to H^n(\hom_R(C_\bullet, M)) \to \hom_R(H_n, M) \to 0,$$

which is the required exact sequence. The sequence is natural as Ext is natural. A splitting can be induced from a choice of a splitting for $0 \to Z_n \to C_n \to B_{n-1} \to 0$; the splitting is natural with respect to the coefficient module, but may not be natural with respect to the chain complex. □

Exercise 17.2: (*UCT for Projectives*)

State and prove the UCT for (co)homology of a chain complex of *projective* R-modules, where R is a PID.

Exercise 17.3: (*Unnaturality of the Splitting in Cohomology UCT*)

Give an example to demonstrate that the splitting in Theorem 17.1 may not be natural with respect to the chain complex.

Remark 17.4: (*The Integral Morphism*)

The natural map $H^n(\hom_R(C_\bullet, M)) \to \hom_R(H_n(C_\bullet), M)$ in Theorem 17.1 is induced by

$$\int : \hom_R(C_n, M) \to \hom_R(H_n(C_\bullet), M)$$
$$\varphi \mapsto \left([\sigma] \mapsto \int_\sigma \varphi = \varphi(\sigma)\right).$$

The reason to denote this map by the *integral* symbol becomes apparent when the cohomology theory is the deRham cohomology and the homology is the singular homology with compact support.

Let us state the universal coefficient theorem for cochain complexes.

Theorem 17.5: (*UCT for Cochain Complex*)

Let R be a PID, $C^\bullet$ be a cochain complex of free R-modules, and M be an arbitrary R-module. Then, there exists a short exact sequence

$$0 \to H^n(C^\bullet) \otimes M \to H^n(C^\bullet \otimes M) \to \mathrm{Tor}(H^{n+1}(C^\bullet), M) \to 0,$$

which is natural in both $C^\bullet$ and the coefficient module M. Moreover, the short exact sequence splits, the splitting is natural in the coefficient module, but may not be natural with respect to the chain complex.

Proof : We can identify the cochain complex (C^n, δ^n) as the chain complex (C_n, ∂_n), where $C_n = C^{-n}$ and $\partial_n : C_n \to C_{n-1}$ as $\delta^{-n} : C^{-n} \to C^{-n+1}$. The proof is then immediate from Theorem 16.18. □

In order to compute the cohomology with coefficients in a module from cohomology without coefficients, we need some extra hypothesis.

Definition 17.6: (*Finite Type Chain Complex*)

A chain complex $C_\bullet$ of R-module is called *finite type* if C_n is a finitely generated R-module for each n.

Exercise 17.7: ($\hom$ *Commutes with Tensor if Finitely Generated*)

Let R be a commutative ring and $C, M \in R\text{-Mod}$. Shoh that there exists a canonical isomorphism

$$\hom_R(C, M) \cong \hom_R(C, R) \otimes M,$$

provided either C or M is finitely generated R-module.

We have the following *rectification* result.

Proposition 17.8: (*Finite Type Homology*)

Let R be a PID, and $C_\bullet$ be a chain complex of free R-modules such that $H_n(C_\bullet)$ is finitely generated for each n. Then, there exists a chain complex $C'_\bullet$ of finite type, such that $C_\bullet$ is chain homotopy equivalent to $C'_\bullet$.

Theorem 17.9: (*UCT for Cohomology with Coefficients*)

Let R be a PID, $C_\bullet$ be a chain complex of free R-modules, and M be an R-module. Asuume that either $H_n(C_\bullet)$ is finitely generated for each n, or that M is finitely generated. Then, there exists a short exact sequence

$$0 \to H^n(\hom_R(C_\bullet, R)) \otimes M \to H^n(\hom_R(C_\bullet, M)) \to \mathrm{Tor}(H^{n+1}(\hom_R(C_\bullet, R)), M) \to 0,$$

which is natural in both $C_\bullet$ and the coefficient module M (whenever the additional hypothesis holds). Moreover, the sequence splits, the splitting is natural in the coefficient module.

Proof : If M is finitely generated, then we have a natural isomorphism

$$\hom_R(C_\bullet, R) \otimes M \cong \hom_R(C_\bullet, M).$$

We can then use Theorem 17.5. If $H_n(C_\bullet)$ is finitely generated for each n, then using Proposition 17.8, we can get a free chain complex $C'_\bullet$ of finite type such that $C'_\bullet$ is chain homotopy equivalent to $C_\bullet$. Again, we have the isomorphism $\hom_R(C'_\bullet, R) \otimes M \cong \hom_R(C'_\bullet, M)$, and using Theorem 17.5 we get the short exact sequence for $C'_\bullet$. Since all the functors involved is invariant for chain homotopy equivalence, we have the required short exact sequence for $C_\bullet$. ☐

17.2 Singular (Co)Homology With Coefficients

Let R be a PID. Then, given a space X, one can define the singular chain complex

$$S_n(X; R) = \text{Free } R\text{-module generated by singular } n\text{-simplices}.$$

This can also be understood as

$$S_n(X; R) = S_n(X) \otimes_{\mathbb{Z}} R.$$

Next, for an R-module M, we have the singular chain complex with coefficients in M as

$$S_n(X; M) := S_n(X; R) \otimes M.$$

The homology groups of $S_n(X; M)$ are called the *singular homology with coefficients in M*, and is denoted as $H_n(X; M)$. Similarly, we have singular cochain complex with coefficients in M defined as

$$S^n(X; M) := \hom_R(S_n(X; R), M).$$

The cohomology groups are called *singular cohomology with coefficients in M*, and is denoted as $H^n(X; M)$. Using the universal coefficient theorems, we can now derive the following theorems.

Theorem 17.10: (*UCT for Singular Homology with Coefficients*)

Let R be a PID, and M be an R-module. Then, for a space X, there exists a short exact sequence

$$0 \to H_n(X; R) \otimes M \to H_n(X; M) \to \operatorname{Tor}(H_{n-1}(X; R), M) \to 0,$$

which is natural in both X and in M. Moreover, the sequence splits, the splitting being natural in M but possibly not in X.

Proof : Proof is immediate from Theorem 16.18. ☐

Theorem 17.11: (*UCT for Singular Cohomology*)

Let R be a PID, and M be an R-module. Then, for a space X, there exists a short exact sequence

$$0 \to \operatorname{Ext}(H_{n-1}(X; R), M) \to H^n(X; M) \to \hom_R(H_n(X; R), M) \to 0,$$

which is natural in both X and in M. Moreover, the sequence splits, the splitting being natural in M but possibly not in X.

Proof : Proof is immediate from Theorem 17.1. ☐

Theorem 17.12: (*UCT for Singular Cohomology with Coefficients*)

Let R be a PID, M be an R-module, and X be a space. Suppose that either M is finitely genrated, or that each $H_n(X;R)$ is finitely generated. Then, for a space X, there exists a short exact sequence

$$0 \to H^n(X;R) \otimes M \to H^n(X;M) \to \operatorname{Tor}(H^{n+1}(X;R), M) \to 0,$$

which is natural in both X and in M. Moreover, the sequence splits, the splitting being natural in M but possibly not in X.

Proof : Proof is immediate from Theorem 17.9. ☐

17.3 Graded Modules and Tensor Product

Let us recall the notions of graded ring and graded modules over it.

Definition 17.13: (*Graded Ring*)

A *$\mathbb{Z}$-graded ring* R is a ring equipped with a direct sum decomposition $R = \bigoplus_{i\in\mathbb{Z}} R_i$, such that each R_i is an ideal, and $R_i \cdot R_j \subset R_{i+j}$.

Example 17.14: (*Example of Graded Rings*)

Here a few examples of graded ring that appears naturally.

- A prototypical example of a graded ring is the ring of polynomials $R[T]$, which is graded by degree of the polynomial (with 0 in the negative degrees); i.e, $R[T] = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} R\langle T^i\rangle$.
- The *Laurent polynomial ring* $R[T, T^{-1}]$ (obtained by localizing $R[T]$ at T) is also a graded ring: $R[T, T^{-1}] = \bigoplus_{i\in\mathbb{Z}} R\langle T^i\rangle$.
- Any ring R can be understood as a graded ring with $R_0 = R$ and $R_i = 0$ for all $i \neq 0$.

Definition 17.15: (*Graded Module*)

Given a graded ring R, a *graded module* over R is an R-module M equipped with a direct sum decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$, such that $R_i \cdot M_j \subset M_{i+j}$ holds. An R-module map $f : M \to N$ of graded modules is given as the sum $f = \sum_{i\in\mathbb{Z}} f_i$, where $f_i : M_i \to N_i$. More generally, a *degree-k graded map* is an R-module map $f : M \to N$, where $f = \sum f_i$ for $f_i : M_i \to N_{i+k}$.

Note that a graded module can be thought of as a chain complex with 0-differential, and then a degree k map is precisely a degree k chain map. The homological algebra we developed so far carries over to the category of graded modules with graded map (of degree 0).

Exercise 17.16: (*Projective and Injective Graded Modules*)

Let R be a ring. Justify the following.

1. A graded R-module $P = \bigoplus_{i\in\mathbb{Z}} P_i$ is a projective R-module if and only if each P_i is a projective R-module
2. A graded R-module $I = \bigoplus_{i\in\mathbb{Z}} I_i$ is a projective R-module if and only if each I_i is an injective R-module

Definition 17.17: (*Tensor Product of Graded Modules*)

Given graded modules M, N over a graded ring R, their tensor product is the R-module $M \otimes_R N$ (computed by forgetting the grading) with the grading given by

$$(M \otimes N)_k = \frac{\bigoplus_{p+q=k} M_p \otimes_{\mathbb{Z}} N_q}{\langle mr \otimes n - m \otimes rn \mid m \in M_a, r \in R_b, n \in N_c, \quad a+b+c=n\rangle}.$$

For a usual ring R (with grading concentrated at 0), this simplifies to

$$(M \otimes N)_k = \bigoplus_{p+q=k} M_p \otimes_R N_q.$$

Now, given two chain complexes, we can take tensor product of the underlying graded modules. We need to define a differential on it; of course we can 0 as the differential, but that does not give it the desired properties!

Definition 17.18: (*Tensor Product of Chain Complexes*)

Let $(C_\bullet, \partial_\bullet^C), (D_\bullet, \partial_n^D)$ be two chain complexes of R-modules (or more generaly, of objects in any monoidal Abelian category). Then, the *tensor product* is defined as the chain complex $\left((C \otimes D)_\bullet, \partial_\bullet^{C\otimes D}\right)$, where

$$(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j,$$

and the differential is defined on *homogeneous* degree by the formula

$$\partial_n^{C\otimes D}(x \otimes y) = \partial_i^C(x) \otimes y + (-1)^i x \otimes \partial_{j(y)}^D, \qquad x \otimes y \in C_i \otimes D_j, \quad i+j=n.$$

Exercise 17.19: (*Boundary of Tensor Product*)

Verify that the boundary map $\partial_\bullet^{C\otimes D}$ squares to zero, so that $\left((C \otimes D)_\bullet, \partial_\bullet^{C\otimes D}\right)$ is indeed a chain complex.

Remark 17.20: (*Tensor-*hom *Adjunction*)

Let R be a module, and $(D_\bullet, \partial_\bullet^D)$ be a chain complex of R-modules. Then the functor $\_ \bigotimes D_\bullet : \mathrm{Ch} \to \mathrm{Ch}$ is left adjoint to the internal hom-functor $[D_\bullet, \_] : \mathrm{Ch} \to \mathrm{Ch}$. That is, there is a natural isomorphism

$$\hom_{\mathrm{Ch}}(C \otimes D, E) \cong \hom_{\mathrm{Ch}}(C, [D, E]), \quad \forall \quad C, E \in \mathrm{Ch}.$$

Thus, the tensor product is actually defined via an universal property.

Let us finally mention a rule that governs the signs appearing everywhere!

Remark 17.21: (*Koszul Sign Rule*)

The boundary map in the tensor product can be defined symbolically as

$$\partial_n^{C \otimes D} = \sum_{i+j=n} \partial_i^C \otimes \mathrm{Id}_{D_j} + \mathrm{Id}_{C_i} \otimes \partial_j^D.$$

The appearance of the sign can be justified by the *Koszul sign rule*. Suppose $A = \bigoplus_{n \in \mathbb{Z}} A_n$, $B = \bigoplus_{n \in \mathbb{Z}} B_n$ are graded objects, and $f, g : A \to B$ are graded maps. We define $f \otimes g : A \otimes A \to B \otimes B$ as the sum $\sum_{i+j=n} f_i \otimes g_j$. The sign appears whenever we *evaluate* a tensor of maps on an element :

$$(f \otimes g)(x \otimes y) := (-1)^{|x||g|} f(x) \otimes g(y).$$

Here, $|x|$ denotes the homogeneous degree of x, and $|g|$ denotes the degree of the map.

Heurestically, whenever g jumps over an element x, it collects the sign $(-1)^{|x||g|}$. It generalizes to multiple tensors as well. The sign rule is designed in such way that boundary map squares to zero in any graded context.

Example 17.22: (*The Tensor Product Boundary Formula*)

As mentioned in Remark 17.21, the boundary formula can be written as

$$\partial_n^{C \otimes D} = \sum_{i+j=n} \partial_i^C \otimes \mathrm{Id}_{D_j} + \mathrm{Id}_{C_i} \otimes \partial_j^D.$$

Let us justify that it squares to 0 using the Koszul sign rule. We have

$$\begin{aligned}
\partial_n^{C \otimes D} \circ \partial_{n+1}^{C \otimes D} &= \left(\sum_{i+j=n} \partial_i^C \otimes \mathrm{Id}_{D_j} + \mathrm{Id}_{C_i} \otimes \partial_j^D \right) \circ \left(\sum_{i+j=n+1} \partial_i^C \otimes \mathrm{Id}_{D_j} + \mathrm{Id}_{C_i} \otimes \partial_j^D \right) \\
&= \sum_{i+j=n} \left[\left(\partial_i^C \otimes \mathrm{Id}_{D_j} \right) \circ \left(\mathrm{Id}_{C_i} \otimes \partial_{j+1}^D \right) + \left(\mathrm{Id}_{C_i} \otimes \partial_j^D \right) \circ \left(\partial_{i+1}^C \otimes \mathrm{Id}_{D_j} \right) \right] \\
&= \sum_{i+j=n} \left[(-1)^{\overbrace{0}^{|\mathrm{Id}_{D_j}|} \cdot \overbrace{0}^{|\mathrm{Id}_{C_i}|}} \partial_i^C \otimes \partial_{j+1}^D + (-1)^{\overbrace{-1}^{|\partial_j^D|} \cdot \overbrace{-1}^{|\partial_{i+1}^C|}} \partial_{i+1}^C \otimes \partial_j^D \right] \\
&= \sum_{i+j=n} \partial_i^C \otimes \partial_{j+1}^D - \sum_{i+j=n} \partial_{i+1}^C \otimes \partial_j^D \\
&= \sum_{i+j=n+1} \partial_i^C \otimes \partial_j^D - \sum_{i+j=n+1} \partial_i^C \otimes \partial_j^D \\
&= 0.
\end{aligned}$$

Note that any other composition is automatically 0, and we can ignore the signs.

As another example, recall the internal hom given as $[C, D]_n := \prod_{k \in \mathbb{Z}} \mathrm{hom}(X_k, Y_{k+n})$, which consists of degree n maps between the underlying graded modules $C_\bullet, D_\bullet$. The boundary map $d_n : [C, D]_n \to [C, D]_{n-1}$ is given via the formula

$$d_n\left((h_k)_{k \in \mathbb{Z}}\right) = \left(\partial^D_{k+n} \circ h_k - (-1)^n h_{k-1} \circ \partial^C_k\right)_{k \in \mathbb{Z}}.$$

The sign is justified since we are swapping the degree n map (h_k) and the degree -1 boundary map.