

Algebraic Topology II (KSM4E02)

Instructor: Aritra Bhowmick

Day 16 : 31st March, 2026

Tor via flat resolution – right derived functors – Ext functors – injective Abelian groups – injective R -modules – universal coefficient theorems

16.1 A Digression : Left Derived Functor by Flat Resolution

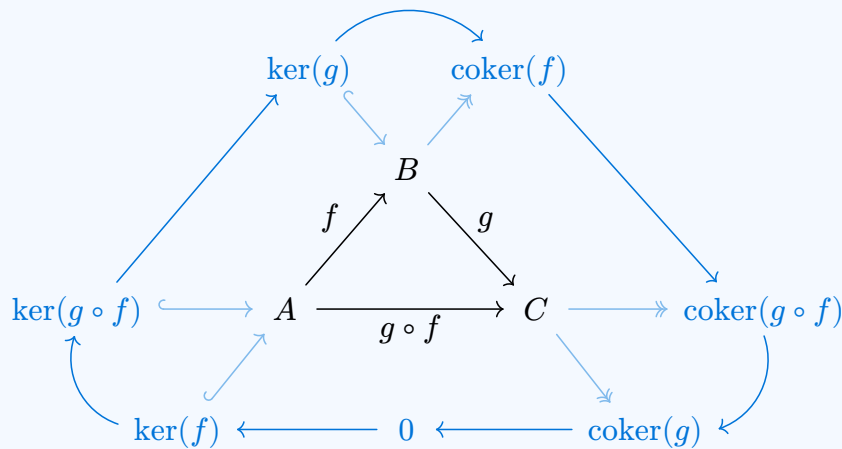
Let us show that Tor can also be computed using flat resolution. We need an algebraic lemma first.

Lemma 16.1: (*Snake Lemma à la Cartier-Weil*)

Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be maps in an Abelian category \mathcal{A} . Then, there exists an exact sequence

$$0 \rightarrow \ker(f) \rightarrow \ker(g \circ f) \rightarrow \ker(g) \rightarrow \operatorname{coker}(f) \rightarrow \operatorname{coker}(g \circ f) \rightarrow \operatorname{coker}(g) \rightarrow 0.$$

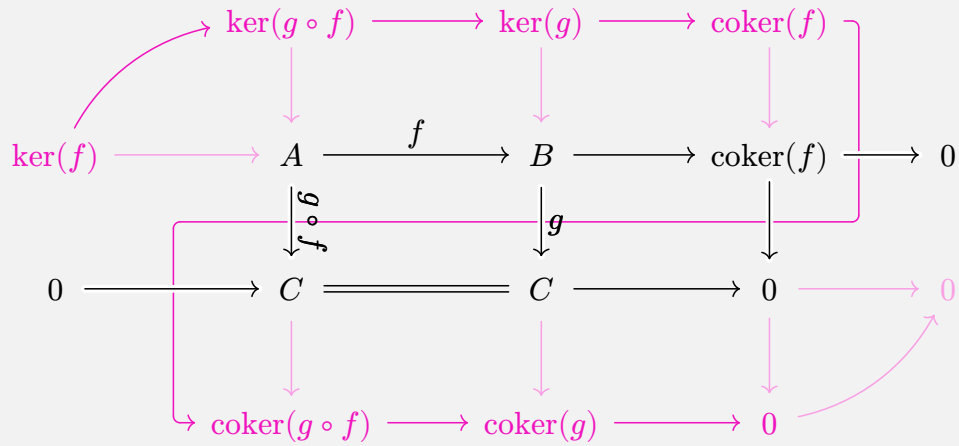
Visualized another way, we have



Proof : Consider the diagram of exact sequences

$$\begin{array}{ccccccc}
 A & \xrightarrow{f} & B & \longrightarrow & \operatorname{coker}(f) & \longrightarrow & 0 \\
 \downarrow f \circ g & & \downarrow g & & \downarrow & & \\
 0 & \longrightarrow & C & \xlongequal{\quad} & C & \longrightarrow & 0
 \end{array}$$

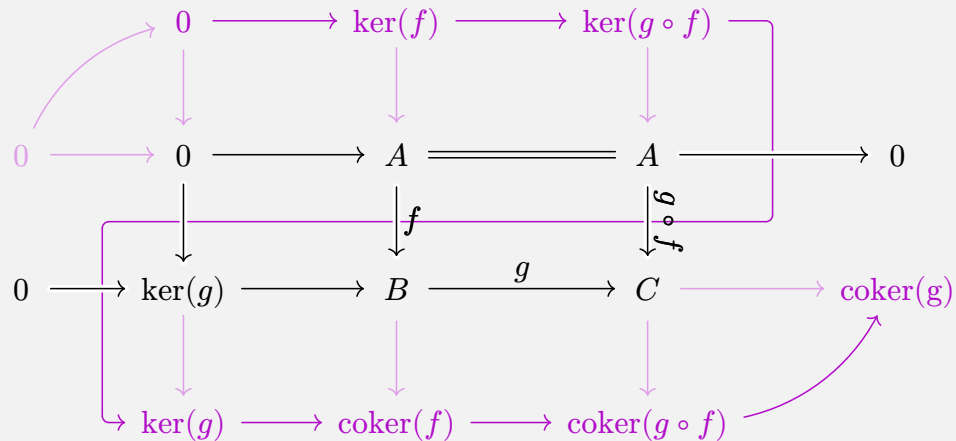
Applying Lemma 6.29, we get the exact sequence



Similarly, consider the diagram of exact sequence,

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \xlongequal{\quad} & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \ker(g) & \longrightarrow & B \xrightarrow{g} C
 \end{array}$$

Again, applying Lemma 6.29, we get the exact sequence



Splicing the two exact sequences, we get

$$0 \rightarrow \ker(f) \rightarrow \ker(g \circ f) \rightarrow \ker(g) \rightarrow \operatorname{coker}(f) \rightarrow \operatorname{coker}(g \circ f) \rightarrow \operatorname{coker}(g) \rightarrow 0,$$

which proves the claim. \square

We now have the following useful fact.

Proposition 16.2: (*Tor via Flat Resolution*)

Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a right exact additive functor between two Abelian categories, and suppose \mathcal{A} has enough projectives. Then the Tor functors can be computed by flat resolutions, i.e, given $M, N \in \mathcal{A}$ and a flat resolution $F_\bullet \rightarrow M$, we have $\operatorname{Tor}_n^R(M, N) \cong H_n(F_\bullet \otimes N)$, and similarly in the second variable.

Proof: Let $M, N \in \mathcal{A}$ be fixed. Suppose $F_\bullet \xrightarrow{\varepsilon} M$ is a flat resolution. Since tensor is right exact, we have an exact sequence

$$F_1 \otimes N \xrightarrow{d_1 \otimes \operatorname{Id}_N} F_0 \otimes N \xrightarrow{\varepsilon \otimes \operatorname{Id}_N} M \otimes N \rightarrow 0.$$

It follows that

$$H_0(F_\bullet \otimes N) = \frac{F_0 \otimes N}{\text{im}(d_1 \otimes \text{Id}_N)} \cong M \otimes N \cong \text{Tor}_0^R(M, N).$$

For $n = 1$, we have the diagram

$$\begin{array}{ccccc} F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{\varepsilon} & M \\ & & & \searrow d'_1 & \nearrow \iota_0 & & \\ & & & & K_0 = \ker \varepsilon & & \end{array}$$

Tensoring by N , we have the commutative diagram

$$\begin{array}{ccc} F_1 \otimes N & \xrightarrow{d_1 \otimes \text{Id}_N} & F_0 \otimes N \\ & \searrow d'_1 \otimes \text{Id}_N & \nearrow \iota_0 \otimes \text{Id}_N \\ & & K_0 \otimes N \end{array}$$

The right exactness of tensor shows that $d'_1 \otimes \text{Id}_N$ is an epimorphism, and hence, $\text{im}(d_1 \otimes \text{Id}_N) = \text{im}(\iota_0 \otimes \text{Id}_N)$ follows. Let

$$f := d'_1 \times \widetilde{\text{Id}_N} : \frac{F_1 \otimes N}{\text{im}(d_2 \otimes \text{Id}_N)} \rightarrow K_0 \otimes N, \quad g = \iota_0 \otimes \text{Id}_N : K_0 \otimes N \rightarrow F_0 \otimes N.$$

The exact sequence $F_2 \rightarrow F_1 \rightarrow K_0 \rightarrow 0$ gives the exact sequence

$$F_2 \otimes N \xrightarrow{d_2 \otimes \text{Id}_N} F_1 \otimes N \xrightarrow{d'_1 \otimes \text{Id}_N} K_0 \otimes N \rightarrow 0,$$

which implies

$$\ker(f) = \frac{\ker(d'_1 \otimes \text{Id}_N)}{\text{im}(d_2 \otimes \text{Id}_N)} = 0.$$

Also, f is an epimorphism, as it is induced by the epimorphism $d'_1 \times \text{Id}_N$. Now, by [Lemma 16.1](#), we have the exact sequence

$$\underbrace{\ker(f)}_0 \rightarrow \ker(g \circ f) \rightarrow \ker(g) \rightarrow \underbrace{\text{coker}(f)}_0.$$

Hence, we have an isomorphism $\ker(g \circ f) \cong \ker(g)$. But $\ker(g \circ f) = \frac{\ker(d_1 \otimes \text{Id}_N)}{\text{im}(d_2 \otimes \text{Id}_N)} = H_1(F_\bullet \otimes N)$. Hence, $H_1(F_\bullet \otimes N) \cong \ker(g) = \ker(\iota_0 \otimes \text{Id}_N)$. Now, applying [Theorem 15.12](#) to the exact sequence $0 \rightarrow K_0 \rightarrow F_0 \rightarrow M \rightarrow 0$ we have (part of) the exact sequence for Tor

$$\text{Tor}_1^R(F_0, N) \rightarrow \text{Tor}_1^R(M, N) \rightarrow K_0 \otimes N \xrightarrow{\iota_0 \otimes \text{Id}_N} F_0 \otimes N.$$

Since F_0 is flat, we have $\text{Tor}_1^R(F_0, N) = 0$, which implies

$$\text{Tor}_1^R(M, N) = \ker(\iota_0 \otimes \text{Id}_N) \cong H_1(F_\bullet \otimes N).$$

Let us now inductively assume that $\text{Tor}_n^R(M, N) \cong H_n(F_\bullet \otimes N)$ for any flat resolution of any object in \mathcal{A} . We have part of the exact sequence

$$\mathrm{Tor}_{n+1}^R(F_0, N) \rightarrow \mathrm{Tor}_{n+1}^R(M, N) \rightarrow \mathrm{Tor}_n^R(K_0, N) \rightarrow \mathrm{Tor}_n(F_0, N).$$

As F_0 is flat, we have $\mathrm{Tor}_{n+1}^R(M, N) \cong \mathrm{Tor}_n(K_0, N)$. Now, $\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow K_0 \rightarrow 0$ is a flat resolution of K_0 , say, $F'_\bullet \rightarrow K_0$. Hence, by induction

$$\mathrm{Tor}_{n+1}^R(M, N) \cong \mathrm{Tor}_n^R(K_0) \cong H_n(F'_\bullet \otimes N) = H_{n+1}(F_\bullet \otimes N).$$

This proves the claim. \square

16.2 Right Derived Functors and Ext

Similar to the left derived functors of a right exact functor, we can compute the right derived functors of a left exact functor. In particular, suppose $F : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor. Suppose \mathcal{A} has enough injectives. Given $A \in \mathcal{A}$, fix some injective resolution $A \xrightarrow{\varepsilon} I^\bullet$. Then, the i^{th} *right derived functor* of F is defined as

$$\mathfrak{R}^i F(A) := H^i(F(I_\bullet)) = H^i(0 \rightarrow F(I^0) \rightarrow F(I^1) \rightarrow \cdots).$$

Just like the left derived functors, it follows that $\mathfrak{R}^i F : \mathcal{A} \rightarrow \mathcal{B}$ is a well-defined functor (up to isomorphism), and $\mathfrak{R}^0 F = F$ naturally. Moreover, we have the following.

Theorem 16.3: (Long Exact Sequence of Right Derived Functors)

Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact additive functor between Abelian categories, where \mathcal{A} has enough injectives. Then, given a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{A} , there exists a natural long exact sequence

$$0 \rightarrow \underbrace{\mathfrak{R}^0 F(A)}_{F(A)} \rightarrow \underbrace{\mathfrak{R}^0 F(B)}_{F(B)} \rightarrow \underbrace{\mathfrak{R}^0 F(C)}_{F(C)} \rightarrow \mathfrak{R}^1 F(A) \rightarrow \mathfrak{R}^1 F(B) \rightarrow \mathfrak{R}^1 F(C) \rightarrow \cdots.$$

The (internal) hom functor in the category of modules, both the covariant and contravariant ones, are well-known left exact functors.

Definition 16.4: (Ext Functor)

For a fixed R -module, the right derived functors of the (covariant) hom-functor $\mathrm{hom}_R(M, _) : R\text{-Mod} \rightarrow R\text{-Mod}$ are called the *Ext functors*, and is denoted as $\mathrm{Ext}_R^n(M, _) : R\text{-Mod} \rightarrow R\text{-Mod}$.

Similarly, for fixed R -module N , the right derived functors of the *contravariant* hom-functor $\mathrm{hom}_R(_, N) : R\text{-Mod} \rightarrow R\text{-Mod}$ are also called Ext functors, denoted as $\mathrm{Ext}_R^n(_, N) : R\text{-Mod} \rightarrow R\text{-Mod}$. We can do this using a left projective resolution as well. Indeed, given a projective resolution $P_\bullet \rightarrow M$, it follows that $\mathrm{Ext}_R^i(M, N) = H_i(\mathrm{hom}(P_\bullet, N))$. As a consequence, if R is a PID, it follows that $\mathrm{Ext}_R^i(M, N) = 0$ for $i \geq 2$, and for $R = \mathbb{Z}$ we simply denote

$$\mathrm{Ext}(M, N) := \mathrm{Ext}_{\mathbb{Z}}^1(M, N), \quad M, N \in \mathbb{Z}\text{-Mod} = \mathrm{Ab}.$$

In general, we have a bi-functor

$$\mathrm{Ext}_R^n(_, _) : R\text{-Mod} \rightarrow R\text{-Mod},$$

which is contravariant on the first variable, and covariant in the second. Of course, this requires *balancing of Ext*, i.e., we need to show that $\mathfrak{R}^i(\mathrm{hom}_R(M, _))(N) \cong \mathrm{Ext}_R^i(M, N) \cong \mathfrak{R}^i(\mathrm{hom}_R(_, N))(M)$.

16.3 Ext over a PID

Over a PID, injective modules are characterized as *divisible* ones.

Definition 16.5: (*Divisible Module*)

An R -module M is called *divisible* if given any regular element $0 \neq r \in R$ (i.e, $r \in R$ is a non-zero-divisor), and any $m \in M$, there is some $x \in M$ such that $rx = m$ holds. In other words, $rM = M$ for any regular element $r \in R$.

In general, when considering divisible modules, we assume that the ring is an *integral domain* (i.e, whenever $r_1 r_2 = 0$ for some $r_1, r_2 \in R$, we have $r_1 = 0$ or $r_2 = 0$). In this case, every nonzero element is a non-zero-divisor. Note that a PID is by definition an integral domain.

Theorem 16.6: (*Injective Modules over PID are Divisible*)

If R is an integral domain, then an injective R -module I is divisible. Moreover, if R is a PID, a divisible R -module D is injective.

Proof : Suppose R is an integral domain, and I is an injective R -module. Let $0 \neq r \in R$ and $m \in I$ be fixed. Consider the map $f : R \rightarrow R$ given by (left) multiplication by r . Since $r \neq 0$ is not a zero-divisor, it follows that f is injective. Also consider the map $\varphi : R \rightarrow I$ given by $\varphi(1) = m$ (and extending linearly). As I is injective ([Definition 14.25](#)), we have an R -linear map $\Phi : R \rightarrow I$ solving the diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & R & \xrightarrow{f} & R \\
 & & \downarrow \varphi & \swarrow \Phi & \\
 & & I & &
 \end{array}$$

Consider $x = \Phi(1)$. Then,

$$rx = r\Phi(1) = \Phi(r \cdot 1) = \Phi \circ f(1) = \varphi(1) = m.$$

As $0 \neq r \in R$ and $m \in I$ are arbitrary, we have I is divisible.

Now, suppose R is a PID and D is a divisible R -module. Let us consider a monomorphism $f : A \rightarrow B$ and a map $\varphi : A \rightarrow D$. To show the existence of $\Phi : B \rightarrow D$ such that $\Phi \circ f = \varphi$, we need to apply Zorn's lemma. As f is monic, without loss of generality, by identifying A with $f(A)$, we can assume that A is a submodule of B . Consider the family

$$\mathcal{F} = \{(C, \psi) \mid A \subset C \subset B \text{ as submodules, } \psi : C \rightarrow D, \psi|_A = \varphi\}.$$

The family is nonempty since $(A, \varphi) \in \mathcal{F}$. We have a partial order \preceq on \mathcal{F} : for $(C_1, \psi_1), (C_2, \psi_2) \in \mathcal{F}$, we have $(C_1, \psi_1) \preceq (C_2, \psi_2)$ if and only if $C_1 \subset C_2$ and $\psi_2|_{C_1} = \psi_1$. Suppose $\mathcal{J} = \{(C_i, \psi_i)\}$ is a chain in \mathcal{F} with respect to \preceq . Let $C = \bigcup C_i$, and define $\psi : C \rightarrow D$ by $\psi|_{C_i} = \psi_i$. It is easy to see that C is a submodule of B with $A \subset C \subset B$, and ψ is a well-defined R -module map, such that $\psi|_A = \varphi$. Thus $(C, \psi) \in \mathcal{F}$ is an upper bound of the chain \mathcal{J} . Then by Zorn's lemma, there exists a maximal element $(C_0, \psi_0) \in \mathcal{F}$. We claim that $C_0 = B$. If not, then there is an element $x \in B \setminus C_0$. Let $C \subset B$ be the submodule generated by $C_0 \cup \{x\}$. Clearly, $A \subset C_0 \subsetneq C \subset B$. Let $I = \{r \in R \mid rx \in C_0\}$. Clearly, I is an ideal in R , and hence, $I = \langle a \rangle$ for some $a \in R$ (as R is a PID).

- If $a \neq 0$, divisibility of D implies that there is some $y \in D$ such that $ay = \psi_0(ax)$.
- If $a = 0$, choose some arbitrary $y \in D$.

In any case, we can now extend ψ_0 to $\psi : C \rightarrow D$ by setting $\psi(x) = y$. That is, define $\psi(c_0 + rx) = \psi_0(c_0) + ry$ for $c_0 + rx \in C$. Let us verify that this is well-defined. Indeed, if $rx \in C_0$ for some $r \in R$, we have $r \in I = \langle a \rangle \Rightarrow r = sa$ for some $s \in R$. Then, $\psi(rx) = ry = say = s\psi_0(ax) = \psi_0(sax) = \psi_0(rx)$. Thus, $\psi : C \rightarrow D$ is well-defined. This contradicts the maximality of (C_0, ψ_0) . Hence, $C_0 = B$, and clearly $\Phi = \psi_0$ is an extension of φ as required. This proves that D is an injective module. \square

Example 16.7: (Injective Abelian Groups)

It is easy to verify that $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{R}$ are divisible Abelian groups. Since \mathbb{Z} is a PID, injective \mathbb{Z} -modules are precisely the divisible groups (Theorem 16.6). Hence, we have $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{R}$ as examples of injective \mathbb{Z} -modules.

Recall the fact that every R -module embeds in an injective module, i.e, $R\text{-Mod}$ has enough injectives (Definition 15.1). Let us give a proof of this for \mathbb{Z} . The general statement is proved in Corollary 16.17.

Proposition 16.8: (Enough Injective Abelian Groups)

Every Abelian group can be embedded in a direct product of \mathbb{Q}/\mathbb{Z} , which is seen to be a divisible (and hence an injective) Abelian group.

Proof : Let A be an Abelian group. Fix $0 \neq a \in A$, and consider the subgroup $\langle a \rangle \hookrightarrow A$.

- If a has infinite order in A , choose arbitrary $0 \neq x_a \in \mathbb{Q}/\mathbb{Z}$.
- If a has finite order, say, $n \geq 2$ in A , choose $0 \neq x_a \in \mathbb{Q}/\mathbb{Z}$ such that $\text{ord}(x_a) \mid n$.

In any case, we have a map $\varphi_a : \langle a \rangle \rightarrow \mathbb{Q}/\mathbb{Z}$ defined by setting $\varphi(a) = x_a$. Since \mathbb{Q}/\mathbb{Z} is divisible and hence injective, extend the map φ_a to a map $\Phi_a : A \rightarrow \mathbb{Q}/\mathbb{Z}$. Then, using the universal property of product, define a map $\Phi : A \rightarrow \prod_{0 \neq a \in A} \mathbb{Q}/\mathbb{Z}$ where each component is Φ_a . Clearly direct product of divisible groups is again divisible, and hence $\prod_{0 \neq a \in A} \mathbb{Q}/\mathbb{Z}$ is injective. Moreover, Φ is clearly injective. Indeed, if for some $0 \neq a \in A$ we have $\Phi(a) = 0$, then in particular, $0 = \varphi_a(a) = x_a$, a contradiction. Thus, we can embed A in to an injective Abelian group. \square

We now note the following interesting result for a PID.

Proposition 16.9: (Quotient of Injective)

Let R be a PID. Then, any quotient of an injective R -module is again injective.

Proof : Since injective modules are precisely the divisible ones for a PID, let us consider D to be a divisible R -module, and let $\varphi : D \rightarrow E$ be an epimorphism. Let us show that E is divisible. Let $0 \neq r \in R$ and $e \in E$ be given. Choose some $d \in D$ such that $\varphi(d) = e$. Since D is divisible, there is some $y \in D$ such that $ry = d$. Let $x = \varphi(y) \in E$. Clearly, $rx = r\varphi(y) = \varphi(ry) = \varphi(d) = e$. Thus, E is divisible, and hence, an injective R -module. \square

As a corollary, we have the following.

Corollary 16.10: ($\text{Ext}_R^n = 0$ for $n \geq 2$ over a PID R)

Let R be a PID. Given any R -module N , there is an injective resolution $0 \rightarrow N \rightarrow I^0 \rightarrow I^1 \rightarrow 0$, and consequently, $\text{Ext}_R^n(M, N) = 0$ for any $M \in R\text{-Mod}$ and $n \geq 2$.

Proof : Let N be an arbitrary R -module. We embed $0 \rightarrow N \xrightarrow{\eta} I^0$, where I^0 is injective. Let $I^1 = \text{coker}(\eta)$, which is then again injective. Clearly, we have an exact sequence $0 \rightarrow N \rightarrow I^0 \rightarrow I^1 \rightarrow 0$, which is then an injective resolution.

As we have an injective resolution of length 2, it follows from the construction that $\text{Ext}_R^n(M, N) = 0$ for any R -module M , and $n \geq 2$. \square

Exercise 16.11: (Computation of Ext)

Compute the following.

1. $\text{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/a\mathbb{Z}, G)$ for any Abelian group G and $a > 1$.
2. $\text{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z})$ for $a, b > 1$.
3. $\text{Ext}_{\mathbb{Z}}^n(G, \mathbb{Q}/\mathbb{Z})$ for any Abelian group G .
4. $\text{Ext}_k^n(V, W)$ for a field k and k -modules V, W .

Example 16.12: (Group Cohomology)

Let us give the dual example to [Example 15.14](#). Let G be a group, R be a ring, and M be an $R[G]$ -module. Define the *invariants* as

$$M^G := \{m \in M \mid g \cdot m = m, \forall g \in G\}.$$

This defines a functor $(\cdot)^G : R[G]\text{-Mod} \rightarrow R[G]\text{-Mod}$, which can be identified with $\text{hom}_{R[G]}(R, \cdot)$. Thus, invariants being a hom functor is left exact. The right derived functors, denoted as $H^i(G, M)$ is called the *group cohomology* of G with coefficients in M . Clearly, $H^i(G, M) = \text{Ext}_{R[G]}^i(R, M)$.

Remark 16.13: (Hochschild (co)Homology)

Another interesting class of derived functors appear in Algebra known as Hochschild (co)homology. In particular, given a possibly noncommutative algebra A over a ring R , and an A -bimodule, one defines the *Hochschild cohomology* as $HH^i(A, M) := \text{Ext}_{A^{\text{op}} \otimes_k A}^i(A, M)$, and the *Hochschild homology* as $HH_i(A, M) := \text{Tor}_i^{A^{\text{op}} \otimes_k A}(A, M)$. These are important tools in deformation theory of algebras.

Exercise 16.14: (Ext and Projective/Injective)

Let A be an R -module. Show that

1. A is projective if and only if $\text{Ext}_R^1(M, A) = 0$ for all R -module M , and
2. A is injective if and only if $\text{Ext}_R^1(A, N) = 0$ for all R -modules N .

Remark 16.15: (Interpreting Ext_R^n As Extensions)

Given R -modules A, C , we call $B \in R\text{-Mod}$ to be an *extension* of C by A if we have a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$. Call two extensions B_1, B_2 *equivalent* if there is an isomorphism $\varphi : B_1 \rightarrow B_2$ such that the following diagram commutes

$$\begin{array}{ccccccc}
 & & & & B_1 & & \\
 & & & & \searrow & & \\
 0 & \longrightarrow & A & & & & C \longrightarrow 0 \\
 & & \nearrow & & \downarrow \varphi & & \nearrow \\
 & & & & B_2 & &
 \end{array}$$

One can verify that the equivalence classes of extensions is in bijection with the set $\text{Ext}_R^1(C, A)$, where 0 corresponds to split extensions. In particular, if $\text{Ext}_R^1(C, A) = 0$, then every extension of C by A splits.

Higher Ext modules are related to n -extensions, namely, an exact sequence $0 \rightarrow A \rightarrow B_1 \rightarrow \dots \rightarrow B_n \rightarrow C$. Consider the equivalence relation on all n -extensions *generated* by the relation that there is a chain map

$$\begin{array}{ccccccccccc}
 0 & \rightarrow & A & \rightarrow & B_1 & \rightarrow & \dots & \rightarrow & B_n & \rightarrow & C & \rightarrow & 0 \\
 & & \parallel & & \downarrow & & & & \downarrow & & \parallel & & \\
 0 & \rightarrow & A & \rightarrow & B_1 & \rightarrow & \dots & \rightarrow & B_n & \rightarrow & C & \rightarrow & 0
 \end{array}$$

Then, $\text{Ext}_R^n(C, A)$ is in bijection with the equivalence classes of all n -extensions of C by A .

16.4 A Digression : Enough Injective R -Modules

Let us give a category theoretic proof of the fact that for any ring R , the category of R -modules has enough injectives! First, we have the following useful result.

Proposition 16.16: (Adjoint and Injectives)

Suppose we have an adjunction $\mathcal{L} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathcal{R}$ of additive functors \mathcal{L}, \mathcal{R} between two Abelian categories \mathcal{A}, \mathcal{B} . Assume that

- \mathcal{L} is *exact*, i.e, takes short exact sequence in \mathcal{A} to short exact sequence in \mathcal{B} (in particular, assume \mathcal{L} is left exact as any left adjoint is always right exact), and
- \mathcal{L} is *faithful*, i.e, given any $A_1, A_2 \in \mathcal{A}$ the induced map $\text{hom}_{\mathcal{A}}(A_1, A_2) \rightarrow \text{hom}_{\mathcal{B}}(\mathcal{L}(A_1), \mathcal{L}(A_2))$ is injective.

Then, \mathcal{A} has enough injectives, provided \mathcal{B} has enough injectives.

Proof : Suppose \mathcal{B} has enough injectives. Let $A \in \mathcal{A}$. Then, there exists an injective object $I \in \mathcal{B}$ and a monomorphism $\iota : \mathcal{L}(A) \hookrightarrow I$. By the adjunction $\text{hom}_{\mathcal{B}}(\mathcal{L}(A), I) \cong \text{hom}_{\mathcal{A}}(A, \mathcal{R}(I))$, we have a morphism $\tilde{\iota} : A \rightarrow \mathcal{R}(I)$. We claim the following.

- **$\mathcal{R}(I)$ is injective in \mathcal{A} :** We show that the right adjoint of an exact functor preserves injectives. Indeed, consider a monomorphism $f : A_1 \hookrightarrow A_2$ and a map $\varphi : A_1 \rightarrow \mathcal{R}(I)$ in \mathcal{A} . Applying the adjunction, and the fact \mathcal{L} is exact (and in particular, preserves monomorphisms), we have the diagram

$$\begin{array}{ccc}
\mathfrak{L}(A_1) & \xrightarrow{\mathfrak{L}(f)} & \mathfrak{L}(A_2) \\
\tilde{\varphi} \downarrow & \swarrow \Psi & \\
I & &
\end{array}$$

The extension Ψ exists since I is injective. But then again by the adjunction, we have a map $\Phi = \tilde{\Psi} : A_2 \rightarrow I$ solving

$$\begin{array}{ccc}
A_1 & \xrightarrow{f} & A_2 \\
\varphi \downarrow & \swarrow \Phi & \\
\mathfrak{R}(I) & &
\end{array}$$

- $\tilde{\iota}$ is a **monomorphism**: We have an exact sequence $0 \rightarrow \ker(\tilde{\iota}) \rightarrow A \xrightarrow{\tilde{\iota}} \mathfrak{R}(I)$. Since \mathfrak{L} is exact, we have an exact sequence

$$0 \rightarrow \mathfrak{L}(\ker(\tilde{\iota})) \rightarrow \mathfrak{L}(A) \xrightarrow{\mathfrak{L}(\tilde{\iota})} \mathfrak{L}(\mathfrak{R}(I)).$$

Now, using the unit $\varepsilon : \mathfrak{L} \circ \mathfrak{R} \Rightarrow \text{Id}_{\mathcal{B}}$ of the adjunction, it follows that the composition is

$$\begin{array}{ccc}
\mathfrak{L}(A) & \xrightarrow{\mathfrak{L}(\tilde{\iota})} & \mathfrak{L}(\mathfrak{R}(I)) & \xrightarrow{\varepsilon(I)} & I \\
& \searrow \iota & & &
\end{array}$$

Since the composition is monic, it follows that $\mathfrak{L}(\tilde{\iota})$ is also monic. But then $\mathfrak{L}(\ker(\tilde{\iota})) \rightarrow \mathfrak{L}(A)$ is the 0 map. Since \mathfrak{L} is assumed to be faithful, it follows that $\ker(\tilde{\iota}) \rightarrow A$ is also the 0 map. In other words, ι is monic.

Thus, we have an embedding $\iota : A \hookrightarrow \mathfrak{R}(I)$, where $\mathfrak{R}(I)$ is injective in \mathcal{A} . Since $A \in \mathcal{A}$ was arbitrary, it follows that \mathcal{A} has enough injectives. \square

Then, as corollary we have that any module category has enough injectives.

Corollary 16.17: (R -Mod has Enough Injectives)

Given any ring R , the category R -Mod has enough injectives.

Proof : Consider the *forgetful functor* $U : R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod} = \text{Ab}$.

- U is evidently additive.
- U has a right adjoint $\mathfrak{R} : \text{Ab} \rightarrow R\text{-Mod}$ given by *coextension of scalars*. Explicitly, we have $\mathfrak{R}(G) = R \otimes_{\mathbb{Z}} G$, where we treat the ring R as a \mathbb{Z} -module; then $\mathfrak{R}(G)$ is an R -module via the multiplication in R . Clearly \mathfrak{R} is additive as well.
- U being a left adjoint, is right exact. Moreover, any monomorphism of R -module is clearly an injective Abelian group map. Hence, U is exact.
- U is clearly faithful since any R -module map being 0 as a group map implies the map itself is 0.

Now, by [Proposition 16.8](#), we have Ab has enough injectives. But then by [Proposition 16.16](#), we have $R\text{-Mod}$ has enough injectives. \square

Unlike [Proposition 16.9](#), over a general ring R it does not follow that a quotient of an injective module is again injective. Thus, for an arbitrary ring R , higher Ext modules may not vanish.

16.5 Universal Coefficient Theorems

The goal of this section is to compute the (co)homology of a free chain complex after tensoring (or taking hom) by an R -module. This leads to the definition of (co)homology with coefficients. We shall always assume that R is a PID; there are analogous theorems for a general ring R , which are usually stated via spectral sequences. If there is no confusion, we shall simply denote $\text{Tor} = \text{Tor}_1^R$ and $\text{Ext} = \text{Ext}_R^1$.

Theorem 16.18: (UCT for Homology with Coefficients)

Let R be a PID, C_\bullet be a chain complex of free R -modules, and M be an arbitrary R -module. Then, there exists a short exact sequence

$$0 \rightarrow H_n(C_\bullet) \otimes M \rightarrow H_n(C_\bullet \otimes M) \rightarrow \text{Tor}(H_{n-1}(C_\bullet), M) \rightarrow 0,$$

which is natural in both C_\bullet and the coefficient module M . Moreover, the short exact sequence splits, the splitting is natural in the coefficient module, but may not be natural with respect to the chain complex.

Proof : Let us denote the module of cycles, boundaries, and the homology as

$$Z_n := \ker\left(C_n \xrightarrow{\partial} C_{n-1}\right), \quad B_n := \text{im}\left(C_{n+1} \xrightarrow{\partial} C_n\right), \quad H_n := H_n(C_\bullet) = B_n/Z_n.$$

Since R is a PID, both Z_n and B_n are free, being submodules of the free R -module C_n ([Proposition 14.7](#)). In particular, we have a short exact sequence

$$0 \rightarrow Z_n \rightarrow C_n \rightarrow B_{n-1} \rightarrow 0,$$

which is split as B_{n-1} is a free (and hence projective) R -module. Hence, tensoring by M , we again get a split exact sequence

$$0 \rightarrow Z_n \otimes M \rightarrow C_n \otimes M \rightarrow B_{n-1} \otimes M \rightarrow 0.$$

Treating $Z_\bullet \otimes M$ and $B_\bullet \otimes M$ as a chain complex with 0-differentials, and $C_\bullet \otimes M$ with the differential $\partial \otimes \text{Id}_M$, we have a short exact sequence of chain complexes

$$0 \rightarrow Z_\bullet \otimes M \rightarrow C_\bullet \otimes M \rightarrow B_\bullet \otimes M[-1] \rightarrow 0,$$

where $B_\bullet \otimes M$ is shifted by degree -1 . Applying [Theorem 7.1](#), we then have a long exact sequence in homology

$$\cdots \rightarrow B_n \otimes M \xrightarrow{\delta} Z_n \otimes M \rightarrow H_n(C_\bullet \otimes M) \rightarrow B_{n-1} \otimes M \xrightarrow{\delta} Z_{n-1} \otimes M \rightarrow \cdots$$

Here δ is the boundary map of the long exact sequence. Recall from [Remark 7.2](#), we have the diagram

$$\begin{array}{ccc}
 C_n \otimes M & \xrightarrow{\partial \otimes \text{Id}_M} & B_{n-1} \otimes M \\
 \downarrow \partial \otimes \text{Id}_M & & \downarrow y \\
 Z_{n-1} \otimes M & \xrightarrow{\quad} & C_{n-1} \otimes M \\
 \uparrow x & \xrightarrow{\quad} & x \\
 & \xrightarrow{\delta} &
 \end{array}$$

Hence, δ is nothing but the map $\iota \otimes \text{Id}_M$, where $\iota : B_n \hookrightarrow Z_n$ is the inclusion.

We also have a short exact sequence

$$0 \rightarrow B_n \rightarrow Z_n \rightarrow H_n \rightarrow 0.$$

Since B_n, Z_n are free, we can treat this as a free (hence projective) resolution of H_n . In particular, it follows that

$$\text{Tor}(H_n, M) \cong \ker(\iota \otimes \text{Id}_M : B_n \otimes M \rightarrow Z_n \otimes M) = \ker(\delta).$$

Also, exactness of $B_n \otimes M \rightarrow Z_n \otimes M \rightarrow H_n \otimes M \rightarrow 0$ implies that

$$\text{coker}(\delta) = \text{coker}(\iota \otimes \text{Id}_M) \cong H_n \otimes M.$$

Then, splicing the homology long exact sequence we get

$$0 \rightarrow H_n \otimes M \rightarrow H_n(C_\bullet \otimes M) \rightarrow \text{Tor}(H_n, M) \rightarrow 0,$$

which is the required short exact sequence.

The naturality of the short exact sequence follows from the functoriality of Tor and the above construction. As for the splitting, choose a splitting $s_n : B_{n-1} \rightarrow C_n$ of the exact sequence $0 \rightarrow Z_n \rightarrow C_n \rightarrow B_{n-1} \rightarrow 0$, which gives a splitting $\sigma_n := s_n \otimes \text{Id}_M$ of the exact sequence after tensoring. In particular, σ_\bullet is a splitting for the short exact sequence of the chain complexes, which induces a splitting in the homology exact sequence. Since s_n was a choice independent of the module M , it is natural with respect to the coefficients, but may not be natural with respect to the chain complex. \square

Let us explain the meaning of the splitting be *not* natural.

Example 16.19: (Unnaturality of the Splitting in Homology UCT)

Let us consider the map of chain complexes

$$\begin{array}{ccccccccccc}
 C_\bullet & : & \overset{(4)}{0} & \longrightarrow & \overset{(3)}{0} & \longrightarrow & \overset{(2)}{\mathbb{Z}} & \xrightarrow{\times 2} & \overset{(1)}{\mathbb{Z}} & \xrightarrow{0} & \overset{(0)}{\mathbb{Z}} & \longrightarrow & 0 \\
 f_\bullet \downarrow & & \downarrow & & \downarrow & & \parallel & & \downarrow & & \parallel & & \\
 D_\bullet & : & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
 \end{array}$$

Note that the homology is given as

$$H_n(C_\bullet) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}/2\mathbb{Z}, & n = 1 \\ 0, & \text{otherwise.} \end{cases} \quad H_n(D_\bullet) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}/2\mathbb{Z}, & n = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the map f_\bullet induces 0 everywhere except Id on $H_0(C_\bullet) \rightarrow H_0(D_\bullet)$. Next, tensoring by $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, we have the diagram

$$\begin{array}{ccccccccccc} C_\bullet \otimes \mathbb{Z}_2: & & \overset{(4)}{0} & \longrightarrow & \overset{(3)}{0} & \longrightarrow & \overset{(2)}{\mathbb{Z}_2} & \xrightarrow{0} & \overset{(1)}{\mathbb{Z}_2} & \xrightarrow{0} & \overset{(0)}{\mathbb{Z}_2} & \longrightarrow & 0 \\ f_\bullet \times \text{Id}_{\mathbb{Z}_2} \downarrow & & \downarrow & & \downarrow & & \parallel & & \downarrow & & \parallel & & \\ D_\bullet \otimes \mathbb{Z}_2: & & 0 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{0} & \mathbb{Z}_2 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 \end{array}$$

We have the homology with coefficients

$$H_n(C_\bullet \otimes \mathbb{Z}_2) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & n = 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases} \quad H_n(D_\bullet \otimes \mathbb{Z}_2) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & n = 0, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

Passing to homology, we now have an extra identity map $\text{Id} : H_2(C_\bullet \otimes \mathbb{Z}_2) \rightarrow H_2(D_\bullet \otimes \mathbb{Z}_2)$ induced by $f_\bullet \otimes \text{Id}_{\mathbb{Z}_2}$.

Looking at the short exact sequence from [Theorem 16.18](#), we have the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \overset{0}{H_2(C_\bullet) \otimes \mathbb{Z}_2} & \longrightarrow & \overset{\mathbb{Z}_2}{H_2(C_\bullet \otimes \mathbb{Z}_2)} & \longrightarrow & \overset{\mathbb{Z}_2}{\text{Tor}(H_1(C_\bullet), \mathbb{Z}_2)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \overset{\mathbb{Z}_2}{H_2(D_\bullet) \otimes \mathbb{Z}_2} & \longrightarrow & \overset{\mathbb{Z}_2}{H_2(D_\bullet \otimes \mathbb{Z}_2)} & \longrightarrow & \overset{0}{\text{Tor}(H_1(D_\bullet), \mathbb{Z}_2)} \longrightarrow 0 \end{array}$$

If the splitting were natural, then we must have a commutative diagram

$$\begin{array}{ccc} H_2(C_\bullet \otimes \mathbb{Z}_2) & \xrightarrow{\cong} & H_2(C_\bullet) \otimes \mathbb{Z}_2 \oplus \text{Tor}(H_1(C_\bullet), \mathbb{Z}_2) \\ \text{Id} \downarrow & & \downarrow 0 \oplus 0 \\ H_2(C_\bullet \otimes \mathbb{Z}_2) & \xrightarrow{\cong} & H_2(C_\bullet) \otimes \mathbb{Z}_2 \oplus \text{Tor}(H_1(C_\bullet), \mathbb{Z}_2) \end{array}$$

where the isomorphisms are induced by the splitting. But clearly the diagram does not commute, which shows that the splitting in UCT is not natural in the chain complexes.