Course notes for

Algebraic Topology II (KSM4E02)

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`resolution — horseshoe lemma — left derived functors — Tor functors — long exact sequence of left derived functors`

15.1 Resolutions

The definition of projective and injective objects make sense in any category via the diagrams. More interestingly, they make sense in any Abelian categories via the exactness of the $\hom$ functors. This leads to the following definition.

Definition 15.1: (*Enough Projective and Enough Injective*)

An Abelian categroy $\mathcal{A}$ is said to have

- *enough projectives* if given any $A \in \mathcal{A}$ there is an epimorphism $\pi : P \twoheadrightarrow A$, where $P \in \mathcal{A}$ is a projective object, and
- *enough injective* if given any $A \in \mathcal{A}$ there is a monomorphism $\iota : A \hookrightarrow I$, where $I \in \mathcal{A}$ is an injective object.

The category $R\text{-}\mathrm{Mod}$ has enough projectives, since we can easily construct epimorphism from free (and hence, projective) modules. $R\text{-}\mathrm{Mod}$ has enough injectives as well, although it is considerably harder to prove and involves the axiom of choice.

The point of *resolving an object* (or a map) in a category is to replace it by something *equivalent* which behaves well with computation. As a concrete example, recall that in Theorem 11.9 we say that given any topology space Y, there is a CW complex X and a map $f : X \to Y$ such that f is a weak homotopy equivalence. CW complexes are well-suited to compute singular homology (Theorem 13.18), and homology remains invariant under weak homotopy equivalence. Thus, the CW approximation theorem gives a *resolution* in the category of topological spaces. To make this precise, one needs the notion of *model category,* where one can perform (co)fibrant replacement of maps and objects. We shall focus on the category $\mathrm{Ch} = \mathrm{Ch}(R)$ of chain complexes of R-modules.

Definition 15.2: (*Left and Right Resolution*)

Given an R-module M, a *left resolution* is an exact sequence

$$\cdots \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} M \to 0,$$

which we express as $C_\bullet \xrightarrow{\varepsilon} M \to 0$. Similarly, a *right resolution* is an exact sequence

$$0 \to M \xrightarrow{\varepsilon} D_0 \xrightarrow{d_0} D_1 \to \cdots \to D_n \xrightarrow{d_n} D_{n+1} \to \cdots,$$

which we express as $0 \to M \xrightarrow{\varepsilon} D_\bullet$. The map ε is called the *augmentation map*.

The R-module M can be realized as a chain complex concentrated at degree 0. Thus, a left resolution $C_\bullet \to M \to 0$ can be understood as the chain map

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r, "d_2"] \arrow[d] & C_1 \arrow[r, "d_1"] \arrow[d] & C_0 \arrow[r, "0"] \arrow[d, "\varepsilon"'] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

The fact that $C_\bullet \xrightarrow{\varepsilon} M \to 0$ is an exact sequence is equivalent to the fact that the chain map $\varepsilon_\bullet : C_\bullet \to M$ is a chain equivalence. Thus, getting a resolution is to replace the object M by a chain equivalent complex, which is exact everywhere except at degree 0. Note that if $C_\bullet \xrightarrow{\varepsilon} M \to 0$ and $D_\bullet \xrightarrow{\eta} N \to 0$ are two resolutions, then $C_\bullet \oplus D_\bullet \to (\varepsilon + \eta) \to M \oplus N \to 0$ is again a resolution.

Definition 15.3: (*Free/Projective/Flat/Injective Resolution*)

Given an R-module, a left resolution $C_\bullet \xrightarrow{\varepsilon} M \to 0$ is called a *free resolution* (resp. *projective resolution*, *flat resolution*, *injective resolution*) if each C_n is a free (resp. projective, flat, injective) module. Similarly, we define free/projective/flat/injective right resolution.

In practice, we are interested in free/projective/flat left resolutions, and injective right resolutions.

Proposition 15.4: (*Existence of Projective and Injective Resolutions*)

Given any R-module M, there exists a free (and hence projective) left resolution and injective right resolution. If R is a a PID, then there is a free left resolution $0 \to F_1 \to F_0 \to M \to 0$.

Proof: There is an epimorphism $\varepsilon : F_0 \to M$, where F_0 is a free module (e.g, take $F_0 = R[M]$). Let $K_0 = \ker(\varepsilon)$ and get an epimorphism $d_1 : F_1 \to K_0$. Then, we have an exact sequence $F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \to 0$. Inductively continuing we have a free resolution. If R is a PID, then K_0 is free being a submodule of a free module (Proposition 14.7), and hence we can simply set $F_1 = K_0$.

Since R-Mod has enough injectives, we have an injection $0 \to M \xrightarrow{\varepsilon} I_0$, where I_0 is an injective module. Set $L_0 = \mathrm{coker}(M \to I_0)$ and get monomorphism $\delta_0 : L_0 \to I_1$, where I_1 is injective. Composing, we have $d_0 : I_0 \to L_0 \to I_1$. Then, $0 \to M \xrightarrow{\varepsilon} I_0 \xrightarrow{d_0} I_1$ is exact. Continuing inductively, we have an injective resolution. □

In general, in any Abelian category with enough projectives, we have projective left resolutions. Similarly, if the category has enough injectives, we have injective right resolutions. As an example, the category Sh of Ab-valued sheaves over a topological space have enough injectives, which leads to the *Godement resolution*, but in general Sh does not have enough projectives.

In any case, the existence of a resolution is not unique as we made choices at each step. Thus, we need to be able to compare different resolutions.

Theorem 15.5: (*Comparison of Resolutions*)

Let $P_\bullet \xrightarrow{\varepsilon} M$ be a projective resolution of M, and $Q_\bullet \xrightarrow{\eta} N$ be any resolution of N. Say, $f' : M \to N$ is a given map. Then, there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ such that the diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f'"'] & 0 \\
\cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"'] & N \arrow[r] & 0
\end{tikzcd}$$

Moreover, any such chain map is unique up to chain homotopy. In fact the statement remains true under the assumption that $P_\bullet$ is just a chain complex of projective modules.

Proof : The proof is via induction. Denote $P_{-1} = M, Q_{-1} = N$ and $f_{-1} = f'$ for notational convenience. Assume that f_n is constructed so that the diagram commutes

$$\begin{tikzcd}
P_{n+1} \arrow[r] \arrow[d, dashed, "f_{n+1}"'] & P_n \arrow[r] \arrow[d, "f_n"'] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f'"] & 0 \\
Q_{n+1} \arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_0 \arrow[r, "\eta"'] & N \arrow[r] & 0
\end{tikzcd}$$

Denoting the cycles $Z_n = \ker(P_n \to P_{n-1})$ and $W_n = \ker(Q_n \to Q_{n-1})$ it follows that we have an induced map

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "f'_n"] & P_n \arrow[r] \arrow[d, "f_n"'] & P_{n-1} \arrow[d, "f_{n-1}"'] \\
0 \arrow[r] & W_n \arrow[r] & Q_n \arrow[r] & Q_{n-1}
\end{tikzcd}$$

Sicne $Q_\bullet$ is a resolution, we have an exact sequence $Q_{n+1} \to W_n \to 0$. Since P_{n+1} is projective, we have a lift

$$\begin{tikzcd}
P_{n+1} \arrow[dd, dashed, "f_{n+1}"'] \arrow[dr, bend left, "d"] & & \\
 & Z_n \arrow[d, "f'_n"] & \\
Q_{n+1} \arrow[r] & W_n \arrow[r] & 0
\end{tikzcd}$$

Clearly, f_{n+1} satisfies the chain map condition. Thus, inductively we can get a lift. Note: we only used the fact that the differential $d : P_{n+1} \to P_n$ maps in to Z_n, i.e, $P_\bullet$ need only be a chain complex.

Next, suppose $g_\bullet : P_\bullet \to Q_\bullet$ be another lift. We inductively construct the chain homotopy $s_n : P_n \to Q_{n+1}$. For $n < 0$, we can set $s_n = 0$. Indeed, for $s_{-1} : P_{-1} = M \to Q_0$, we check $\eta 0 = 0 = f' - f'$. Consider the diagram

$$\begin{tikzcd}
& P_0 \arrow[r, "\varepsilon"] \arrow[d, bend right, "f_0"'] \arrow[d, bend left, "g_0"] \arrow[dl, dashed, "s_0"'] & M \arrow[r] \arrow[d, "f'"'] \arrow[dl, dashed, "0"'] & 0 \\
Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"'] & N \arrow[r] & 0
\end{tikzcd}$$

Note that $\eta(f_0 - g_0) = (f' - f')\varepsilon = 0$, and thus, $f_0 - g_0 : P_0 \to Z_0 = Z_0 := \ker(\eta)$. Since P_0 is projecive, and since $Q_1 \to Z_0 \to 0$ is exact, we have a lift $s_0 : P_0 \to Q_1$ such that $ds_0 = f_0 - g_0$. Inductively assume that $s_k : P_k \to Q_{k+1}$ has been constructed for $k \leq n$. In particular, we have

$$f_n - g_n = ds_n + s_{n-1}d \Rightarrow ds_n = f_n - g_n - s_{n-1}d.$$

Let us compute

$$d(f_{n+1} - g_{n+1} - s_n d) = (f_n - g_n)d - ((f_n - g_n) - s_{n-1}d)d = 0.$$

Thus, $f - g - sd$ lands in $W_n = \ker(d : Q_{n+1} \to Q_n)$. But since P_{n+1} is projective and $Q_\bullet$ is a resolution, we have a lift $s_{n+1} : P_{n+1} \to Q_{n+2}$. Clearly this fits together as a chain homotopy. This completes the proof. ☐

As a corollary, we have the following.

Corollary 15.6: (*Resolutions are Unique Upto Chain Homotopy Equivalence*)

Given two projective resolutions $P_\bullet \overset{\varepsilon}{\to} M$ and $Q_\bullet \overset{\varepsilon}{\to} M$, there is a chain homotopy equivalence $f_\bullet : P_\bullet \to Q_\bullet$.

Proof : Using Theorem 15.5, we can lift Id_M to get two chain maps $f : P_\bullet \to Q_\bullet$ and $g : Q_\bullet \to P_\bullet$. Then, $g \circ f : P_\bullet \to P_\bullet$ lifts Id_M as well. But $\mathrm{Id}_{P_\bullet} : P_\bullet \to P_\bullet$ is trivially a lift. Hence, again by Theorem 15.5, we have $g \circ f$ is chain homotopic to $\mathrm{Id}_{P_\bullet}$. By the same argument, we have $f \circ g$ is chain homotopic to $\mathrm{Id}_{Q_\bullet}$. Consequently, f is a chain homotopy equivalence, with inverse g. ☐

The next lemma gets its name from the shape of the diagram!

Lemma 15.7: (*Horseshoe Lemma*)

Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence of R-modules. Suppose $P'_\bullet \xrightarrow{\varepsilon'} A$ and $P''_\bullet \xrightarrow{\varepsilon''} A''$ are two projective resolutions of A' and A'' resppectively. Then, setting $P_n := P'_n \oplus P''_n$ yields a projective resolution $P_\bullet \xrightarrow{\varepsilon} A$. Moreover, we get the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P'_2 \arrow[r] \arrow[d, "\iota"] & P'_1 \arrow[r] \arrow[d, "\iota"] & P'_0 \arrow[r, "\varepsilon'"] \arrow[d, "\iota"] & A' \arrow[r] \arrow[d, "\iota"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\pi"] & P_1 \arrow[r] \arrow[d, "\pi"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi"] & 0 \\
\cdots \arrow[r] & P''_2 \arrow[r] \arrow[d] & P''_1 \arrow[r] \arrow[d] & P''_0 \arrow[r, "\varepsilon''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

where each column is a short exact sequence, and each row is exact.

Proof : Since P''_0 is projective, we can lift ε'' to a map $P''_0 \to A$. As P_0 is the direct sum $P'_0 \oplus P''_0$, we can take the sum of the map $P''_0 \to A$ with the map $\iota\varepsilon'$ to get the map $\varepsilon : P_0 \to A$. Then, we have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(\varepsilon') \arrow[r] \arrow[d, dashed] & P'_0 \arrow[r, "\varepsilon'"] \arrow[d, "\iota"] & A' \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & \ker(\varepsilon) \arrow[r] \arrow[d, dashed] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & \ker(\varepsilon'') \arrow[r] \arrow[d] & P''_0 \arrow[r, "\varepsilon''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The left column is induced naturally and it is a short exact sequence by the snake lemma (Lemma 6.29). Since $P'_\bullet$ and $P''_\bullet$ are exact everywhere, we have the diagram, where the column is exact.

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
\cdots \arrow[r] & P'_1 \arrow[r] & \ker(\varepsilon') \arrow[r] \arrow[d, dashed] & 0 \\
 & & \ker(\varepsilon) \arrow[d, dashed] & \\
\cdots \arrow[r] & P''_1 \arrow[r] & \ker(\varepsilon'') \arrow[r] \arrow[d] & 0 \\
 & & 0 &
\end{tikzcd}$$

We can now continue inductively. This completes the proof. ☐

Remark 15.8: (*Explicit Formula for $P_\bullet \to A$*)

Let us justify an explicit formula for the differential appearing in the resolution $P_\bullet$. Let us write in the matrix form

$$d_n = \begin{pmatrix} f_n & \lambda_n \\ \mu_n & g_n \end{pmatrix} : \begin{matrix} P'_n \\ \oplus \\ P''_n \end{matrix} \longrightarrow \begin{matrix} P'_{n-1} \\ \oplus \\ P''_{n-1} \end{matrix}.$$

Following the construction in Lemma 15.7, we see that $f_n = d'_n, \mu_n = 0$ and $g_n = d''_n$. In other words, $d = \left(\begin{smallmatrix} d' & \lambda \\ 0 & d'' \end{smallmatrix}\right)$ for some $\lambda_n : P''_n \to P'_{n-1}$.

Exercise 15.9: (*Comparison and Horseshoe Lemma of Injective Resolution*)

State and proof the comparison theorem and the horseshoe lemma for injective right resolutions.

15.2 Left Derived Functors and Tor

Let us fix two Abelian categories $\mathcal{A}, \mathcal{B}$, so that $\mathcal{A}$ has enough projectives. In particular, we can consider $\mathcal{A} = R\text{-Mod}$ and $\mathcal{B} = S\text{-Mod}$. Let $F : \mathcal{A} \to \mathcal{B}$ be a *right* exact additive functor. Given $A \in \mathcal{A}$, fix some projective resolution $P_\bullet \xrightarrow{\varepsilon} A$. Then, the i^{th} *left derived functor* of F is defined as

$$\mathfrak{L}_i F(A) := H_i(F(P_\bullet)) = H_i(\cdots \to F(P_2) \to F(P_1) \to F(P_0) \to 0).$$

In other words, if we consider a projective resolution to be a chain complex $\cdots \to P_1 \to P_0 \to 0$ with a weak equivalence $\varepsilon : P_\bullet \to M$, then the left derived functor is precisely the homology of $P_\bullet$. Note that a priori, the definition depends on the choice of the projective resolution.

Proposition 15.10: (*$\mathfrak{L}_i F$ is a Well-defined Functor*)

$\mathfrak{L}_i F : \mathcal{A} \to \mathcal{B}$ is a well-defined functor (up to isomorphism).

Proof : Suppose $Q_\bullet \xrightarrow{\eta} A$ is another projective resolution. Then, by Corollary 15.6, we can lift $\mathrm{Id}_A : A \to A$ to a chain map $f : P_\bullet \to Q_\bullet$, which is unique up to chain homotopy (hence gives *same map* at homology), and moreover, which is a chain homotopy equivalence (hence gives homology isomorphism). In particular, $H_\bullet(f)$ is a uniquely defined isomorphism, which justifies that $\mathfrak{L}_i F(A)$ is well-defined.

Given any $\varphi : A \to B$, we can again use Theorem 15.5 to get a chain map $\Phi : P_\bullet \to Q_\bullet$, which induces a map $H_i(\Phi) : \mathfrak{L}_i F(A) \to \mathfrak{L}_i F(B)$. As Φ is unique up to chain homotopy, it follows that $H_i(\Phi)$ is uniquely defined. Thus, we have $\mathfrak{L}_i F(\varphi) = H_i(\Phi) : \mathfrak{L}_i F(f) \to \mathfrak{L}_i F(B)$. Let us check $\mathfrak{L}_i F$ is actually a functor.

- Since Id_A lifts to identity chain map, it follows that $\mathfrak{L}_i F(\mathrm{Id}_A) = \mathrm{Id}_{\mathfrak{L}_i F(A)}$.
- Say, $A \xrightarrow{f} B \xrightarrow{g} C$ is given. Get reolutions $P_\bullet \to A, Q_\bullet \to B, R_\bullet \to C$, and lifts $F : P_\bullet \to Q_\bullet, G : Q_\bullet \to R_\bullet$ and $H : P_\bullet \to R_\bullet$ of f, g, and $g \circ f$ respectively. Now, $G \circ F$ is also a lift of $g \circ f$, and hence, $G \circ F$ is chain homotopic to H. Thus, $\mathfrak{L}_i F(g \circ f) = \mathfrak{L}_i F(g) \circ \mathfrak{L}_i F(f)$.

Thus, we see that $\mathfrak{L}_i F(A)$ is defined up to a canonical isomorphism. Moreover, the composition also holds true, compatible with these isomorphisms. ☐

Remark 15.11: (*Derived Functor and Choice*)

As noted, $\mathfrak{L}_i F$ is only defined up to isomorphism. To make it a true functor, we need to *fix* a projective resolution for each object, which is moreover functorial. This is possible in most model categories. As an example, when working with $R\text{-}\mathrm{Mod}$, given any R-module M, we have a *canonical* free resolution $0 \to F_1 = \ker(\varepsilon) \to F_0 \xrightarrow{\varepsilon} M \to 0$, where $F_0 = F(M)$ is the free module generated by M (as a set). It is easy to see that any $f : M \to N$ induces a natural map

$$\begin{tikzcd}
0 \arrow[r] & F_1(M) \arrow[r] \arrow[d, "f_1"'] & F_0(M) \arrow[r] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & F_1(N) \arrow[r] & F_0(N) \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Thus, $\mathfrak{L}_i F$ is a uniquely defined functor here.

Since F is right exact, given any resolution $P_\bullet \xrightarrow{\varepsilon} M$, we have $F(P_1) \to F(P_0) \to F(A) \to 0$ is exact. Hence,

$$\begin{aligned}
\mathfrak{L}_0 F(A) = H_0(\cdots \to F(P_1) \to F(P_0) \to 0) &= \frac{\ker(F(P_0) \to 0)}{\operatorname{im}(F(P_1) \to F(P_0))} \\
&= \frac{F(P_0)}{\ker\left(F(P_0) \xrightarrow{F(\varepsilon)} F(A)\right)} = F(A).
\end{aligned}$$

That is, $\mathfrak{L}_0 F = F$ naturally. The goal of derived functor is to measure the failure of exactness of the functor F.

Theorem 15.12: (*Long Exact Sequence of Left Derived Functors*)

Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact additive functor between Abelian categories, where $\mathcal{A}$ has enough projectives. Then, given any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, there exists a natural long exact sequence

$$\cdots \to \mathfrak{L}_i F(A) \to \mathfrak{L}_i F(B) \to \mathfrak{L}_i F(C) \xrightarrow{\delta} \mathfrak{L}_{i-1} F(A) \cdots \to \mathfrak{L}_1 F(C) \xrightarrow{\delta} \underbrace{\mathfrak{L}_0 F(A)}_{F(A)} \to \underbrace{\mathfrak{L}_0 F(B)}_{F(B)} \to \underbrace{\mathfrak{L}_0 F(C)}_{F(C)} \to 0$$

Proof: Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence. Get projective resolutions $P'_\bullet \xrightarrow{\varepsilon'} A'$ and $P''_\bullet \xrightarrow{\varepsilon''} A''$. Using Lemma 15.7, we get the projective resolution $P_\bullet \xrightarrow{\varepsilon} A$, so that $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ fits together as a short exact sequence. Now, $0 \to P'_n \to P_n \to P''_n \to 0$ splits by Proposition 14.16. Hence, $0 \to F(P'_n) \to F(P_n) \to F(P''_n) \to 0$ is a (split) exact sequence. Thus, we have a short exact sequence $0 \to F(P'_\bullet) \to F(P_\bullet) \to F(P''_\bullet) \to 0$ of chain complexes. By Lemma 6.29, we get the long exact sequence of homology

$$\cdots \to \mathfrak{L}_i F(A) \to \mathfrak{L}_i F(B) \to \mathfrak{L}_i F(C) \xrightarrow{\delta} \mathfrak{L}_{i-1} F(A) \to \cdots$$

We need to justify he naturaility of the boundary map δ. Consider the commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}$$

Get projective resolutions $P'_\bullet \xrightarrow{\varepsilon'} A'$, $P''_\bullet \xrightarrow{\varepsilon''} A''$, $Q'_\bullet \xrightarrow{\eta'} B'$, $Q''_\bullet \xrightarrow{\eta''} B''$ by Proposition 15.4. Next, using Theorem 15.5, lift f' and f'' to chain maps $F'_\bullet : P'_\bullet \to Q'_\bullet$ and $F''_\bullet : P''_\bullet \to Q''_\bullet$. Using Lemma 15.7, get the projective resolutions $P_\bullet \xrightarrow{\varepsilon} A$ and $Q_\bullet \xrightarrow{\eta} B$, so that we have the short exact sequences $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$, and $0 \to Q'_\bullet \to Q_\bullet \to Q''_\bullet \to 0$. We need to fill in $F_\bullet : P_\bullet \to Q_\bullet$ lifting $f : A \to B$ so that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[rr] & & P'_\bullet \arrow[rr] \arrow[dr, "\varepsilon'"] \arrow[dd, "F'_\bullet"] & & P_\bullet \arrow[rr] \arrow[dr, "\varepsilon"] \arrow[dd, dashed, "F_\bullet"] & & P''_\bullet \arrow[rr] \arrow[dr, "\varepsilon''"] \arrow[dd, "F''_\bullet"] & & 0 & \\
& 0 \arrow[rr] & & A' \arrow[rr, "\iota_A"] \arrow[dd, "f'"] & & A \arrow[rr, "\pi_A"] \arrow[dd, "f"] & & A'' \arrow[rr] \arrow[dd, "f''"] & & 0 \\
0 \arrow[rr] & & Q'_\bullet \arrow[rr] \arrow[dr, "\eta'"] & & Q_\bullet \arrow[rr] \arrow[dr, "\eta"] & & Q''_\bullet \arrow[rr] \arrow[dr, "\eta''"] & & 0 & \\
& 0 \arrow[rr] & & B' \arrow[rr, "\iota_B"] & & B \arrow[rr, "\pi_B"] & & B'' \arrow[rr] & & 0
\end{tikzcd}$$

Note that every square in the above diagram commutes, provided $F_\bullet$ have been constructed. We construct a map $\gamma_n : P''_n \to Q'_n$ so that written in a matrix form, we have

$$F_n = \begin{pmatrix} F'_n & \gamma_n \\ 0 & F''_n \end{pmatrix} : \begin{array}{c} P'_n \\ \oplus \\ P''_n \end{array} \longrightarrow \begin{array}{c} Q'_n \\ \oplus \\ Q''_n \end{array}$$

Recall that $P_n = P'_n \oplus P''_n$ and $Q_n = Q'_n \oplus Q''_n$. Now F needs to be a lift of f. So, $\eta \circ F = f \circ \varepsilon$ must hold. Restricting to the summands, we get the equations

$$f\varepsilon|_{P'_0} = \iota_B \eta' F'_0, \qquad\qquad f\varepsilon|_{P''_0} = \iota_B \eta' \gamma_0 + \eta|_{Q''_0} \, F''_0 .$$

The first relation is already satisfied as F'_0 is a lift of f', and hence,

$$f\varepsilon|_{P'_0} = f\iota_A \varepsilon' = \iota_B f' \varepsilon' = \iota_B \eta' F'_0 .$$

For the second relation, we need to find $\gamma_0 : P''_0 \to Q'_0$ such that

$$\iota_B \eta' \gamma_0 = f\varepsilon|_{P''_0} - \eta|_{Q''_0} \, F''_0 .$$

Note that

$$\pi_B \left(f\varepsilon|_{P''_0} - \eta|_{Q''_0} \, F''_0 \right) = f'' \pi_A \varepsilon|_{P''_0} - \pi_B \eta|_{Q''_0} \, F''_0 = f''\varepsilon'' - \eta'' F''_0 = 0,$$

and hence, we have a well-defined map

$$\beta := \iota_B^{-1} \left(f\varepsilon|_{P''_0} - \eta|_{Q''_0} \, F''_0 \right) : P''_0 \to B' .$$

As P''_0 is projective, we can now get a lift γ_0 in the diagram

$$\begin{tikzcd}
& P''_0 \arrow[dl, dashed, "\gamma_0"'] \arrow[d, "\beta"] & \\
Q'_0 \arrow[r, "\eta'"'] & B' \arrow[r] & 0
\end{tikzcd}$$

This immediately gives $\iota_B\eta'\gamma_0 = \iota_B\beta = f\varepsilon|_{P_0''} - \eta|_{Q_0''} F_0''$, as required. Next, for F to be lifted to a chain map, we require $dF = Fd$. Using Remark 15.8, we get

$$\begin{aligned} 0 = dF - Fd &= \begin{pmatrix} d' & \lambda \\ 0 & d'' \end{pmatrix}\begin{pmatrix} F' & \gamma \\ 0 & F'' \end{pmatrix} - \begin{pmatrix} F' & \gamma \\ 0 & F'' \end{pmatrix}\begin{pmatrix} d' & \lambda \\ 0 & d'' \end{pmatrix} \\ &= \begin{pmatrix} d'F' & d'\gamma + \lambda F'' \\ 0 & d''F'' \end{pmatrix} - \begin{pmatrix} F'd' & F'\lambda + \gamma d'' \\ 0 & F''d'' \end{pmatrix} \\ &= \begin{pmatrix} d'F' - F'd' & d'\gamma - \gamma d'' + \lambda F'' - F'\lambda \\ 0 & d''F'' - F''d'' \end{pmatrix} \end{aligned}$$

Thus, inductively, we need to solve for $\gamma_n : P_n'' \to Q_n'$ such that

$$d'\gamma_n = \gamma_{n-1}d'' + \lambda_n F_n'' - F_{n-1}'\lambda_n.$$

We compute,

$$\begin{aligned} d'(\gamma_{n-1}d'' + \lambda_n F_n'' - F_{n-1}'\lambda_n) &= (\gamma_{n-2}d'' + \lambda_{n-1}F_{n-1}'' - F_{n-2}'\lambda_{n-1})d'' \\ &\quad + d'\lambda_n F_n'' - F_{n-2}'d'\lambda_n \\ &= \lambda_{n-1}d''F_n'' - F_{n-2}'\lambda_{n-1}d'' + d'\lambda_n F_n'' - F_{n-2}'d'\lambda_n \\ &= (\lambda_{n-1}d'' + d'\lambda_n)F_n'' - F_{n-2}'(\lambda_{n-1}d'' + d'\lambda_n) \\ &= (d'\lambda_n + \lambda_{n-1}d'')(F_n'' - F_{n-2}') = 0, \end{aligned}$$

where the last equality follows from $0 = d^2 = \left(\begin{smallmatrix} d' & \lambda_n \\ 0 & d'' \end{smallmatrix}\right)\left(\begin{smallmatrix} d' & \lambda_{n-1} \\ 0 & d'' \end{smallmatrix}\right) \Rightarrow d'\lambda_{n-1} + \lambda_n d''$. Thus, we have the map

$$\beta_n := \gamma_{n-1}d'' + \lambda_n F_n'' - F_{n-1}'\lambda_n : P_n'' \to \ker(d') \subset Q_{n-1}'.$$

Now, there is a lift $\gamma_n : P_n'' \to Q_n'$ such that the diagram commutes

$$\begin{tikzcd} & P_n'' \arrow[ld, dashed, "\gamma_n"'] \arrow[d, "\beta_n"] & \\ Q_n' \arrow[r, "d'"'] & \operatorname{im}(d') = \ker(d') \arrow[r] & 0. \end{tikzcd}$$

Clearly, this lets us define $F_\bullet : P_\bullet \to Q_\bullet$ so that we have a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & P_\bullet' \arrow[r] \arrow[d, "F_\bullet'"] & P_\bullet \arrow[r] \arrow[d, "F_\bullet"] & P_\bullet'' \arrow[r] \arrow[d, "F_\bullet''"] & 0 \\ 0 \arrow[r] & Q_\bullet' \arrow[r] & Q_\bullet \arrow[r] & Q_\bullet'' \arrow[r] & 0 \end{tikzcd}$$

But then the naturality of the boundary map in Lemma 6.29 shows that the boundary map in the long exact sequence of derived functors is also natural. ☐

One of the most useful right exact functor that appears in the category of R-modules are the tensor functors.

Definition 15.13: (*Tor Functor*)

The left derived functors of the (right exact) tensor functor $\_ \otimes_R N : R\text{-Mod} \to R\text{-Mod}$ is called the *Tor functors*, and is denoted as $\operatorname{Tor}_n^R(\_, N) : R\text{-Mod} \to R\text{-Mod}$.

Example 15.14: (*Group Homology*)

Let G be a group, and R be a ring. One defines the *group ring* $R[G]$ as the free R-module generated by G (as a set), along with the multiplication $\left(\sum_{g\in G} r_g g\right)\cdot\left(\sum_{h\in G} s_h h\right) = \sum_{g\in G}\sum_{g_1g_2=g}\left(r_{g_1}s_{g_1}\right)g$. This makes $R[G]$ in to an algebra over R. An $\mathbb{Z}[G]$-module is known as a *G-module*. Now, given an $R[G]$ -module M, define the *coinvariants* as

$$M_G := M/\{m\in M \mid g\cdot m = m, \forall g\in G\}.$$

This defines a functor $(.)_G : R[G]\text{-Mod}\to R[G]\text{-Mod}$, which can be verified to be *right* exact. The left derived functors of the coinvariants is denoted as $H_i(G,M)$, the *group homology* of G with coefficients in the $R[G]$-module M. One can naturally identify $M_G = R\otimes_{R[G]} M$, where R is given the trivial $R[G]$-module structure: $\left(\sum r_g g\right)\cdot r = \sum r_g r$. In other words, taking coinvariants is same as taking the tensor product $R\otimes_{R[G]}$ _. Thus, group homology is essentially a Tor functor, i.e, $H_i(G,M) = \mathrm{Tor}_i^{R[G]}(R,M)$.

Exercise 15.15: (*Tor over PID*)

Let R be a PID. Given R-modules M,N, show that $\mathrm{Tor}_n^R(M,N) = 0$ for $n\geq 2$. In particular, over a PID R we can simply denote $\mathrm{Tor}^R(\_,\_) := \mathrm{Tor}_1^R(\_,\_)$. Moreover, when $R=\mathbb{Z}$, we can simply right $\mathrm{Tor}(\_,\_) := \mathrm{Tor}_1^{\mathbb{Z}}(\_,\_)$.

Since $M\otimes\_ : R\text{-Mod}\to R\text{-Mod}$ is also right exact (by the symmetry of the tensor when R is commutative), we can left derive $M\otimes\_$ as well and get another definition of the Tor functor. It turns out the two definitions match, and is known as *balancing the Tor*, i.e, $\mathfrak{L}_i(\_\otimes_R N)(M)\cong \mathrm{Tor}_i^R(M,N)\cong \mathfrak{L}_i(M\otimes_R\_)(N)$. Commutativity of R also leads to the isomorphism $\mathrm{Tor}_n^r(M,N)\cong\mathrm{Tor}_n^R(M,N)$ for all $n\geq 0$.

Exercise 15.16: (*Computation of Tor*)

Compute the following.

1. $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z},G)$ for any Abelian group G.
2. $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z},G)$ for any Abelian group G and $a>1$.

> **<u>Hint</u> :** Given an Abelian group G and an integer a, we have two subgroups:
>
> - $aG = \{g\in G \mid g = ah \text{ for some } h\in G\}$, i.e, the subgroup of elements divisible by a.
> - ${}_aG = \{g\in G \mid ag = 0\}$, i.e, the subgroup of elements of order dividing a.
>
> Check that $\mathbb{Z}/a\mathbb{Z}\otimes G = aG$ and $\mathrm{Tor}(\mathbb{Z}/a\mathbb{Z},G) = {}_aG$.

3. $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z},\mathbb{Z}/b\mathbb{Z})$ for $a,b>1$.
4. $\mathrm{Tor}_n^{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z})$ where $\mathbb{Z}/2\mathbb{Z}$ is given a $\mathbb{Z}/4\mathbb{Z}$-module structure in the obvious way.
5. $\mathrm{Tor}_n^{\mathbb{Z}/r\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z},\mathbb{Z}/b\mathbb{Z})$, where $a,b,r>1$ are integers, and $\mathrm{lcm}(a,b)\mid r$.
6. $\mathrm{Tor}_n^R(M,N)$ for a flat R-module M, and arbitrary R-module N.
7. $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Q},G)$ for any Abelian group G.
8. $\mathrm{Tor}_n^k(V,W)$ for a field k, and k-modules V,W.

Remark 15.17: (*Tor over Noncommutative Ring*)

Let R, S, T be arbitrary rings (possibly noncommutative). We can treat the tensor as a bifunctor

$$\_ \otimes_S \_ : R\text{-}S\text{-BiMod} \times S\text{-}T\text{-BiMod} \to R\text{-}T\text{-BiMod}.$$

It follows that $\_ \otimes_S \_$ is right exact in both places, and one can define the (left) derived Tor functor as

$$\mathrm{Tor}_n^S : R\text{-}S\text{-BiMod} \times S\text{-}T\text{-BiMod} \to R\text{-}T\text{-BiMod},$$

which is also *balanced*.

Corollary 15.18: (*Long Exact Sequence of Tor*)

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules, and N be a fixed R-module. Then, there exists a natural long exact sequence

$$\cdots \to \mathrm{Tor}_1^R(A, N) \to \mathrm{Tor}_1^R(B, N) \to \mathrm{Tor}_1^R(C, N) \to A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0.$$

Moreover, if R is a PID, then we have the 6-term long exact sequence

$$0 \to \mathrm{Tor}_1^R(A, N) \to \mathrm{Tor}_1^R(B, N) \to \mathrm{Tor}_1^R(C, N) \to A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0.$$

Proof : The existence of the long exact sequence is immediate from Theorem 15.12. When R is a PID, given any R-module M, we have a long exact sequence $0 \to 0 \to P_1 \to P_0 \xrightarrow{\varepsilon} M$, which shows that $\mathrm{Tor}_n^R(M, N) = 0$ for $n \geq 2$. This justifies the 6-term exact sequence. $\square$