Course notes for

Algebraic Topology II (KSM4E02)

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`rings — modules — tensor product — free module — projective module — injective module — flat module`

14.1 Rings and Modules

Let us recall some basic definition from (commutative) algebra.

Definition 14.1: (*Ring*)

A *ring* R is a set R equipped with two binary operations, usually denoted as *addition* and *multiplication*, such that $(R, +)$ is an *Abelian group,* $(R, \cdot)$ is a *monoid,* and the multiplication distributes over the addition, i.e, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in R$.

The above definition assumes that R has a multiplicative identity, usually denoted 1 (or 1_R), since a *monoid* is just a set M equpped with an associative binary operation and a both-sided identity for the same operation. In the lieterature, a ring without a multiplicative identity is sometimes called a *rng*.

Remark 14.2: (*Rings as Monoid Objects*)

In category theoretic language, one can say that a ring is just a monoid object in the monoidal category of Abelian groups! This abstract generalization leads to the notion of *ring spectra,* which are monoid object in the category of *spectra.*

Example 14.3: (*Endomorphism Ring*)

Given an Abelian group G, consider $E = \mathrm{End}(G, G)$ to be the collection of endomorphisms $G \to G$. Then, E is clearly a ring, where the multiplication is given by composition and the unit is the idenity map $1_G : G \to G$.

We have a natural notion of *ring homomorphism* $f : R \to S$ which preseves both the Abelian group structure, and the monoid structure. Explicitly, we have $f(r_1 - r_2) = f(r_1) - f(r_2)$, $f(r_1 r_2) = f(r_1) f(r_2)$ and $f(1_R) = 1_S$.

Definition 14.4: (*Left Module over a Ring*)

Given a ring R, a *left module* over R is an Abelian group M, equipped with a ring homomorphism $\varphi : R \to \mathrm{End}(M, M)$.

Unpacking the defintion, we are given an action map $\bullet : R \times M \to M$ given by $r \bullet m := \varphi(r)(m)$, such that the following holds.

1. $r \bullet \_$ is a group homomorphism, i.e, $r \bullet (m_1 + m_2) = r \bullet m_1 + r \bullet m_2$.
2. $r \mapsto r \bullet \_$ is a ring map, i.e, $(r_1 \cdot r_2) \bullet m = r_1 \bullet (r_2 \bullet m)$, and $1_R \bullet m = m$.

Definition 14.5: (*Right Module over a Ring*)

A *right module* over R is an Abelian group M equipped with a ring *anti*-homomorphism $R \to \mathrm{End}(M, M)$, i.e, there is a right action $M \times R \to M$ satisfying $m \bullet (r_1 \cdot r_2) = (m \bullet r_1) \bullet r_2$. The category of left (resp. right) R-modules is denoted as $R\text{-}\mathrm{Mod}$ (resp. $\mathrm{Mod}\text{-}R$).

Given a ring R, we have the notion of the *opposite ring* R^{op}, where the multiplication is flipped, i.e, $r \tilde{\cdot} s := s \cdot r$. Now, given a left R-module M, we can define a natural right R^{op}-module structure on M given by $m \tilde{\bullet} r := r \bullet m$. If R is a commutative ring, then R^{op} is naturally isomorphic to R, and hence, every left R-module is naturally a right R-module and vice versa. In this case, we only consider modules without specifying left or right, and denote the category of R-modules as $R\text{-}\mathrm{Mod}$.

Example 14.6: (*Modules over Rings*)

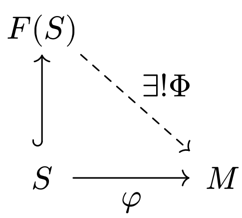
Here are some example of modules over some known rings.

1. Every ring R is a module over itself. A submodule of R is precisely an *ideal* of R.
2. A module over $\mathbb{Z}$ is precisely an Abelian group.
3. A module over a field k is precisely a vector space over k.
4. A module over the polynomial ring $k[X]$ is a vector space over k, equipped with a linear endomorphism.
5. (Restriction of scalars) If $R \to S$ is a ring map, then every S-module M is naturally an R-module by considering the composition $R \to S \to \mathrm{End}(M, M)$.

The category $R\text{-}\mathrm{Mod}$ is additive (Definition 6.19). It has arbitrary *coproduct*: explicitly, given a collection $\{M_\alpha\}_{\alpha \in \Lambda}$ of R-modules, the direct sum $\bigoplus_{\alpha \in \Lambda} M_\alpha$ is defined as the collection of formal sums $\sum_{\alpha \in \Lambda} r_\alpha m_\alpha$, where $r_\alpha \in R, m_\alpha \in M_\alpha$, and all but finitely many $r_\alpha = 0$. We also have arbitrary *product,* namely the direct product $\prod_{\alpha \in \Lambda} M_\alpha$, where the action is component-wise. In fact, $R\text{-}\mathrm{Mod}$ has all limits and colimits, and moreover, it is Abelian (Definition 6.27).

14.2 Free Module

We have the *forgetful functor* $U : R\text{-}\mathrm{Mod} \to \mathrm{Sets}$, which admits a right adjoint $F : \mathrm{Sets} \to R\text{-}\mathrm{Mod}$, called the *free R-module* functor. Explicitly, given a set S, the free R-module $F(S)$ can be constructed as the direct sum $\bigoplus_S R$. We can canonically identify $S \hookrightarrow F(S)$ by considering each s as the formal sum $1_R \cdot s \in \bigoplus_S R$. Recall that the free module has the following universal property: given any R-module M and a set map $\varphi : S \to M$, there exists a unique R-module map $\Phi : F(S) \to M$ such that the following commutes



One can think of the free module admitting a basis. In particular, as a consequence of the axiom of choice, it follows that for a field k every k-module is a free module (i.e, every k-vector space admits a basis). Recall that a ring is called a *PID* (i.e, a *principal ideal domain*) if it is an integral domain such that every ideal is generated by a single element.

Proposition 14.7: (*Submodule of Free Module over a PID*)

Let R be a PID and F be a free R-module. If $M \subset F$ is a submodule then M is again free.

Proof : Let $\{e_i \mid i \in I\}$ be a basis of F over R. By the well-ordering principal, get an well-order on the index set I. Let $F_i \subset F$ be the (free) submodule generated by all $\{e_j \mid j \in I, j \leq i\}$, and set $U_i := U \cap F_i$. Denote by $\pi_i : F \to R$ the projection along the basis element e_i. Then, $\pi_i(U_i)$ is a submodule of R, i.e, an ideal of R. Since R is a PID, we have $\pi_i(U_i) = Ra_i$ for some $a_i \in R$. Choose some $u_i \in U_i$ such that $\pi_i(u_i) = a_i$; if $a_i = 0$ choose $u_i = 0$.

By transfinite induction, one can show that $\{u_i \mid u_i \neq 0, i \in I\}$ is a basis of U. Hence, U is a free R-module $\square$

14.3 Tensor Product

Given two R-modules M, N, the hom set $\hom_R(M, N)$ is naturally an R-module. In particular, we have a functor

$$\hom_R : R\text{-Mod} \times R\text{-Mod} \to R\text{-Mod},$$

which is contravariant on the first position, and covariant on the second position. Given $N \in R\text{-Mod}$, the *tensor product* functor $\_ \otimes_R N$ is defined as the left adjoint to the covariant $\hom$ functor $\hom_R(N, \_)$. In particular, we have a natural R-module isomorphism

$$\hom_R(M \otimes_R N, P) \cong \hom_R(M, \hom_R(N, P))$$

for any $M, N, P \in R\text{-Mod}$. In particular, every bilinear map $M \times N \to P$ admits a unique lift to a linear map $M \otimes_R N \to P$. Explicitly, one constructs $M \otimes_R N$ as the quotient of the free module generated by formal symbols $m \otimes n$ via the bilinearity relations. The tensor product is naturally associative, and moreover tensoring by R (on both sides) is naturally isomorphic to the identity functor. If R is commutative, then $M \otimes_R N$ is naturally isomorphic to $N \otimes_R M$. In particular, $(R\text{-Mod}, \otimes_R, R)$ is a monoidal category, which is a *symmetric* monoidal category if R is commutative.

Exercise 14.8: (*Tensor Product as Cokernel*)

Let $f : R \to S$ be a ring map, and $M, N \in S\text{-Mod}$. Justify that we have an exact sequence

$$M \otimes_R S \otimes_R N \xrightarrow{\varphi} M \otimes_R N \to M \otimes_S N \to 0,$$

where $\varphi(m \otimes s \otimes n) := ms \otimes n - m \otimes sn$. Here we are identifying M, N, S as R-modules by restriction of scalars via the map $f : R \to S$.

When the underline ring is understood, we shall simply denote the tensor product as $\otimes$.

Exercise 14.9: (*Tensor Product of Cyclic Groups*)

Compute the tensor product $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ for integers $m, n \geq 1$. Given the $\mathbb{Z}/4\mathbb{Z}$-module structure on $\mathbb{Z}/2\mathbb{Z}$ via restriction of scalar by the canonical ring map $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, compute $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}/4\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.

In general, if $I \subset R$ is an ideal such that $MI = 0 = IN$, it follows that $M \otimes_R N \cong M \otimes_{R/I} N$. Tensor pruduct is additive, even for arbitrary sum. In fact, $\_ \otimes N$ being a left adjoint functor, preserves all colimits.

Proposition 14.10: (*Tensor Product and Arbitrary Coproduct*)

Given an arbitrary direct sum $\bigoplus_{i\in I} M_i$ of R-modules, we have $\left(\bigoplus_{i\in I} M_i\right) \otimes N = \bigoplus_{i\in I}(M_i \otimes N)$ and $N \otimes \left(\bigoplus_{i\in I} M_i\right) = \bigoplus_{i\in I}(N \otimes M_i)$

<u>**Proof**</u> **:** We proof one of them via the universal property. Let $\varphi : \left(\bigoplus_{i\in I} M_i\right) \times N \to \bigoplus_{i\in I}(M_i \otimes N)$ be given as $(\sum m_i, n) \mapsto \sum m_i \otimes n$, which is easily seen to be an R-bilinear map. Hence, there is a unique lift $\Phi : \left(\bigoplus_{i\in I} M_i\right) \otimes N \to \bigoplus_{i\in I}(M_i \otimes N)$. On the other hand, for each $j \in I$, we have a bilinear map $\psi_j : M_j \times N \to \left(\bigoplus_{i\in I}\right) \times N$ given by $(m_j, n) \mapsto m_j \otimes n$, which lifts to a map $\Psi_j : M_j \otimes N \to \left(\bigoplus_{i\in I} M_i\right) \otimes N$. Taking direct sum, we have $\Psi : \bigoplus_{i\in I}(M_i \otimes N) \to \left(\bigoplus_{i\in I} M_i\right) \otimes N$. It is easy to see that Φ and Ψ are inverses of each other, proving the claim. □

14.4 Projective, Injective and Flat Modules

Definition 14.11: (*Exact Functor*)

An additve functor $F : R\text{-Mod} \to R\text{-Mod}$ is called *left exact* if given any short exact sequence $0 \to M \to N \to P \to 0$ we have $0 \to F(M) \to F(N) \to F(P)$ is again exact, and is called *right exact* if $F(M) \to F(N) \to F(P) \to 0$ is exact.

In the definition of the exactness, we do not need to start with a *short* exact sequence. Indeed, F is left exact if and only for any exact sequence $0 \to M \to N \to P$, we have $0 \to F(M) \to F(N) \to F(P)$ is exact, i.e, if F preserves kernel. Similarly, righ exactness can be characterized as F is right exact if and only if it preserves cokernel.

Exercise 14.12: (*Exactness and Adjoint*)

Verify that given an adjoint pair of additive functors $R\text{-Mod} \underset{\mathcal{R}}{\overset{\mathcal{L}}{\rightleftarrows}} S\text{-Mod}$, we have $\mathcal{L}$ is *right* exact, and $\mathcal{R}$ is *left* exact. In particular, the covariant $\hom$ functor $\hom(N, \_)$ is left exact, and the tensor product functor $\_ \otimes N$ is right exact.

Exercise 14.13: (*Exactness of Hom*)

Verify that both the covariant and contravariant $\hom$ functors are left exact.

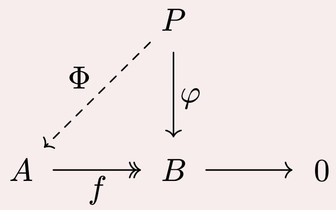
Exercise 14.14: (*Exactness of Tensor*)

Verify that both $M \otimes \_$ and $\_ \otimes M$ are right exact functors.

Observe that if F is a free R-module, then $\hom(F, \_)$ is right exact as well. In general, $\hom$ is *not* right exact. This leads to the following important notion.

Definition 14.15: (*Projective Module*)

A module $P \in R\text{-Mod}$ is called *projective* if the $\hom$ functor $\hom(P, \_)$ is right exact. In other words, given any epimorphism $f : A \to B$ and any map $\varphi : P \to B$, there is a lift $\Phi : P \to A$ such that the diagram commutes



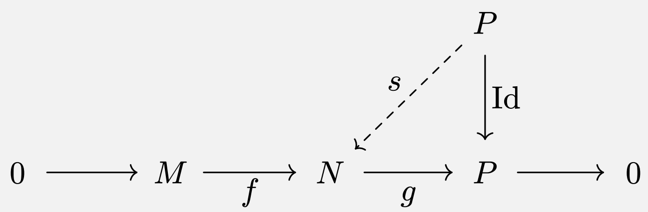
As noted earlier, every free module is projective. If R is a PID, then every projective module is also free.

Proposition 14.16: (*Characterization of Projective Modules*)

Given an R-module P, the following are equivalent.

1. P is projective.
2. Any short exact sequence $0 \to M \to N \to P \to 0$ splits.
3. P is a direct summand of a free R-module, i.e, there exists an R-module Q such that $P \oplus Q$ is free.

Proof : Suppose P is projective. Let $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ be a short exact sequence. Then, we have a map $s : P \to N$ in the commutative diagram



Clearly, s is then a splitting.

Next, assume that any short exact sequence $0 \to M \to N \to P \to 0$ splits. Let $S \subset P$ be a set of generators $F = F(S)$ be the free R-module. Then, we have a epimorphism $p : F \twoheadrightarrow P$. Let $Q = \ker(p)$, so that we have a short exact sequence $0 \to Q \to F \to P \to 0$. But then this splits, and we have $F = P \oplus Q$.

Finally, suppose there is a module Q such that $F = P \oplus Q$. We have the isomorphism $\hom(F, C) = \hom(P \oplus Q, C) = \hom(P, C) \oplus \hom(Q, C)$ for any $C \in R\text{-Mod}$. Now, given an epimorphism $B \to C$, we have the commutative diagram

$$\begin{array}{ccc} \hom(F,B) & \longrightarrow & \hom(F,C) \\ \cong \downarrow & & \downarrow \cong \\ \hom(P,B)\oplus\hom(Q,B) & \longrightarrow & \hom(P,C)\oplus\hom(Q,C) \end{array}$$

Since F is free, the blue arrow is surjective, and hence, the green arrow is also an epimorphism. But the green arrow is surjective precisely when each component arrow is surjective. In particular, it follows that P (and Q as well) are projective. □

Exercise 14.17: (*Direct Summand of Projective*)

Let $P = \bigoplus_{i\in I} P_i$ be an arbitrary direct sum of R-modules. Show that P is projective if and only if each P_i is projective.

Remark 14.18: (*Eilenberg-Mazur Swindle*)

Given a projective R-module P, one can get *free* module A such that $P \oplus A$ if free. Indeed, by Proposition 14.16, we have some module Q such that $F := P \oplus Q$ is free. Then, consider A to be infinite sum

$$A := (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots = F \oplus F \oplus \cdots,$$

which is clearly a free module. Now,

$$P \oplus A = P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots = (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots = F \oplus F \oplus \cdots,$$

which is again free. This *rearranging* is counterintuitive but completely valid, and is called Mazur's swindle! The same does not work in real analysis with infinite series sum, as evident by Riemann rearrangement theorem. Indeed, it leads to the *proof* that $1 = 1 + (-1+1) + (-1+1) + \cdots = (1 - 1) + (1-1) + \cdots = 0$.

As an immediate corollary to the above characterization, we have the following example.of a projective module which is *not* free.

Example 14.19: (*Projective $\not\Rightarrow$ Free*)

Consider $R = \mathbb{Z}/6\mathbb{Z}$. Then, R itself is trivially free over R. Now, $R = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is a direct sum decomposition. Hence, both $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are projective R-modules. But neither can be free, since any (nonzero) free R-module must have at least 6 elements.

A consequence of Proposition 14.7 and Proposition 14.16 is the following.

Corollary 14.20: (*Projective over PID*)

Let R be a PID. Then any projective R-module is free.

Similarly, the failure of left exactness of the tensor functor leads to the following definition.

Definition 14.21: (*Flat Module*)

An R-module M is called *flat* if the functor $\_ \otimes M$ is left exact. In other words, M is flat if given any monomorphism $f : A \hookrightarrow B$, the induced map $f \otimes \mathrm{Id}_M : A \otimes M \to B \otimes M$ is injective.

It follows that any projective module is a flat module.

Proposition 14.22: (*Projective $\Rightarrow$ Flat*)

A projective module is flat.

<u>**Proof**</u> **:** Let $f : A \to B$ be a monomorphism. Let us first consider a free module $F = F(S)$, where S is the generating set. Consider the map $\Phi := f \otimes \mathrm{Id}_F : A \otimes F \to B \otimes F$. Now, $F = \bigoplus_{s \in S} R$, so that $A \otimes F = A \otimes \left(\bigoplus_{s\in S} R\right) = \bigoplus_{s \in S}(A \otimes R) = \bigoplus_{s \in S} A$, and similarly, $B \otimes F = \bigoplus_{s \in S} B$. The map Φ is seen to be identified with the map $\bigoplus_{s \in S} f$. Since f is monic, it follows that Φ is monic, proving that F is flat. Next, let P be a projective R-module. Then, there exists a module Q such that $F = P \oplus Q$ is free. Now, we have the commutative diagram

$$\begin{array}{ccc} A \otimes F & \xrightarrow{\quad f \otimes \mathrm{Id}_F \quad} & B \otimes F \\ \cong \downarrow & & \downarrow \cong \\ (A \otimes P) \oplus (A \otimes Q) & \xrightarrow[f \otimes \mathrm{Id}_P + f \otimes \mathrm{Id}_Q]{} & (B \otimes Q) \oplus (B \otimes Q) \end{array}$$

The vertical isomorphisms follow from Proposition 14.10. Now, F being a free module is a flat module and hence the top row is a monomorphism. But then the bottom row is a monomorphism as well. This is possible if and only if each component is a monomorphism. In particular, P is a flat module. $\square$

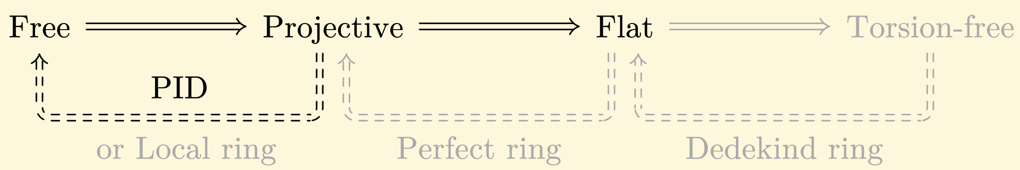
Just as Exercise 14.17, we have the following fact: given a direct summand $M = \oplus_{i \in I} M_i$, the module M is flat if and only if M_i is flat.

Example 14.23: (*Flat $\not\Rightarrow$ Projective*)

Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module (i.e, an Abelian group). It follows that $\mathbb{Q}$ is flat, but not a projective module.

Remark 14.24:

We have the following (strict) implications, where the reverse inclusion holds when the underlying ring has certain extra properties.



Finally, the failure of the right exactness of the contravariant $\hom$ functor leads to the following definition.

Definition 14.25: (*Injective Module*)

An R-module I is called ***injective*** if $\hom(\_, I)$ is right exact. In other words, given any monomorphism $f : A \to B$, and any map $\varphi : A \to I$, there is a map $\Phi : B \to I$ such that the diagram commutes

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B \\ & & \varphi \big\downarrow & \swarrow_{\Phi} & \\ & & I & & \end{array}$$

Just as in Proposition 14.16, it follows that I is an injective R-module if and only if any short exact sequence $0 \to I \to M \to N \to 0$ splits. As an example, $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. Unlike the projective modules, injective modules are harder to construct.