Course notes for

Algebraic Topology II (KSM4E02)

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`homological degree – degree of suspension – local degree – cellular boundary map – cellular and singular homology`

13.1 Homological Degree and Cellular Boundary Formula

In order to compute the cellular boundary map, we need to understand the notion of the *degre* of a self map of a sphere.

Definition 13.1: (*Degree of a Self-map of a Sphere*)

Given a map $f : S^n \to S^n$, the *homological degree* of the map $d(f)$ is defined to be the integer such that $\tilde{H}_\star(f) : \tilde{H}_\star(S^n) \to \tilde{H}_\star(S^n)$ is given as the multiplication by $d(f)$. Recall, $\tilde{H}_\star(S^n) = \begin{cases} \mathbb{Z}, & \star = n \\ 0, & \text{otherwise.} \end{cases}$

Exercise 13.2: (*Properties of Homological Degree*)

Verify the following.

1. If $f \sim g : S^n \to S^n$ are homotopic, then $d(f) = d(g)$. In paarticular, if f is null-homotopic then $d(f) = 0$.
2. Given $S^n \overset{f}{\to} S^{\overset{g}{\to}} S^n$, we have $d(g \circ f) = d(g) \cdot d(f)$.
3. If $f : S^n \to S^n$ is a homeomorphism (or even a homotopy equivalence), then $d(f) = \pm 1$.
4. What are the possible degrees of a map $S^0 \to S^0$?

Let us show some other interesting properties of the degree. Let us first re-interpret the suspension isomorphism. Given the sphere S^n, denote the two hemi-spheres by $D^n_\pm$. Then, we have two maps

$$s_+ : (D^n_-, S^{n-1}) \hookrightarrow (S^n, D^n_+), \quad s_- : (D^n_+, S^{n-1}) \hookrightarrow (S^n, D^n_-),$$

given by reflection about the equator. These give rise to the *suspension isomorphisms*

$$\sigma_\pm : \tilde{H}_{k-1}(S^{n-1}) \xleftarrow[\cong]{\partial} H_k(D^n_\mp, S^{n-1}) \xrightarrow[\cong]{H_k(s_\pm)} H_k(S^n, D^n_\pm) \xleftarrow[\cong]{j_\star} \tilde{H}_k(S^n).$$

Exercise 13.3:

Verify that $\partial, j_\star$ and $H_k(s_\pm)$ are indeed isomorphisms!

Consider the diagram

$$\begin{tikzcd}
 & & \tilde{H}_k(S^n) \arrow[dll, "H_k(l_+)"'] \arrow[drr, "H_k(l_-)"] \arrow[dd, "H_k(j_0)"'] & & \\
H_k(S^n, D^n_+) & & & & H_k(S^n, D^n_-) \\
 & & H_k(S^n, S^{n-1}) \arrow[ull, "H_k(j_+)"] \arrow[urr, "H_k(j_-)"'] \arrow[dd, "\partial"'] & & \\
H_k(D^n_-, S^{n-1}) \arrow[uu, "H_k(s_+)", "\cong"'] \arrow[urr, "H_k(\iota_-)"] \arrow[drr, "\partial_+"'] & & & & H_k(D^n_+, S^{n-1}) \arrow[uu, "H_k(s_-)"', "\cong"] \arrow[ull, "H_k(\iota_+)"'] \arrow[dll, "\partial_-"] \\
 & & \tilde{H}_k(S^{n-1}) & &
\end{tikzcd}$$

Using Lemma 5.1, it follows that $\sigma_+ = -\sigma_-$.

Proposition 13.4: (*Degree of the Antipodal Map*)

Degree of any reflection map $S^n \to S^n$ about an hyperplane through origin is -1, degree of the antipodal map $\eta : S^n \to S^n$ is $(-1)^{n+1}$.

Proof : Consider the map $t : S^n \to S^n$ which reflects about S^{n-1}, i.e, $t(x_1, ..., x_n, x_{n+1}) = (x_1, ..., x_n, -x_{n+1})$. It is easy to see that $t_\star\sigma_+ = \sigma_- = -\sigma_+$, and hence, $d(t) = -1$. In particular, degree of any reflection about an hyperplane through origin is -1, since any two such reflections are homotopic. The antipodal map $\eta : S^n \to S^n$ can be written as $(n+1)$-fold composition of reflections. Hence, it follows that $d(\eta) = (-1)^{n+1}$. ☐

Next, we compute the degree of the suspension of a map.

Proposition 13.5: (*Degree of Suspension*)

Let $f : S^n \to S^n$ be a map, and $\Sigma f : S^{n+1} = \Sigma S^n \to \Sigma S^n = S^{n+1}$ be the suspension. Then, $d(\Sigma f) = d(f)$.

Proof : Consider the pair (CS^n, S^n) and the quotient map $q : (CS^n, S^n) \to \Sigma S^n = S^{n+1}$. The induced map of (Cf, f) in the quotient is precisely the suspension Σf. By Theorem 10.14, $\tilde{H}_\star(q)$ is an isomorphism, also the boundary map for the pair is the suspension isomorphism. Naturality of the boundary map gives the commutative diagram

$$\begin{tikzcd}
\tilde{H}_{k+1}(S^{n+1}) \arrow[d, "\tilde{H}_k(\Sigma f)"'] & H_{k+1}(CS^n, S^n) \arrow[l, "H_{k+1}(q)"'] \arrow[r, "\partial"] \arrow[d, "\tilde{H}_k(Cf, f)"'] & \tilde{H}_k(S^n) \arrow[d, "\tilde{H}_k(f)"] \\
\tilde{H}_{k+1}(S^{n+1}) & H_{k+1}(CS^n, S^n) \arrow[l, "H_{k+1}(q)"] \arrow[r, "\partial"'] & \tilde{H}_k(S^n)
\end{tikzcd}$$

The commutativity forces that $d(f) = d(\Sigma f)$. ☐

Exercise 13.6: (*Constructing Map of Specified Degree*)

Given any integer $k \in \mathbb{Z}$ and $n \geq 1$, construct a map $f : S^n \to S^n$ such that $d(f) = k$.

Hint : Start with the k-fold map $S^1 \to S^1$ and then suspend it $(n-1)$-times.

Next, we prove a formula to compute the degree of a map via *local degree*. Let $f : S^n \to S^n$ be a map. Suppose, there exists $y \in S^n$ such that $f^{-1}(y)$ is a finite set, say, $f^{-1}(y) = \{x_1, .., x_k\} \subset S^n$. By continuity of f, we can fix open sets $y \in\subset V \subset S^n$ and $x_i \in U_i \subset f^{-1}(V) \subset S^n$, such that $V_i \cap V_j = \emptyset$ for $i \neq j$. Now, $f(U_i \setminus x_i) \subset V \setminus i$ for $1 \leq i \leq k$. We have a commutative diagram

$$\begin{tikzcd}
 & H_n(U_i, U_i \setminus x_i) \arrow[r, "f_\star"] \arrow[dl, "\cong"'] \arrow[d, "q_i"] & H_n(V, V \setminus y) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus x_i) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_\star"] & H_n(S^n, S^n \setminus y) \\
 & H_n(S^n) \arrow[u, "j"'] \arrow[ul, "\cong"] \arrow[r, "f_\star"'] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

The blue isomorphisms are via excision, and the red isomorphisms follow from long exact sequence as $S^n \setminus p$ is contractible for any $p \in S^n$. In particular, the two homology groups on the top row are then isomorphic to $H_n(S^n) = \mathbb{Z}$ via the isomorphisms. The other maps are induced from obvious space level maps, which gives the commutativity.

Definition 13.7: (*Local Degree*)

The local degree of $f : S^n \to S^n$ at x_i, denoted as $d(f)|_{x_i}$ is the integer that gives the map $H_n(U_i, U_i \setminus x_i) \to H_n(V, V \setminus y)$ via multiplication.

Local degree is independent of choice of U_i and V, and thus, it is indeed local. If f maps U_i homeomorphically onto V, then $d(f)|_{x_i} = \pm 1$. In practice, this situation appears many times, which helps us to compute the degree of a map using the next proposition (e.g Exercise 13.9).

Proposition 13.8: (*Degree is Sum of Local Degree*)

Let $f : S^n \to S^n$ be such that $f^{-1}(y) = \{x_1, ..., x_k\}$. Then, $d(f) = \sum d(f)|_{x_i}$.

Proof : We again consider the diagram

$$\begin{tikzcd}
& H_n(U_i, U_i \setminus x_i) \arrow[r, "f_\star"] \arrow[d, "q_i"] \arrow[dl, "\varphi_i"'] & H_n(V, V \setminus y) \arrow[d, "\varphi"] \\
H_n(S^n, S^n \setminus x_i) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_\star"] & H_n(S^n, S^n \setminus y) \\
& H_n(S^n) \arrow[u, "j"] \arrow[ul, "\psi_i"] \arrow[r, "f_\star"'] & H_n(S^n) \arrow[u, "\psi"]
\end{tikzcd}$$

It follows via excision that

$$H_n(S^n, S^n \setminus f^{-1}(y)) \cong H_n\left(\bigsqcup_{i=1}^{k} U_i, \bigsqcup_{i=1}^{k} (U_i \setminus x_i)\right) \cong \bigoplus_{i=1}^{k} H_n(U_i, U_i \setminus x_i).$$

Moreover, the inclusion of the i^{th} summand is via the map q_i, and the projection to the i^{th} summand is via the map p_i. Thus, we have the equation

$$\sum q_i \varphi_i^{-1} p_i = \text{Id}.$$

Fix a generator $z \in H_n(S^n)$. This gives generators $z_i := \varphi_i^{-1}\psi_{i(z)}$ for $H_n(U_i, U_i \setminus x_i)$, and generator $z_0 = \varphi^{-1}\psi(z)$ of $H_n(V, V \setminus y)$. By definition of (local) degree

$$f_\star(z) = d(f) \cdot z, \quad f_\star(z_i) = d(f)|_{x_i} \cdot z_0.$$

We compute,

$$\begin{aligned} d(f) \cdot \psi(z) = \psi f_\star(z) = f_\star j(z) = f_\star\Big(\sum q_i \varphi_i^{-1} p_i\Big) j(z) = \sum \varphi f_\star \varphi_i^{-1} \psi_i(z) = \varphi\Big(\sum f_\star(z_i)\Big) \\ = \varphi\Big(\sum d(f)|_{x_i} \cdot z_0\Big) = \sum d(f)|_{x_i} \cdot \varphi(z_0) = \sum d(f)|_{x_i} \, \psi(z). \end{aligned}$$

Since ψ is an isomorphism, we get the formula $d(f) = \sum d(f)|_{x_i}$. ☐

Exercise 13.9: (*Constructing Map of Specified Degree*)

Consider the fold map $\vee_{i=1}^{k} S^n \to S^n$, and the "diagonal" map $S^n \to \vee_{i=1}^{k} S^n$. One way to construct the diagonal map is to first fix k many distinct points and their disjoint neighborhoods, and then collapsing the complement to a point. Verify that the composition $S^n \to \vee_{i=1}^{k} S^n \to S^n$ has degree k. Composing with a reflection about a hyperplane gives a map with degree $-k$.

Example 13.10: (*Cell Decomposition of* $\mathbb{RP}^n$)

Let us give a cellular decomposition of $\mathbb{RP}^n$, similar to Example 12.5. Recall, $\mathbb{RP}^n$ is the collection of lines in $\mathbb{R}^{n+1}$, i.e, $\mathbb{RP}^n = \mathbb{R}^{n+1}/_\sim$, where $x \sim \lambda x$ for $\lambda = \pm 1$. Restricting to unit vectors, we equivalently get $\mathbb{RP}^n = S^n/_\sim$, where $x \sim y$ if and only if (x, y) are antipode points, i.e, $y = -x$. We can realize $\mathbb{RP}^n$ as the orbit space of the S^0 action on S^n, where the nontrvial map is the antipode map, i.e, $\mathbb{RP}^n = S^n/S^0$. Clearly $\mathbb{RP}^0 = \{\star\}$ is a singleton.

In order to get a cell decomposition, observe that there is a canonical inclusion $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$ induced by the inclusion $\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+2}$ as the hyperplane with last coordinate 0. The points in $\mathbb{RP}^{n+1} \setminus \mathbb{RP}^n$ represents the lines that are *uniquely* determined by a point in the, say, upper hemishpere $D_+^{n+1} = \{(x_0, ..., x_{n+1}) \in S^{n+1} \subset \mathbb{R}^{n+2} \mid x_{n+1} > 0\}$. Thus, we see that $\mathbb{RP}^{n+1}$ is obtained from $\mathbb{RP}^n$ by attaching a single $(n+1)$-cell. Explicitly, we have the characteristic map

$$\begin{aligned} \Phi : D^{n+1} &\longrightarrow \mathbb{RP}^{n+1} \\ \mathbf{x} &\longmapsto \left[x_0 : \cdots : x_n : \sqrt{1 - \|\mathbf{x}\|^2}\right], \end{aligned}$$

where $\mathbf{x} = (x_0, ..., x_n)$, and $\|\mathbf{x}\|^2 = \sum x_i^2$. The attaching map $\varphi = \Phi|_{S^{n+1}}$ is in fact the universal (double) cover. Immediately, it follows that $\mathbb{RP}^1 = \mathbb{RP}^0 \cup_\varphi D^1 = S^1$, where $\varphi = \Phi|_{S^0 = \partial D^1}$ is the attaching map, which is just the constant map.

13.2 Cellular Boundary Formula

Using Proposition 12.1, for each $(n+1)$-cell E_α^{n+1}, and each n-cell E_β^n, we have components of the map ∂, denoted as

$$m(\alpha, \beta) : H_{k+1}(D_\alpha^{n+1}, S_\alpha^n) \to H_k\left(D_\beta^n, S_\beta^{n-1}\right).$$

Observe that we have the composition

$$\iota(\alpha, \beta) : S_\alpha^n \xrightarrow{\varphi_\alpha} X^n \xrightarrow{q_\beta} X^n / \left(X^n \setminus E_\beta^n\right) \xleftarrow[\cong]{\Phi_\beta} D_\beta^n / S_\beta^{n-1}.$$

Fixing a homeomorphism $\kappa_n : D^n / S^{n-1} \to S^n$, we have the composed map

$$\kappa_n \circ \iota(\alpha, \beta) : S_\alpha^n \longrightarrow S_\beta^n$$

as a self-map of spheres. We now give a formula for the cellular boundary map in terms of the *degree* of self-map of spheres. Indeed, we identify $m(\alpha, \beta)$ as the degree $d(\kappa_n \circ \iota(\alpha, \beta))$.

It is easy to see that $m(\alpha, \beta) \circ \sigma : H_n(D_\alpha^n, S_\alpha^{n-1}) \to H_{n+1}(D_\alpha^{n+1}, S_\alpha^n) \to H_n\left(D_\beta^n, S_\beta^{n-1}\right)$ is nothing but multiplication by $d(\alpha, \beta)$. Here σ is the *relative suspension isomorphism*.

Lemma 13.11: (*Relative Suspension Isomorphism*)

Given a pair (X, A), we have isomorphisms

$$H_n(X, A) \to H_n(X \times \partial I \cup A \times I, X \times \{0\} \cup A \times I) \overset{\partial}{\leftarrow} H_{n+1}(X \times I, X \times \partial I \cup A \times I),$$

where the first map is given by identifying (X, A) with $(X \times \{1\}, A \times \{1\})$, and the boundary map is from the triple $(X \times I, X \times \partial I \cup A \times I, X \times 0 \cup A \times I)$. The composition is denoted as $\sigma_{X,A}$

Proof : The first map is an isomorphism since we can excise out $X \times \{0\}$, and then use a strong deformation retract. The boundary map is part of the exact sequence, where the groups $H_\bullet(X \times I, X \times \{0\} \cup A \times I) = 0$ since the inclusion $X \times \{0\} \cup A \times I \hookrightarrow X \times I$ is a homotopy equivalence (both deformation retracts onto $X \times \{0\}$). But then by exactness, ∂ is an isomorphism ☐

Let us now describe the cellular boundary formula via homological degree.

Theorem 13.12: (*Cellular Boundary Formula*)

We have a commutative diagram

$$\begin{tikzcd}
H_{k+1}(D^{n+1}_\alpha, S^n_\alpha) \arrow[r, "m(\alpha,\beta)"] \arrow[d, "\partial"'] & H_n\left(D^n_\beta, S^{n-1}_\beta\right) \arrow[d, "(q_\beta)_\star"] \\
\tilde{H}_k(S^n_\alpha) \arrow[r, "\iota(\alpha,\beta)_\star"'] & \tilde{H}_k\left(D^n_\beta / S^{n-1}_\beta\right)
\end{tikzcd}$$

Consequently, given a relative suspension isomorphism $\sigma : H_k(D^n, S^{n-1}) \to H_{k+1}(D^{n+1}, S^n)$, we have $m(\alpha,\beta) \circ \sigma$ is multiplication by the degree of the map $\kappa_n \circ \iota(\alpha,\beta)$, provided $\partial \circ \sigma = (\kappa_n)_\star \circ (q_\beta)_\star$ holds.

Proof : We have the following diagram, where the solid arrows are commutative

$$\begin{tikzcd}
H_{k+1}(X^{n+1}, X^n) \arrow[r, blue, "\partial"] & \tilde{H}_k(X^n) \arrow[r, blue, "j_\star"] & H_k(X^n, X^{n-1}) \arrow[r, "q_\star"] \arrow[d, blue] & \tilde{H}_k(X^n/X^{n-1}) \arrow[d] \\
H_{k+1}(D^{n+1}_\alpha, S^n_\alpha) \arrow[u, blue, "(\Phi_\alpha)_\star"', "\cong"] \arrow[r, "\partial"'] \arrow[rrd, blue, bend right, "m(\alpha,\beta)"'] & \tilde{H}_k(S^n_\alpha) \arrow[u, "(\varphi_\alpha)_\star"'] \arrow[r, red, "(\varphi_\alpha)_\star"'] & H_k\left(X^n, X^n \setminus e^n_\beta\right) \arrow[r, red, "(q_\beta)_\star"] & \tilde{H}_k\left(X^n / X^n \setminus e^n_\beta\right) \\
 & & H_k\left(D^n_\beta, S^{n-1}_\beta\right) \arrow[u, blue, "(\Phi_\beta)_\star", "\cong"'] \arrow[r, "(q_\beta)_\star"'] \arrow[llu, dashed, bend left, "\sigma_{D^n, S^{n-1}}"] & \tilde{H}_k\left(D^n_\beta / S^{n-1}_\beta\right) \arrow[u, red, "\cong", "(\varphi_\beta)_\star"'] \arrow[d, dashed, "(\kappa_n)_\star"] \\
 & & & \tilde{H}_k\left(S^n_\beta\right)
\end{tikzcd}$$

Note that $m(\alpha,\beta)$ is given by following the blue arrows, whereas $\iota(\alpha,\beta)$ is given by following the red arrows. This gives the commutativity of the diagram in the statement.

Next, fix a relative suspension isomorphism $\sigma = \sigma_{D^n, S^{n-1}}$ and a homeomoephism $\kappa_n : D^n/S^{n-1} \to S^n$, such that $\partial \circ \sigma = (\kappa_n)_\star \circ (q_\beta)_\star$. Here, we identify $S^n_\alpha = S^n = S^n_\beta$ etc. Then, we have a diagram

$$\begin{tikzcd}
H_k(D^n, S^{n-1}) \arrow[r, blue, "\sigma"] \arrow[d, blue, "q_\star"'] & H_{k+1}(D^{n+1}, S^n) \arrow[r, red, "m(\alpha,\beta)"] \arrow[d, blue, "\partial"] & H_k(D^n, S^{n-1}) \arrow[r, blue, "\sigma"] \arrow[d, blue, "q_\star"] & H_{k+1}(D^{n+1}, S^n) \arrow[d, blue, "\partial"] \\
\tilde{H}_k(D^n/S^{n-1}) \arrow[r, blue, "(\kappa_n)_\star"'] & \tilde{H}_k(S^n) \arrow[r, red, "\iota(\alpha,\beta)_\star"'] & \tilde{H}_k(D^n/S^{n-1}) \arrow[r, blue, "(\kappa_n)_\star"'] & \tilde{H}_k(S^n)
\end{tikzcd}$$

All the blue arrows are isomorphisms. Since the left and right squares (they are identical) are commutative by assumption, it follows that $m(\alpha,\beta) \circ \sigma$ is same multiplying by the degree of the map $\kappa_n \circ \iota(\alpha,\beta)$. ☐

Note that the cellular boundary formula depends on the choice of the suspension isomorphism σ, as we saw earlier we can get another isomorphism which differs by a sign. On the other hand, if we fix them once and use it for all the cells, then the cellular boundary map is determined up to a sign. Indeed, we can fix $\sigma : H_0(D^0, S^{-1}) \to H_1(D^1, S^0)$, and then use its iterates. Thus, the cellular homology computation will be unaffected.

Example 13.13: (*Cellular Homology of* $\mathbb{RP}^n$)

In Example 13.10, we say a cell decomposiont $\mathbb{RP}^{n+1} = \mathbb{RP}^n \cup_\varphi e^{n+1}$, where the attaching map $\varphi : S^n \to \mathbb{RP}^n$ is given as the restriction of the map

$$\begin{aligned} \Phi : D^{n+1} &\longrightarrow \mathbb{RP}^n \\ (x_0, ..., x_n) &\longmapsto \left[x_0 : \cdots : x_n : \sqrt{1 - \sum x_i^2}\right]. \end{aligned}$$

We compute the cellular boundary map for this decomposition. By Theorem 13.12, we need to understand the degree of the map $\iota : S^n \xrightarrow{\varphi} \mathbb{RP}^n \to \mathbb{RP}^n/(\mathbb{RP}^n \setminus e^n) \xrightarrow{\cong} S^n$, where φ is the universal covering map. Note that $\mathbb{RP}^n/(\mathbb{RP}^n \setminus e^n) = \mathbb{RP}^n/\mathbb{RP}^{n-1}$. Fix some $y \in \mathbb{RP}^n \setminus \mathbb{RP}^{n-1}$. Then, $y = [x_0 : ... : x_n : t]$, where $t = \sqrt{1 - \sum x_i^2} \neq 0$. Then, $\iota^{-1}(y) = \{\pm z\}$, where $z = \left(\frac{x_0}{t}, ..., \frac{x_n}{t}\right)$. Fix the open hemi-spheres $D^n_\pm$, and without loss of generality assume that $z \in D^n_+$ (otherwise relabel z by $-z$). It follows that φ is identity on D^n_+, and we can get φ on D^n_- by by first applying the antipode map on D^n_- (which maps it to D^n_+), followed by φ. Thus, from Proposition 13.8, it follows that the degree of ι is $1 + (-1)^{n+1}$. Hence, denoting $C_n := C_n^{\text{cell}}(\mathbb{RP}^n) = H_n(\mathbb{RP}^n, \mathbb{RP}^{n-1})$, it follows that the cellular boundary map $\partial_n : C_n \to C_{n-1}$ is given by $1 + (-1)^{(n-1)+1} = 1 + (-1)^n$. In particular, we have

$$H_k(\mathbb{RP}^{2n+1}) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/2\mathbb{Z}, & 0 < k < 2n+1, k \text{ odd} \\ \mathbb{Z}, & k = 2n+1 \\ 0, & k > 2n+1. \end{cases} \qquad H_k(\mathbb{RP}^{2n}) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/2\mathbb{Z}, & 0 < k < 2n, k \text{ odd} \\ 0, & k \geq 2n. \end{cases}$$

Exercise 13.14: (*Homology of* $S^p \times S^q$)

Extend Exercise 12.6 by computing the ceullar (and hence the singular) homology of $S^p \times S^q$ for any p, q.

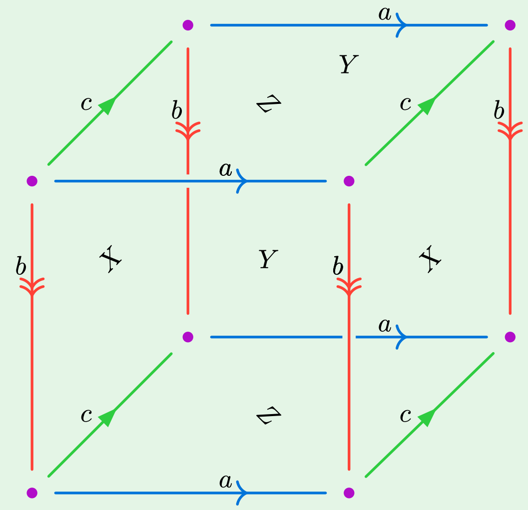
Hint : Justify $S^p \times S^q = (S^p \vee S^q) \cup_\varphi e^{p+q}$, where $\varphi : S^{p+q-1} \to S^p \vee S^q$ is the attaching map.

Exercise 13.15: (*Cellular Homology of* S^n)

Compute the celluar homology of the n-shpere with exactly two k-cells in each dimension, $0 \leq k \leq n$. Compute the cellular homology of S^∞.

Example 13.16: (*Cellular Homology of* 3*-Torus* $S^1 \times S^1 \times S^1$)

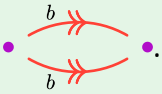
The three torus $\mathbb{T}^3 = S^1 \times S^1 \times S^1$ can be given the following CW structure.



There is **one** 0-cell, say, v, **three** 1-cells, a, b, c, **three** 2-cells, say, X, Y, Z, and **one** 3-cell, say, Δ. The two cell X denotes the left-right face, Y denotes the front-back face, and Z denotes the top-bottom face, identified accordingly. The cellular chain complex $C_n := C_n^{\text{cell}}(X)$ is given as

$$\cdots \to \underset{0}{C_4} \to \underset{\mathbb{Z}\langle \Delta \rangle}{C_3} \xrightarrow{\partial_3} \underset{\mathbb{Z}\langle X, Y, Z \rangle}{C_2} \xrightarrow{\partial_2} \underset{\mathbb{Z}\langle a, b, c \rangle}{C_1} \xrightarrow{\partial_1} \underset{\mathbb{Z}\langle v \rangle}{C_0} \to 0.$$

We compute ∂_i by Theorem 13.12.

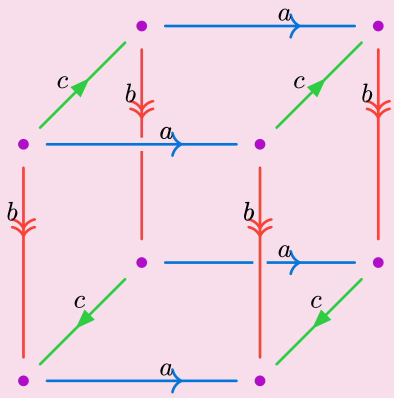
- We have $\partial_1 : H_1(X_1, X_0) \to H_0(X_0, X_{-1})$. We have X_1 is wedge of three circles and $X_{-1} = \emptyset$, and hence, $\partial_1 : H_1(X_1, \{v\}) \to H_0(\{v\}) \to H_0(\{v\})$ is simply the boundary map of the pair $(X_1, \{v\})$. As X_1 is path connected, it follows that $\partial_1 = 0$.
- To compute $\partial_2(X) = a_X a + b_X b + c_X c$, it is easy to see that $a_X = 0$. To compute b_X, we consider the map $S^1 \to S^1$ given by the attaching map . Since the two edges differ by a reflection, it follows from Proposition 13.8 and Proposition 13.4 that $b_X = 0$. Similarly, $c_X = 0$, which makes $\partial_2(X) = 0$. Repeating this argument for Y and Z, we see that $\partial_2 = 0$.
- Finally, we compute $\partial_3(\Delta)$. Say, $\partial_3(\Delta) = uX + vY + wZ$. In order to compute u, we consider the map $S^2 \to S^2$ which identfies the hemispheres via a reflection. Thus, $u = 0$, and similarly $v = w = 0$. Thus, $\partial_3 = 0$.

Since all the boundary maps are 0, we have the cellular homology groups as

$$H_k(X) = \begin{cases} \mathbb{Z}, & k = 0, 3 \\ \mathbb{Z}^3, & k = 1, 2 \\ 0, & k > 3. \end{cases}$$

Exercise 13.17: (*Cellular Homology of $S^1 \times K$, where K is the Klein's Bottle*)

Let K be the Klein's bottle, and $X = S^1 \times K$. Consider the CW structure on X given as



Compute the cellular homology of X.

13.3 Cellular Homology as Homology Theory

Let us consider $H_\bullet$ to be an ordinary homology theory, such that $H_0(\star) = \mathbb{Z}$. In the next theorem, we prove that $H_\bullet$ is naturally isomorphic to $H_\bullet^{\text{cell}}$ for CW complexes. For convenience, we are only considering singular homology theory.

Theorem 13.18: (*Cellular Homology Equals Singular Homology*)

Let X be a CW complex. Then, $H_\bullet^{\text{cell}}(X)$ is naturally isomorphic to the singular homology $H_\bullet(X)$.

Proof : Let us make some observations.

1. $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$.
2. $H_k(X^n) = 0$ for $k > n$. For $n = 0$, this is immediate by the dimension axiom. Inductively, assume it to be true for some $n \geq 0$. Say, $k > n + 1$. Then, we have part of the exact sequence
$$H_k(X^n) \to H_k(X^{n+1}) \to H_k(X^n, X^{n-1}),$$
where the two endpoints are 0 : left one by induction, and the right by previous observation. So, $H_k(X^{n+1}) = 0$.
3. As $H_{n-1}(X^{n-2}) = 0$, the map $H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ is injective. Hence, the cellular n-cylces Z_n^{cell} is same as the kernel $\ker(\partial : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}))$.
4. The exact sequence $0 \to H_n(X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$ then gives an isomorphism $H_n(X^n) \cong Z_n^{\text{cell}}$.
5. $H_k(X, X^n) = 0$ for $k \leq n$. One proves by induction that $H_k(X^{n+\ell}, X^n) = 0$ for $\ell \geq 0$ and $k \leq n$. Then, passing to colimit, we get $H_k(X, X^n) = 0$. This is easy to see for singular homology, but remains true for any homology theory.
6. The map $H_n(X^{n+1}) \to H_n(X)$ is an isomorphism, which follows from the long exact sequence of the pair (X, X^n) and the fact $H_k(X, X^n) = 0$ for $k \leq n$.

Finally, we consider the following diagram.

$$\begin{tikzcd}
H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial"] \arrow[d, dash, "="] & Z_n^{\text{cell}} \arrow[r] & H_n^{\text{cell}}(X) \arrow[r] & 0 \\
H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial"] & H_n(X^n) \arrow[r] \arrow[u, "\cong"'] & H_n(X^{n+1}) \arrow[r] \arrow[u, dashed] \arrow[d, "\cong"'] & 0 \\
 & & H_n(X) &
\end{tikzcd}$$

The commuttivity of the left square frollows from the definition of the cellular boundary map. Then, we have an induced map $H_n(X^{n+1}) \to H_n^{\text{cell}}(X)$, which is an isomorphism by 5-lemma. But then it follows that $H_n(X) \cong H_n^{\text{cell}}(X)$. Clearly, the isomorphism is natural since all the maps and diagrams involved are natural. $\square$

Thus, cellular homology is a powerful tool to compute many homology groups.