Course notes for

Algebraic Topology II (KSM4E02)

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`cellular chain – cellular homology – cellular homology of ℂℙ∞`

12.1 Cellular Chain Complex and Homology

Let $H_\star$ be an additive ordinary homology theory with $H_0(\star) = \mathbb{Z}$. As example, we can assume $H_\bullet$ to be the singular homology (with $\mathbb{Z}$-coefficient). Suppose X is obtained from $A \subset X$ by attaching n-cells via the map

$$(\Phi, \varphi) : \left(\bigsqcup_{\alpha \in \Lambda} D^n_\alpha, \bigsqcup_{\alpha \in \Lambda} S^{n-1}_\alpha \right) \to (X, A).$$

Proposition 12.1: (*Homology Isomorphism and Attaching Cells*)

The induced map

$$\sum H_\star(\Phi_\alpha) : \bigoplus_{\alpha \in \Lambda} H_\star(D^n_\alpha, S^{n-1}_\alpha) \to H_\star(X, A)$$

is an isomorphism.

Proof : Recall that Φ is a *relative homeomorphism*, i.e, a homeomorphism after taking quotient on both sides. Thus, we have the diagram

$$\begin{tikzcd}
\bigoplus H_\star(D^n_\alpha, S^{n-1}_\alpha) \arrow[r, "\cong"] & H_\star(\bigsqcup D^n_\alpha, \bigsqcup S^{n-1}_\alpha) \arrow[r, "H_\star(\Phi)"] \arrow[d, "\cong"'] & H_\star(X, A) \arrow[d, "\cong"'] \\
 & \tilde{H}_\star(\bigvee D^n_\alpha / S^{n-1}_\alpha) \arrow[r, "\cong", "\tilde{H}_\star(\tilde{\Phi})"'] & \tilde{H}_\star(X/A)
\end{tikzcd}$$

By Theorem 10.14, the vertical quotient maps induce isomorphism in homology. The commutativity shows that $H_\star(\Phi)$ is an isomorphism. The proof then follows since $H_\star$ is assumed to be an additive homology, which gives the isomorhpim on the left. □

Let us describe an inverse map as well. For $\alpha \in \Lambda$, we have an inclusion $p_\alpha : (X, A) \hookrightarrow (X, X \setminus E^n_\alpha)$, where $E^n_\alpha = \Phi\left(\mathring{D}^n_\alpha\right)$, and the relative homeomorphism $\Phi_\alpha : (D^n_\alpha, S^{n-1}_\alpha) \to (X, X \setminus E^n_\alpha)$. Thus, we have the maps

$$\begin{tikzcd}
H_k(X, A) \arrow[r, "H_k(p_\alpha)"] & H_k(X, X \setminus E^n_\alpha) & H_k(D^n_\alpha, S^{n-1}_\alpha). \arrow[l, "H_k(\Phi_\alpha)"', "\cong"]
\end{tikzcd}$$

The image under this map (after inverting the isomorphism), gives a map $z \mapsto (z_\alpha)_{\alpha\in\Lambda}$. It follows that this is precisely the inverse of the isomorphism in Proposition 12.1.

Now, suppose X is a CW complex. From the triple (X^{n+1}, X^n, X^{n-1}), we have the boundary map

$$\partial : H_{k+1}(X^{n+1}, X^n) \to H_k(X^n, X^{n-1}).$$

It is easy to see that $\partial^2 = 0$. Indeed, the boundary maps of the triples are obtained from the diagram

$$\begin{tikzcd}
& H_k(X^n) \arrow[rd] & & & \\
H_{k+1}(X^{n+1}, X^n) \arrow[ru] \arrow[rr, "\partial"'] & & H_k(X^n, X^{n-1}) \arrow[rr, "\partial"] \arrow[rd] & & H_{k-1}(X^{n-1}, X^{n-2}) \\
& & & H_{k-1}(X^{n-1}) \arrow[ru] &
\end{tikzcd}$$

Clearly, the consecutive red and blue maps compose to 0 being part of the long exact sequence of (X^n, X^{n-1}). Thus, $\partial^2 = 0$.

Definition 12.2: (*Cellular Chain Complex and Homology*)

Given a CW complex X, the *cellular chain complex* is defined as $C_n^{\text{cell}}(X) := H_n(X^n, X^{n-1})$, with the boundary map $\partial_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ obtained from the long exact sequence of the triple (X^{n+1}, X^n, X^{n-1}). The homology of the chain complex is called the *cellular homology* of X, and is denoted as $H_\bullet^{\text{cell}}(X)$.

Exercise 12.3: (*Cellular Chain as Free Groups on Cells*)

Verify that the the cellular chain group $C_n^{\text{cell}}(X) = H_n(X^n, X^{n-1})$ is canonically identified with the free Abelian group generated by the n-cells in X. That is, verify that $C_n^{\text{cell}}(X) = \bigoplus_{\{\alpha\in\Lambda\}} \mathbb{Z}\langle e_\alpha^n\rangle$, where $\{e_\alpha^n \mid \alpha \in \Lambda\}$ is the collection of n-cells in the CW decomposition of X.

Proposition 12.4: (*Cellular Homology is Functorial and Homotopy Invariant*)

A cellular map $f : X \to Y$ induces a chain map $C_\bullet^{\text{cell}}(X) \to C_\bullet^{\text{cell}}(Y)$. Homotopic chain maps induce a chain homotopy.

Proof : Let $f : X \to Y$ be a chain map, i.e, $f(X^n) \subset X^n$. Then, we have induced maps $H_n(f) : H_n(X^{n+1}, X^n) \to H_n(Y^{n+1}, Y^n)$ which commutes with bounary by the naturality of the long exact sequence of triple (Theorem 3.8).

Next, suppose $h : X \times I \to Y$ is a homotopy between two cellular maps $f, g : X \to Y$. Since $X \times I$ is a CW complex, by Theorem 11.2, we can assume that $\varphi : X \times I \to Y$ is a cellular map as well. Note that $(X \times I)^n = X^n \times \partial I \cup X^{n-1} \times I$. Consider the map

$$s_n : H_n(X^{n+1}, X^n) \overset{\sigma}{\to} H_{n+1}(X^{n+1} \times I, X^{n+1} \times \partial I \cup X^n \times I) \overset{H_{n+1}(\varphi)}{\to} H_{n+1}(Y^{n+1}, Y^n),$$

where the first σ is the isomorphism from Lemma 13.11 for the pair (X^{n+1}, X^n). Working through the diagrams involving the triple (X^{n+1}, X^n, X^{n-1}), one can show that $\partial s_n = H_n(g) - H_n(f) - s_{n-1}\partial$. That is, $s_\bullet$ is a chain homotopy. $\square$

Example 12.5: (*Cellular Homology of* $\mathbb{CP}^n$)

Recall the complex projective space $\mathbb{CP}^n$ is defined as the space of complex lines (i.e, 1-dimensional complex vector subspaces) of $\mathbb{C}^{n+1}$. In other words, $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus 0)/_{\sim}$, where

$$(z_0, \cdots, z_n) \sim (\lambda z_0, \cdots, \lambda z_n), \quad 0 \neq \lambda \in \mathbb{C}.$$

We denote the equivalence class as $[z_0 : \cdots : z_n]$, which gives rise to the *homogeneous coordinates* on $\mathbb{CP}^n$, making it into an n-dimensional complex manifold. Clearly, $\mathbb{CP}^0 = \{\star\}$ is a singleton.

Since every line is determined by the unit vector (up to sign) on the line, restricting to norm one vectors, we can equivalently get $\mathbb{CP}^n = S^{2n+1}/S^1$. Note that the unit vectors in $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is the $2n+1$ -sphere, and $S^1 = \left\{e^{\iota\theta} \mid 0 \leq \theta < 2\pi\right\}$ acts freely as an (Abelian) group. This is generalization of the antipode action which gives $\mathbb{RP}^n = S^n/\mathbb{Z}_2 = S^n/S^0$.

It is easy to see that the inclusion $\mathbb{C}^{n+1} \hookrightarrow \mathbb{C}^{n+2}$ as the hyperplane $\left\{z_{n+2} = 0\right\}$ gives rise to an inclusion

$$\begin{aligned} \mathbb{CP}^n &\lhook\joinrel\longrightarrow \mathbb{CP}^{n+1} \\ [z_0 : \cdots : z_n] &\longmapsto [z_0 : \cdots : z_n : 0]. \end{aligned}$$

Thus, we have

$$\{\star\} = \mathbb{CP}^0 \hookrightarrow \mathbb{CP}^1 \hookrightarrow \cdots \hookrightarrow \mathbb{CP}^\infty,$$

where $\mathbb{CP}^\infty$ is given the weak topology from the inclusions.

Let us now give a CW decomposition of $\mathbb{CP}^n$. We claim $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a $2n$ -cell. Note that the inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ gives the inclusion $S^{2n-1} \hookrightarrow S^{2n+1}$, which induces the same map $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$. Then, any point $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$ is *uniquely* written as $\left[z_0 : \cdots : z_{n-1} : \sqrt{1 - \|\mathbf{z}\|^2}\right]$, where $\mathbf{z} = (z_0, \ldots, z_{n-1}) \in \mathbb{C}^n$ satisfies $\|\mathbf{z}\|^2 := \sum_{j=0}^n z_j \overline{z}_j \lneq 1$. Treating $\mathbb{D}^{2n} \subset \mathbb{C}^n$ as the closed unit ball, we have a continuous map

$$\begin{aligned} \Phi : \mathbb{D}^{2n} &\longrightarrow \mathbb{CP}^n \\ \mathbf{z} &\longmapsto \left[z_0 : \cdots : z_n : \sqrt{1 - \|\mathbf{z}\|^2}\right]. \end{aligned}$$

The boundary map $\varphi := \Phi|_{S^{2n-1}}$ maps to $\mathbb{CP}^{n-1}$ since the last coordinate is 0. In the interior, Φ is a homeomorphism, since we can define the inverse map

$$\begin{aligned} \mathbb{CP}^n \setminus \mathbb{CP}^{n-1} &\longrightarrow \mathring{\mathbb{D}}^{2n} \\ [z_0 : \cdots : z_n] &\longmapsto \left(\frac{z_0}{z_n}, \ldots, \frac{z_{n-1}}{z_n}\right). \end{aligned}$$

In other words, $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a $2n$-cell via the attaching map φ. In particular, note that $\mathbb{CP}^1 = \{\star\} \cup_\varphi D^2 = S^2$ is the 2-sphere.

Now, we can immediately write the cellular chain complex $C_n = C_n^{\mathrm{cell}}(\mathbb{CP}^n)$. It follows that

$$0 \to \underset{\mathbb{Z}}{C_{2n}} \to \underset{0}{\mathbb{C}_{2n-1}} \to \cdots \to \underset{\mathbb{Z}}{C_2} \to \underset{0}{C_1} \to \underset{\mathbb{Z}}{C_0} \to 0$$

The boundary maps are forced to be 0. Hence, it follows that

$$H_k^{\text{cell}}(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}, & 0 \leq k \leq 2n, k \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

This pattern continues for $\mathbb{CP}^\infty$ as well, and we have,

$$H_k^{\text{cell}}(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

Exercise 12.6: (*Cellular Homology of* $S^n \times S^n$)

Compute the cellular homology $X = S^n \times S^n$.

Hint : There is a CW decomposition of X with exactly 4 cells!