Course notes for

Algebraic Topology II (KSM4E02)

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`cellular maps — cellular approximation theorem — weak homotopy equivalence — Whitehead theorem — connectivity and compressibility`

11.1 Cellular Map and Cellular Approximation

Definition 11.1: (*Cellular Map*)

Let $(X, A), (Y, B)$ be two relative CW complexes. A map $f : (X, A) \to (Y, B)$ is called a *cellular map* if $f(X^n) \subset Y^n$ holds for each $n \geq -1$.

Given a CW complex X, a *subcomplex* $A \subset X$ is a subspace which consists of a sub-collection of cells of X. In particular, a subcomplex $A \subset X$ is a CW complex on its own, and the inclusion map $A \hookrightarrow X$ is cellular. We have the following important theorem, proof of which follows from inductively constructing the map cell-by-cell.

Theorem 11.2: (*Cellular Approximation Theorem*)

Let X, Y be CW complexes. Then, any map $f : X \to Y$ is homotopic to a cellular map $g : X \to Y$. Suppose $B \subset X$ is a subcomplex. If $f|_B : B \to Y$ is already cellular, then the homotopy $f \simeq g$ can be chosen to be relative to B, i.e, the homotopy stays constant on B.

Let us see an important consequence of the cellular approximation theorem.

Proposition 11.3: (*Maps $S^n \to S^{n+k}$ are Null-homotopic for $k > 0$*)

Any map $f : S^n \to S^{n+k}$ is null-homotopic provided $k > 0$. Consequently, $\pi_n(S^{n+k}) = 0$ for $k > 0$.

Proof : Let $f : S^n \to S^{n+k}$ be given. Fix basepoint $x_0 \in S^n$, and $y_0 = f(x_0) \in S^{n+k}$. Fix CW structure on S^k (resp. on S^{n+k}) consisting of a single 0-cell x_0 (resp. y_0), and a single n-cell (resp. $(n+k)$-cell). By Theorem 11.2, f is homotopic relative to the basepoint to a cellular map $g : S^n \to S^{n+k}$. In the CW structure considered, the n-skeleton of S^n is S^n itself, whereas that of S^{n+k} consists of the 0-cell $\{y_0\}$ since $n + k > n$. But then being cellular, we have $g(S^n) = \{y_0\}$. Hence, f is null-homtopic. Since it is a base-point preserving homotopy, it follows that $\pi_n(S^{n+k}) = 0$ for $k > 0$. □

Exercise 11.4: (*Homotopy Groups of Infinite Sphere*)

Define the infinite sphere S^∞ as the union of

$$S^0 \hookrightarrow S^1 \hookrightarrow \cdots S^n \hookrightarrow S^{n+1} \hookrightarrow \cdots,$$

where S^{n+1} is obtained from S^n by attaching two $(n+1)$-cells along the equator. This makes S^∞ into a CW complex, with two n-cells in each dimension. Show that $\pi_k(S^\infty) = 0$ for all $k \geq 0$.

11.2 Weak Homotopy Equivalence

Definition 11.5: (*Weak Homotopy Equivalence*)

A map $f : X \to Y$ is called a *weak homotopy equivalence* if

- $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is group isomorphism for each $n \geq 1$ and each $x \in X$, and
- $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.

Recall that the 0^{th} homotopy "group" $\pi_0(X)$ is the set of all path components of X, in general there is no group structure on it. Moreover, $\pi_0(X)$ is independent (i.e, defined up to set bijection) of choice of basepoint. The condition $\pi_0(f)$ is a bijection essentially means that f induces a bijection between the path components. Let us now see an example of an *weak* homotopy equivalence, which cannot be a homotopy equivalence.

Example 11.6: (*Pseudocircle or Digital Circle*)

Consider the four point space $X = \{N, E, W, S\}$ with the following topology

$$\{\emptyset, \{N\}, \{S\}, \{N, S\}, \{N, E, S\}, \{N, W, S\}, X\}.$$

This space is *not* T_2 as E and W cannot be separated by open sets (in fact, it is T_0 but not T_1). Let us now consider a map $f : S^1 \to X$ which maps the open upper semicirlce (resp. lower semicircle) to N (resp. to S), and the two remaining points to E and W respectively. Clearly, f is a continuous map (Check!). In fact, X is obtained from S^1 as the identification space which maps the two open semicircles to two distinct points, and f is precisely the quotient map.

Surjectivity of f shows that X is a path connected space. Moreover, one can construct a contractible universal cover of X, and show that $\pi_1(X) = \mathbb{Z}$, $\pi_k(X) = 0$ for $k > 1$. It follows that $f : S^1 \to X$ is a weak homotopy equivalence. The space X is called the *pseudocircle*.

Note that any $g : X \to S^1$ is necessarily a constant map, as the image must be path connected and can have at most finitely many points. Thus, f (or any other map $S^1 \to X$) cannot have a homotopy inverse since a constant map $S^1 \to S^1$ is not homotopic to the identity.

Thus, we cannot in general invert (even up to homotopy) a weak homotopy equivalence. But when the space is CW, we can!

Theorem 11.7: (*Whitehead Theorem*)

Let X, Y be CW complexes. Then, any weak homotopy equivalence $f : X \to Y$ is actually a homotopy equivalence.

The next example says that having isomorphic homotopy groups is not sufficient for being homotopy equivalent (or in fact even being weak homotopy equivalent). It assumes some results that we shall see later.

Example 11.8: (*Spaces with Isomorphic Homotopy Groups*)

Consider the spaces $Y_1 = \mathbb{RP}^2 \times S^3$ and $Y_2 = S^2 \times \mathbb{RP}^3$. It is easy to see that $X = S^2 \times S^3$ is a double cover of both Y_i, and moreover, Y is an universal cover (being simply connected). Then, $\pi_1(Y_i) = \mathbb{Z}_2$. On the other hand, from the long exact sequence in homotopy groups for the fibration $\mathbb{Z}_2 \hookrightarrow X \to Y_i$ it follows that $\pi_n(X) = \pi_n(Y_i)$ for $n \geq 2$. Thus, $\pi_n(Y_1) = \pi_n(Y_2)$ for all n.

On the other hand, we can compute homology with $\mathbb{Q}$-coefficients : $H_2(Y_1; \mathbb{Q}) = 0$ and $H_2(Y_2; \mathbb{Q}) = \mathbb{Q}$. Hence, Y_1 and Y_2 are not homotopy equivalence (and moreover, not weakly homotopy equivalent).

We have the following important theorem.

Theorem 11.9: (*CW Approximation Theorem*)

Given any space Y, there exists a CW complex X and weak homotopy equivalence $\alpha : X \to Y$, which is called a *CW approximation* of Y. Moreover, if $f : Y_1 \to Y_2$ is a map, and $\alpha_i : X_i \to Y_i$ are CW approximations for $i = 1, 2$, then there exists a (cellular) map $g : X_1 \to X_2$, defined unique up to homotopy, such that $\alpha_2 \circ g \simeq f \circ \alpha_1$.

Note that CW approximation is applicable for any space, but this does not let us replace a space. However, when computing algebraic invariants like homotopy or (co)homology, it is good enough to replace any space by a CW approximation.

Remark 11.10: (*CW Type*)

A space X is said to be of *CW type* if X is homotopy equivalent to a space Y, where Y admits a CW decomposition. We have noted that the Hawaiian earring does not admit a CW decomposition, moreover it does not have CW type. Consider the *Hedgehog space*

$$X := \left\{ re^{\iota\theta} \mid 0 \leq r \leq 1, \theta \in \mathbb{Q} \right\} \subset \mathbb{C}$$

which is a dense collection of spokes. Again, X is not locally contractible at any point other than the origin, and hence, X does not admit a CW decomposition. On the other hand, X is contractible, and so, it is homotopy equivalent to a CW complex (the singleton). So, X is of CW type. It is easy to see that Whitehead theorem (Theorem 11.7) generalizes to CW tpye : a weak homotopy equivalence between spaces of CW type is a homotopy equivalence.

11.3 Connectivity and Compressibility

Let us interpret the weak homotopy equivalence in terms of connectivity.

Definition 11.11: (*Connectivity of a Map*)

A map $f : X \to Y$ is said to be *n-connected* if the following holds : for all $x \in X$, $\pi_i(f) : \pi_i(X, x) \to \pi_i(Y, f(y))$ is an isomorphism for each $i < n$ and an epimorphism for $i = n$. A pair (X, A) is called *n-connected* if the inclusion map $\iota : A \hookrightarrow X$ is n-connected.

In the above definition, it is understood that for $n = 0$, an isomorphism $\pi_0(f) : \pi_0(X, x) \to \pi_0(Y, f(x))$ is just a bijection of pointed sets. In particular, for $n > 0$, we can simply ask $\pi_0(f)$ to induce bijection on path components. Clearly, $f : X \to Y$ is a weak homotopy equivalence if and only if it is n-connected for all $n \geq 0$.

Suppose (X, A) is a pair. If there is a homotopy $\Phi_t : (D^i, S^{i-1}) \to (X, A)$ of pairs, such that Φ_1 lands in A, then we would like to have a homotopy Ψ_t of Φ which is constant relative to the boundary, and Ψ_1 lands in A. More generally, we observe the following technical lemma.

Lemma 11.12: (*Homotopy of Pair and Relative Homotopy*)

Given a map $f : X \to Y$, suppose we have a commuting diagram

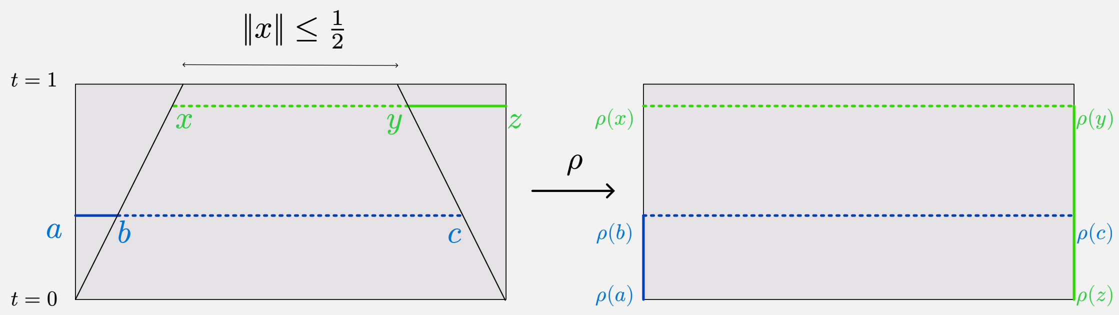
$$\begin{array}{ccc} S^{i-1} & \xrightarrow{\varphi} & X \\ \big\downarrow & & \big\downarrow f \\ D^i & \xrightarrow[\Phi]{} & Y \end{array}$$

Suppose we have homotopies $\Phi_t : D^i \to Y$ and $\varphi_t : S^{i-1} \to X$, such that $\Phi_0 = \Phi$, $\Phi_1 = f \circ \zeta$ for some $\zeta : D^i \to X$, and $\Phi_t|_{S^{i-1}} = f \circ \varphi_t$. Then, there exists a homotopy $\Psi_t : D^i \to Y$ and a map $\psi : D^i \to X$, such that $\Psi_0 = \Phi$, $\Psi_1 = f \circ \psi$ and $\Psi|_{S^{i-1}} = f \circ \varphi$.

Proof : The proof is immediate if one can define a conntinuous map $\rho : D^i \times I \to D^i \times I$ such that $\rho_0 = \mathrm{Id}_{D^i \times \{0\}}$ and $\rho|_{S^{i-1} \times I} = \rho_0|_{S^{i-1}}$. Here is an explicit formula :

$$\rho(x, t) := \left(\frac{2x}{\max(2\|x\|, 2 - t)}, 2 - \max(2\|x\|, 2 - t) \right),$$

where we are considering the euclidean norm. It is easy to see that $\Psi = \Phi \circ \rho$ is the desired homotopy. One can visualize the map as follows.

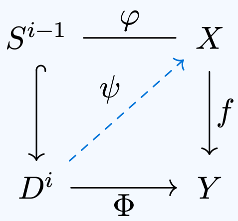


Visualizing deformation of $D^i \times I$ to make homotopy relative to boundary. □

Next, we compare n-connectednes of a map with a lifting problem of discs and spheres.

Lemma 11.13: (*Connectivity and Compressibility*)

A map $f : X \to Y$ is n-connected if and only if for any $i \leq n$, and given any commutative diagram



there exists a map $\psi : D^i \to X$ such that $\psi|_{S^{i-1}} = \varphi$ and $f \circ \psi \underset{\text{rel } S^{i-1}}{\simeq} \Phi$ holds. This equivalent condition is also known as *i-compressibility*, especially when we are considering inclusion maps $\iota : A \hookrightarrow X$.

Proof : Suppose f is n-connected. Given a diagram as above, since $\Phi|_{S^{i-1}} = f \circ \varphi$, it follows that $f \circ \varphi$ is null-homotopic in Y. But $\pi_{i-1}(f) : \pi_{i-1}(X) \to \pi_{i-1}(Y)$ is injective for $i \leq n$, and hence, φ is null-homotopic in X. Thus, we have a null-homotopy $\varphi_t : S^{i-1} \to X$ with $\varphi_0 = \varphi$ and φ_1 constant. As $S^{i-1} \hookrightarrow D^i$ is a cofibration, and since $f \circ \varphi_0 = f \circ \varphi = \Phi|_{S^{i-1}}$, extending we get a homotopy $\Phi_t : D^i \to Y$ with $\Phi_0 = \Phi$ and $\Phi_t|_{S^{i-1}} = f \circ \varphi_t$. In particular, $\Phi_1 : D^i \to Y$ has $\Phi_1|_{S^{i-1}}$ constant. But then Φ_1 induces a map $\zeta : S^i \to Y$. Since ζ represents a homotopy class in $\pi_i(Y)$, by the surjectivity of $\pi_i(f)$, we have a map $\xi : S^i \to X$ with $f \circ \xi \simeq \zeta$ in Y. Let us consider this homotopy $\zeta_t : D^i \to Y$ with $\zeta_t|_{S^{i-1}}$ constant, and $\zeta_0 = \zeta, \zeta_1 = f \circ \xi$. Finally, concatenating with the homotopy Φ_t, we have a homotopy $\Psi_t : D^i \to Y$ such that $\Psi_0 = F$ and $\Psi_1 = f \circ \xi$. Moreover, $\Psi_t|_{S^{i-1}}$ always factors through f, and Ψ_1 factors through f as well. But then by Lemma 11.12 we can get a homotopy $\psi_t : D^i \to Y$ such that $\psi_0 = \Phi, \psi_t|_{S^{i-1}} = f \circ \varphi$ and ψ_1 factors through X. That is, $\psi_1 = f \circ \psi$ for some $\psi : D^i \to X$. This ψ solves desired lifting.

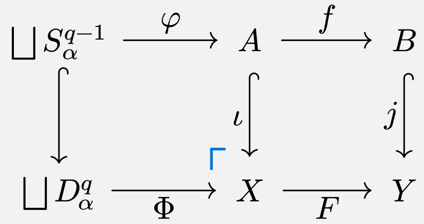
Conversely, suppose $f : X \to Y$ is i-compressible for $0 \leq i \leq n$. First consider the path components. Let $x_0, x_1 \in X$ be in different path components. Then, $f(x_0), f(x_1) \in Y$ are also in different path components, since otherwise by the diagram above we will have a path $D^1 = [0, 1] \to X$ joining x_0 to x_1. Thus, $\pi_0(f)$ is injective on path components. Also, taking $i = 0$ (whence $D^i = D^0 = \{\star\}$ and $S^{i-1} = S^{-1} = \emptyset$), it follows that $\pi_0(f)$ is surjective on path components. Next, suppose $0 < i \leq n$. Let $\alpha : S^i \to Y$ be given. Consider the quotient $q : D^i \to D^i/S^{i-1} = S^i$. Get a point $x_0 \in X$ such that $f(x_0)$ is path connected to the point $\alpha q(S^{i-1}) \in Y$. By modifying the quotient map (think of a collar around the boundary that is collapsed to the path like a balloon with a string), we have map $\Phi : D^i \to Y$, and set $\varphi : S^{i-1} \to X$ to be the constant map at x_0. Then, get a lift $\psi : D^i \to X$ such that $f \circ \psi \simeq \Phi$. Clearly, the restriction $\psi|_{S^i}$ gives a homotopy class that is mapped to $[\alpha]$ by $\pi_i(f)$. This proves that $\pi_i(f)$ is surjective for $i \leq n$. A similar argument using shows the injectivity for $i < n$ as well. □

As a consequence, we have the following important result.

Theorem 11.14: (*Connectivity and Lift Against Relative CW Complex*)

Let (Y, B) be an n-connected pair, and (X, A) be a relative CW-complex of dimension $\leq n$. Then, any map $f : (X, A) \to (Y, B)$ is homotopic to a map into B, relative to A. Moreover, if $\dim(X, A) < n$, then the homotopy class of $X \to B$ is unique relative to A.

Proof : The proof is by induction over skeleton filtration of (X, A). Suppose first X is obtained from A by attaching q-cells, with $q \leq n$. Let $(F, f) : (X, A) \to (Y, B)$ be a map. Then, we have a diagram



As j is n-connected, we see by Lemma 11.13 that $F \circ \Phi$ is homotopic relative to $\sqcup S_\alpha^{q-1}$ to a map in to B. Since this homotopy is relative, by the pushout square on the left, we then have a homotopy of F into B, and they are clearly constantly equal f on A. Thus, we have homotopy of F into B relative to A.

Now, consider a relative CW complex of dimension $\leq n$. Let $f : (X, A) \to (Y, B)$ be a map. Suppose we have a homotopy of f relative to A to a map g such that $g(X^k) \subset B$. Since X^{k+1} is obtained from X^k by attaching $(k+1)$-cells, by the above argument, we have a homotopy of $g : (X^{k+1}, X^k) \to (Y, B)$ relative to X^k which sends X^{k+1} into B. Since $X^{k+1} \hookrightarrow X$ is a cofibration, we can extend this homotopy, relative to X^k, to all of X. Continuing this, we get the desired homotopy.

The above argument will work for $n < \infty$. In case of infinite dimension, at each step, we need to scale the homotopies suitably to, say, $\left[\frac{1}{n}, \frac{1}{n+1}\right]$, and then concatenate to get the desired homotopy.

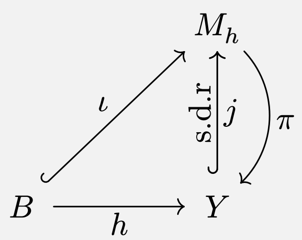
Uniqueness of the homotopy class when $\dim(X, A) < n$ follows by observing that the pair $\dim(X \times I, X \times \partial I \cup A \times I) = n$, and we can apply the above argument to get a homotopy of homotopies. □

As a consequence, we can now prove the Whitehead theorem for CW complexes. Let us first observe the effect of composing by an n-connected map on homotopy class.

Proposition 11.15: (*Composing n-connected Map and Bijection of Homotopy Class*)

Let $h : B \to Y$ be an n-connected map. Then, the induced map $h_\star : [X, B] \to [X, Y]$ is a bijection (resp. surjection) if X is a CW complex with $\dim X < n$ (resp. $\dim X \leq n$). If h is a map of pointed space, then the same conclusion hold for a pointed CW complex X and basepoint preserving homotopy classes.

Proof : Let us apply the *mapping cylinder construction* to replace h by an inclusion. Indeed, we have the following diagram



Here, the mapping cylinder $M_h = Y \cup_h B \times [0, 1]$ deforms to Y via the projection $\pi : M_h \to Y$. Clearly, we have the composition

$$h_\star : [X, B] \overset{\iota_\star}{\to} [X, M_h] \overset{\pi_\star}{\to} [X, Y].$$

Since π is a deformation retract, it follows that $\pi_\star$ is a bijection. So we only need to consider $\iota_\star$. Without loss generality, let us assume that $h : B \hookrightarrow Y$ is an inclusion map.

Now, applying Theorem 11.14 for the CW-complex $(X, \emptyset)$, we immediately have the surjectivity when $\dim(X, \emptyset) = \dim X \leq n$. For injectivity, we again use Theorem 11.14 for the CW-complex $(X \times I, X \times \partial I)$. This concludes the claim. $\square$

We can now give a proof of the Whitehead theorem.

Proof of Theorem 11.7 : Suppose X, Y are CW complexes and $f : X \to Y$ is a weak homotopy equivalence. Then, f is n-connected for all n. Hence, as in Proposition 11.15, we have $f_\star : [Z, X] \to [Z, Y]$ is a bijection for any CW-complex Z. In particular, taking $Z = Y$, we have $f_\star : [Y, X] \to [Y, Y]$ is a bijection. But then there exists $g : Y \to X$ such that $f_\star([g]) = [\mathrm{Id}_Y]$, i.e, $f \circ g \simeq \mathrm{Id}_Y$. Next, consider $Z = X$ so that $f_\star : [X, X] \to [X, Y]$ is a bijection. We have

$$f_\star[g \circ f] = [f \circ (g \circ f)] = [(f \circ g) \circ f] = [\mathrm{Id}_Y \circ f] = [f] = f_\star[\mathrm{Id}_X].$$

By injectivity, we have $[g \circ f] = [\mathrm{Id}_X]$, that is, $g \circ f \simeq \mathrm{Id}_X$. Hence, $f : X \to Y$ is a homotopy equivalence, with g as a homotopy inverse. $\square$