Course notes for

Algebraic Topology II (KSM4E02)

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`Hurewicz isomorphism — reduced singular homology — cofibration — signular homology of cofiber — cellular decomposition`

10.1 Hurewicz Isomorphism

Let us now show that the Hurewicz map is an isomorphism for path connected spaces.

Theorem 10.1: (*Hurewicz Isomorphism*)

If X is a path connected space with a basepoint x_0, then the Hurewicz map $\eta : \pi_1(X, x_0)^{\text{ab}} \to H_1(X)$ is an isomorphism.

Proof : For each $x \in X$, fix a path $u_x : x_0 \to x$, with $\overline{u}_x$ as the reversed pat $x \to x_0$. Assume u_{x_0} to be the constant path c_{x_0}. Next, for any $\sigma : \Delta^1 \to X$, denote $\sigma_0 = \sigma d_0 = \sigma(0,1)$ and $\sigma_1 = \sigma d_1 = \sigma(1,0)$. Then, $\left(u_{\sigma_1} \star \sigma\right) \star \overline{u}_{\sigma_0}$ is a loop at x_0. Let us define the linear map

$$\begin{aligned} \xi' : S_1(X) &\to \pi_1(X, x_0)^{\text{ab}} \\ \sigma &\mapsto \left[\left(u_{\sigma_1} \star \sigma\right) \star \overline{u}_{\sigma_0}\right]. \end{aligned}$$

This is possible since $\pi_1(X, x_0)^{\text{ab}}$ is Abelian. Now, for any 2-simplex $\omega : \Delta^2 \to X$, denote the faces $\omega_j = \omega d_j : \Delta^1 \to X$, and then observe that $\omega_2 \star \omega_0 \simeq \omega_1$. Also, it is easy to see that

$$\begin{aligned} \xi'(\omega_1) &= \left[\left(u_{\omega_{11}} \star \omega_1\right) \star \overline{u}_{\omega_{10}}\right] = \left[\left(u_{\omega_{11}} \star (\omega_2 \star \omega_0)\right) \star \overline{u}_{\omega_{10}}\right] \\ &= \left[\left(\left(u_{\omega_{21}} \star \omega_2\right) \star \overline{u}_{\omega_{20}}\right) \star \left(\left(u_{\omega_{01}} \star \omega_0\right) \star \overline{u}_{\omega_{00}}\right)\right], \text{since } \overline{u}_{\omega_{20}} \star u_{\omega_{01}} \simeq c_{x_0} \\ &= \left[\left(u_{\omega_{21}} \star \omega_2\right) \star \overline{u}_{\omega_{20}}\right] + \left[\left(u_{\omega_{01}} \star \omega_0\right) \star \overline{u}_{\omega_{00}}\right] \\ &= \xi'(\omega_2) + \xi'(\omega_0) \end{aligned}$$

Hence, $\xi'(\partial\omega) = 0$ which implies that ξ' factors as a map $\xi : \frac{S_1(X)}{B_1(X)} \to \pi_1(X)^{\text{ab}}$. Clearly, $\xi \circ \eta = \text{Id}$, and hence, η is injective. To show that η is surjective, let $\alpha \in H_1(X)$ be represented by some 1-cycle $a := \sum n_\sigma \sigma$ with $\partial a = 0$. By linearity, we have a map $u : S_0(X) \to S_1(X)$. Then, for any $\sigma : \Delta^1 \to X$, we have $u(\partial\sigma) = u(\sigma d_0 - \sigma d_1) = u_{\sigma_0} - u_{\sigma_1} = u_{\sigma_0} + \overline{u}_{\sigma_1}$. In particular, we have

$$\eta\Big(\sum n_\sigma \xi(\sigma)\Big) = \sum n_\sigma\Big(\left[u_{\sigma_1}\right] + [\sigma] + \left[\overline{u}_{\sigma_1}\right]\Big) = \sum n_\sigma([\sigma] + [u\partial\sigma]) = \alpha + u([\partial a]) = \alpha,$$

as $\partial a = 0$. This implies η is surjective. □

Exercise 10.2:

Compute the homology groups of $S_g := \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g}$, the torus with g-holes.

> **Hint :** Recall, $\pi_1(S_g) = \langle a_1, b_1, \cdots, a_g, b_g \mid \prod [a_i b_i] = 1 \rangle$ by Van Kampen's theorem.

Remark 10.3: (*The Hurewicz Isomorphism*)

The more general Hurewicz isomorphism states the following. Suppose X is a $(n-1)$-connected, i.e, $\pi_i(X) = 0$ for $i < n$. Then, the Hurewicz map $h : \pi_n(X) \to H_n(X)$ is an isomorphism. As the Hurewicz map commutes with the suspension map, it follows moreover that $h : \pi_{n+1}(X) \to H_{n+1}(X)$ is an epimorphism. The isomorphism lets us inductively show that

$$\pi_i(S^n) = \begin{cases} 0, & i < n \\ \mathbb{Z}, & i = n. \end{cases}$$

In general, computing the higher homotopy groups of the sphere is a nontrivial problem.

10.2 Reduced Singular Homology

Suppose X is a space, and $c : X \to \star$ is the constant map. Recall, the reduced homology is defined as the kernel $\tilde{H}_n(X) := \ker\left(H_n(X) \xrightarrow{H_n(c)} H_n(\star) \right)$. Let us interpret it in a different way for singular homology. Consider the chain map $\varepsilon_\bullet : S_\bullet(X) \to S_\bullet(\star)$ given by $\varepsilon_n = 0$ for $n \neq 0$, and

$$\varepsilon_0\left(\sum n_\sigma \sigma\right) = \left(\sum n_\sigma\right)\sigma_0,$$

where $\sigma_0 : \Delta^0 \to \star$ is the unique 0-simplex.

Exercise 10.4:

Using Proposition 6.10, verify that $H_\bullet(c) = H_\bullet(\varepsilon) : B_\bullet(X) \to H_\bullet(\star)$.

Let us now define a chain homotopy $s_n : S_n(X) \to S_{n+1}(\star)$ by setting $s_n(\sigma) = (-1)^n \sigma_{n+1}$ on generators and extending linearly. Here, $\sigma_{n+1} : \Delta^n \to \star$ is the unique $(n+1)$-simplex. Note that

$$\begin{aligned} s\partial(\sigma) + \partial s(\sigma) &= s\left(\sum_0^n (-1)^i \sigma d_i\right) + (-1)^n \partial\sigma_{n+1} \\ &= (-1)^{n-1} \sum_0^n (-1)^i \sigma_n + (-1)^n \sum_0^{n+1} (-1)^i \sigma_n \\ &= (-1)^n \left[-\sum_0^n (-1)^i \sigma_n + \sum_0^{n+1} (-1)^i \sigma_n \right] \\ &= (-1)^n (-1)^{n+1} \sigma_n = -\sigma_n = \varepsilon_n(\sigma) - S_n(c)(\sigma). \end{aligned}$$

Then, $H_\bullet(\varepsilon) = H_\bullet(c)$ follows.

Definition 10.5: (*Augmented Singular Chain Complex*)

Given a space X, the *augmented singular chain complex* is defined as the chain complex $S_2(X) \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, where $\varepsilon(\sum n_\sigma \sigma) = \sum n_\sigma$. This map ε is called the *augmentation map*.

Observe that considering $\mathbb{Z}$ as a chain complex concentrated at degree 0, we can look at the augmentation map as a chain map by setting it zero every nozero degree.

Exercise 10.6:

Verify that the reduced singular homology is the homology of the augmented singular chain complex.

10.3 Cofibration and Homology

Let us recall the notion of cofibration.

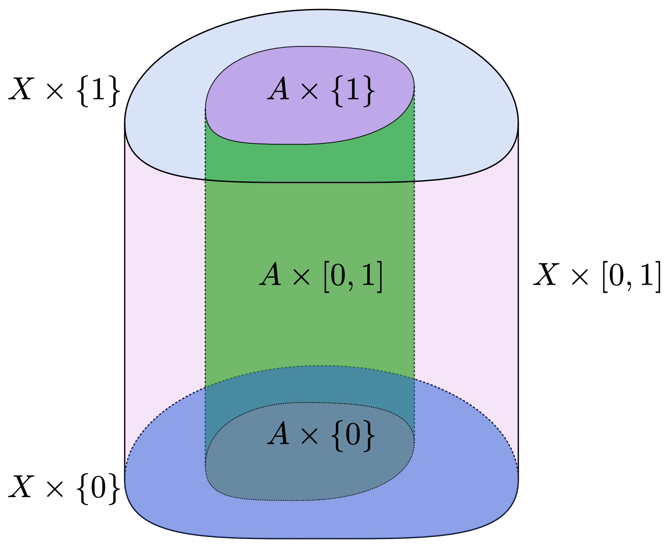
Definition 10.7: (*Cofibration*)

A map $f : A \to X$ is called a *cofibration* if it has the *homotopy extension property* against any space, i.e, given a homotopy $h : A \times [0,1] \to Y$ and a map $F : X \to Y$ with $F \circ f = h|_{A \times 0}$, there exists a homotopy $H : X \times [0,1] \to Y$ such that $H|_{X \times \{t\}} \circ f = h|_{X \times \{t\}}$. In other words, the commuting diagram

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] & Y^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r, "F"'] \arrow[ur, dashed, "H"] & Y
\end{tikzcd}$$

admits a solution H which commutes with other maps.

The homotopy extension is easier to visualize if we only consider an inclusion map $\iota : A \hookrightarrow X$.



Homotopy extension for $\iota : A \hookrightarrow X$

Most of the times, we will only concern ourselves about an inclusion being a cofibration. In fact, given any function $f : A \to X$, consider the *mapping space*

$$M_f := X \cup_f (A \times [0,1]),$$

where we identify $(a,0) \sim f(a)$. It is easy to see that M_f (strongly) deformation retracts onto X by collapsing $A \times [0,1]$. On the other hand, we have the inclusion $\iota : A \hookrightarrow M_f$ given by $a \mapsto (a,1)$. We then have a *homtopy* commutative diagram

$$\begin{tikzcd} & M_f \\ A \arrow[ur, hook, "\iota"] \arrow[r, "f"'] & X \arrow[u, "j"', "\text{s.d.r}"] \end{tikzcd}$$

Exercise 10.8: (*Mapping Cylinder is a Cofibration*)

Given any map $f : A \to X$, show that $\iota : A \to M_f$ is a cofibration.

Let us now observe a few easy topological consequences about cofibraions.

Lemma 10.9: (*Composition of Cofibration*)

Composition of cofibrations is again a cofibration.

<ins>**Proof**</ins> **:** Suppose $f : X \to Y, g : Y \to Z$ are cofibrations. Let us consider the homotopy extension diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"] \arrow[dd, bend right, "g \circ f"'] & W^I \arrow[dd, "\mathrm{ev}_0"'] \\ Y \arrow[d, "g"] & \\ Z \arrow[r, "F"] & W \end{tikzcd}$$

We first get a lift $H_1 : Y \to W^I$ such that $\mathrm{ev}_0 \circ H_1 = F \circ g$ and $H_1 \circ f = h$. Then, we get another lift $H_2 : Z \to W^I$ such that $H_2 \circ g = H_1$ and $\mathrm{ev}_0 \circ H_2 = F$. Then, we have $H_2 \circ (g \circ f) = H_1 \circ f = h$, and $\mathrm{ev}_0 \circ H_2 = F$. Thus, H_2 is the required extension, proving that $g \circ f$ is a cofibration. □

Next, we consider pushout of a cofibration. Let us now recall the defintion first.

Definition 10.10: (*Pushout of Spaces*)

Given continuous maps $X \xleftarrow{f} A \xrightarrow{g} Y$, the *pushout* is defined via the following universal property.

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "p"] \arrow[ddr, bend left, "\varphi"] & \\
X \arrow[r, "q"'] \arrow[drr, bend right, "\psi"'] & Z \arrow[dr, dashed, "\exists! h"] & \\
& & W
\end{tikzcd}
$$

Given any map $\varphi : Y \to W, \psi : X \to W$ with $\varphi \circ g = \psi \circ f$, there exists a unique map $h : Z \to W$ such that $h \circ p = \varphi, h \circ q = \psi$.

Explicitly, one can construct the pushout as the identification space $Z = (X \sqcup Y)/_\sim$, where $\sim$ is the smallest equivalence relation such that $f(a) \sim g(a)$ for $a \in A$.

Example 10.11:

Given a map $f : A \to X$, the mapping cone $C(f)$ is the pushout of the diagram $CA \hookleftarrow A \xrightarrow{f} X$. The pushout of the sphere as the boundary $D^n \hookleftarrow S^{n-1} \hookrightarrow D^n$ is the sphere S^n.

Lemma 10.12: (*Pushout of a Cofibration*)

Suppose $f : A \to X$ is a cofibration. Then, any pushout of f is again a cofibration.

__Proof__ **:** Consider a pushout diagram and a homtopy extension problem for the middle arrow

$$
\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "f"'] & Z \arrow[r, "h"] \arrow[d, "g"] & Y^I \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r, "q"'] & W \arrow[r, "F"'] & Y
\end{tikzcd}
$$

As f is a cofibration, we have a lift

$$
\begin{tikzcd}
A \arrow[r, "h \circ p"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & Y^I \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r, "F \circ q"'] \arrow[ur, dashed] & Y
\end{tikzcd}
$$

Next, we use the pushout to get a unique map

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "f"'] & Z \arrow[d, "g"] \arrow[ddr, bend left, "h"] & \\
X \arrow[r, "q"] \arrow[drr, bend right, "G"'] & W \arrow[dr, dashed, "H"] & \\
& & Y^I
\end{tikzcd}$$

We claim that H is the solution in

$$\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d, "g"'] & Y^I \arrow[d, "\mathrm{ev}_0"] \\
W \arrow[r, "F"'] \arrow[ur, dashed, "H"] & Y
\end{tikzcd}$$

We alredy have $H \circ g = h$ from the pushout diagram. Next, we again look at the pushout

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "f"'] & Z \arrow[d, "g"] \arrow[ddr, bend left, "\mathrm{ev}_0 \circ h"] & \\
X \arrow[r, "q"] \arrow[drr, bend right, "\mathrm{ev}_0 \circ G"'] & W \arrow[dr, dashed, "\mathrm{ev}_0 \circ H", "F"'] & \\
& & Y
\end{tikzcd}$$

It follows that both $\mathrm{ev}_0 \circ H$ and F solves the pushout diagram. Hence, by the uniqueness, we have $\mathrm{ev}_0 \circ H = F$. This proves that H is a homotopy extension, as required. Thus, the pushout $g : Z \to W$ is a cofibration. $\square$

Lemma 10.13: (*Contractible Cofibered Subspace*)

Suppose $A \subset X$ is contractible, and the inclusion $\iota : A \hookrightarrow X$ is a cofibration. Then, the quotient $q : X \to X/A$ is a homotopy equivalence.

Proof : Since A is contractible, there exists a homotopy $h : A \times [0,1] \to A$ such that $h|_{A\times\{0\}} = \mathrm{Id}_A$ and $h|_{A\times\{1\}} = \{a_0\}$ for some $a_0 \in A$. As $\iota : A \hookrightarrow X$ is a cofibration, we have a homotopy extension, $H : X \times [0,1] \to X$ with $H_0 := H|_{X\times\{0\}} = \mathrm{Id}_X$. Observe that $H_1 := H|_{X\times\{1\}} : X \to X$ maps A to a point $\{a_0\}$, and thus, we have a map $r : X/A \to X$ such that $r \circ q = H_1$. By construction, $H : r \circ q \simeq \mathrm{Id}_X$ is a homotopy. On the other hand, the homotopy $H_t : X \to X$ maps $H_t(A) \subset A$, and hence, passing to quotient we have the homotopy $\tilde{H}_t : X/A \to X/A$. Clearly $\tilde{H}_0 = \mathrm{Id}_{X/A}$. Also, $qr([x]) = qrq(x) = qH_1(x) = \tilde{H}_1 q(x) = \tilde{H}_1([x])$, i.e, $\tilde{H}_1 = q \circ r$. Thus, $\tilde{H} : q \circ r \simeq \mathrm{Id}_{X/A}$ is a homotopy. This proves that $q : X \to X/A$ is a homotopy equivalence. $\square$

Theorem 10.14: (*Homology of Quotient by a Cofibered Subspace*)

Suppose $A \subset X$ is a subspace such that the inclusion $\iota : A \hookrightarrow X$ is a cofibration. Then, the quotient map $q : X \to X/A$ induces an isomorphism $H_n(X, A) \to H_n(X/A, \star) \cong \tilde{H}_n(X/A)$.

Proof : Conider the mapping $C(\iota)$ which is given as the pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "\iota"'] & CA \arrow[d, "j"] \\ X \arrow[r] & C(\iota) \end{tikzcd}$$

As ι is a cofibration, by Lemma 10.12, we have j is a cofibration. But CA is contractible. Hence, by Lemma 10.13, we have $q : C(\iota) \to C(\iota)/CA = X/A$ is a homotopy equivalence. Now, the inclusion map $(X, A) \to (C(\iota), CA)$ induces an isomorphism in homology since we can first excise out the coning point, and then apply a deformation retract. Also, $(C(\iota), CA) \to (X/A, \star)$ induces and isomorphism as the quotient map is a homotopy equivalence. Hence, composing, it follows that the quotient $(X, A) \to (X/A, \star)$ induces an isomorphism in homology.

The isomorphism $H_n(X/A, \star) \cong \tilde{H}_n(X/A)$ follows from applying the quotient map $(Y, \emptyset) \to (Y/\emptyset, \emptyset/\emptyset) = (Y, \star)$ for any space Y. $\square$

Example 10.15: (*CW pair*)

Let X be a CW-complex, and $A \subset X$ a subcomplex. Then the inclusion $A \hookrightarrow X$ is a cofibration. In fact, if X is a locally finite CW complex (i.e, only finitely many cells of each dimension), then we can drop the assumption that A is a sub-complex and only require that $A \subset X$ is closed and a CW complex on its own.

10.4 Cellular Decomposition

Let us recall the definition of a (relative) CW complex. Given a pair (X, A), we say X is obtained from by *attaching an n-cell* if there is a pushout diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "\varphi"] \arrow[d, hook] & A \arrow[d, hook, "\iota"] \\ D^n \arrow[r, "\Phi"'] & X \end{tikzcd}$$

Since the inclusion $S^{n-1} \hookrightarrow D^n$ is trivially a cofibration, it follows from Lemma 10.12 that $\iota : A \hookrightarrow X$ is also a cofibration. The map $\varphi : S^{n-1} \to A$ is called the *attaching map* for the n-cell, and Φ is called the *characteristic map*. Note that A is necessarily closed in X, and Φ is a homeomorphism of the interior $\mathring{D}^n$ onto $X \setminus A$. Indeed, X is homeomorphic to the mapping cone $C(\varphi) = A \cup_\varphi \underbrace{C(S^{n-1})}_{D^n}$.

Example 10.16:

The n-sphere is obtained from the point $\{\star\}$ by attaching an n-cell, where the attaching map is the constant map. On the other hand, we can also realize the n-sphere from the n-disc by attaching a single n-cell, where the attaching map is the identity map on the boundary. The real projective space $\mathbb{RP}^n$ is obtained from $\mathbb{RP}^{n-1}$ by attaching an n-cell. The complex projective space $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a $2n$-cell.

Remark 10.17:

Suppose X is a T_2-space with a closed subspace $A \subset X$. Let $\Phi : D^n \to X$ be continuous map, which restricts to a homeomorphism of $\mathring{D}^n$ onto $X \setminus A$. Then, X is obtained from A by attaching an n-cell, via the attaching map $\varphi := \Phi|_{S^{n-1}=\partial D^n}$.

More generally, we can attach multiple n-cells at once. Given a pair (X, A), we say X is obtained from A by *attaching n-cells* if there exists a pushout diagram

$$\begin{tikzcd} \bigsqcup_{\alpha \in J} S^{n-1}_\alpha \arrow[r, "\varphi"] \arrow[d, hook] & A \arrow[d, hook, "\iota"] \\ \bigsqcup_{\alpha \in J} D^n_\alpha \arrow[r, "\Phi"'] & X \end{tikzcd}$$

Here, J is an arbitrary (possibly empty) indexing set. The restriction $\varphi_\alpha := \varphi|_{S^{n-1}_\alpha}$ is called the *attaching map* for the n-cell $E^n_\alpha := \mathring{D}^n_\alpha$. It follows that A is closed in X, and Φ induces a homeomorphism of $\sqcup E^n_\alpha$ with $X \setminus A$, in particular $X \setminus A$ is a union of components, each being an n-cell. In case $J = \emptyset$, we have $X = A$.

Exercise 10.18:

Suppose X is obtained from A by attachin n-cells, with attaching maps $\{\varphi_\alpha : S^{n-1}_\alpha \to A\}_{\alpha \in J}$. Show that a map $f : A \to Y$ extends to a map $F : X \to Y$ if and only if each $f \circ \varphi_\alpha$ is null-homotopic (whence the extension is determined by the choice of null-homotopies).

Definition 10.19: (*CW Complex*)

Let $A \subset X$ be given. A *CW decomposition* of the pair (X, A) is a sequence of subspaces

$$A = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X,$$

such that the following holds.

1. $X = \cup_{i \geq 0} X^i$, with the colimit topology (i.e, a map $f : X \to Y$ is continuous if and only if $f|_{X^n} : X^n \to X$ is continuous for all n).
2. For each $n \geq 0$, the space X^n is obtained from X^{n-1} by attaching n-cells.

The pair (X, A) equipped with a CW decomposition is called a *relative CW complex*. When $A = \emptyset$, we say X is a *CW complex*. The space X^n is called the *n-skeleton* of the CW complex. We say (X, A) is a *finite CW* *complex* (resp. *countable CW complex*) of there are only finitely (resp. countably) many cells attached. We say (X, A) is a *locally finite CW complex* if for each n, only finitely many n-cells are attached.

Exercise 10.20:

Suppose (X, A) is a relative CW complex. Show that the inclusion $A \hookrightarrow X$ is a cofibration. More generally, given a CW complex X, and a subcomplex $Y \subset X$ (i.e, Y is closed and consists of cells of X), the inclusion $Y \hookrightarrow X$ is a cofibration.

Hint : Use Lemma 10.12 to the cofibration $\sqcup S^{n-1} \hookrightarrow \sqcup D^n$, and observe that the composition of cofibrations is again a cofibration (Lemma 10.9). Inductively extend the homotopy $X^n \times [0,1]$ to $X^{n+1} \times [0,1]$. The final homotopy is continuous as the topology on X is a colimit topology.

In the above definition, there is no restriction on the topology on A. If A is a T_2-space, then X is a T_2-space, moreover the topology on X is given by the colimit topology with respect to the family consisting of A and the closure of cells. This leads to the following definition.

Definition 10.21: (*Whitehead Complex*)

A *Whitehead complex* is a space X, along with a *cell decomposition* $\{e_\lambda \mid \lambda \in \Lambda\}$, where each e_λ is homeomorphic to $\mathring{D}^n$ for some n, such that the following holds.

1. X is a T_2-space.
2. For each n-cell, there is a map $\Phi_\lambda : D^n \to X$ such that $\Phi|_{\mathring{D}^n} : \mathring{D}^n \to e_\lambda$ is a homeomorphism, and Φ maps the boundary ∂D^n into X^{n-1}, which is the union of cells of dimension $\leq n-1$.
3. (**C**losure Finiteness) The closure $\overline{e}_\lambda$ of each cell intersects only a finitely many cells.
4. (**W**eak Topology) The topology on X is induced by the colimit topology of the family $\{\overline{e}_\lambda \mid \lambda \in \Lambda\}$, i.e, $U \subset X$ is open if and only if $U \cap \overline{e}_\lambda$ is open for all $\lambda \in \Lambda$.

It follows that any Whitehead complex carries a CW decomposition, and conversely, a CW complex is a Whitehead complex.

Remark 10.22:

Note that the phrase "X is a CW complex" technically means that "X is a space equipped with a CW decomposition, and the corresponding colimit topology". There are spaces which does not admit any CW decomposition, e.g., the Hawaiian earring. To see this, one first proves that a space admitting a CW decomposition must be locally contractible, but Hawaiian earring is not.

Definition 10.23: (*Cellular Map*)

Let $(X, A), (Y, B)$ be two relative CW complexes. A map $f : (X, A) \to (Y, B)$ is called a *cellular map* if $f(X^n) \subset Y^n$ holds for each $n \geq -1$. Given a CW complex X, a subspace $A \subset X$ is called a CW subcomplex if A is a CW complex and the inclusion map $A \hookrightarrow X$ is cellular.

We have the following important theorem, proof of which follows from inductively constructing the map cell-by-cell.

Theorem 10.24: (*Cellular Approximation Theorem*)

Let X, Y be CW complexes. Then, any map $f : X \to Y$ is homotopic to a cellular map $g : X \to Y$. Suppose $B \subset X$ is a subcomplex, i.e, B is a CW complex and $\iota B \hookrightarrow X$ is cellular. If $f|_B : B \to Y$ is already cellular, then the homotopy $f \simeq g$ can be chosen to relative to B, i.e, the homotopy stays constant on B.

Next, recall the notion of weak homotopy equivalence.

Definition 10.25: (*Weak Homotopy Equivalence*)

A map $f : X \to Y$ is called a *weak homotopy equivalence* if

- $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is group isomorphism for each $n \geq 1$ and each $x \in X$, and
- $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.

Example 10.26: (*Digital Circle*)

Consider the four point space $X = \{N, E, W, S\}$ with the following topology

$$\{\emptyset, \{N\}, \{S\}, X\}.$$

This space is *not* T_2. Let us consider a map $f : S^1 \to X$ which maps the open upper semicirlce (resp. lower semicircle) to N (resp. to S), and the two remaining points to E and W respectively. Clearly, f is a continuous map. Surjectivity of f shows that X is a path connected space. Moreover, one can construct a contractible universal cover of X, and show that $\pi_1(X) = \mathbb{Z}$ and $\pi_k(X) = 0$ for $k > 1$. It follows that $f : S^1 \to X$ is a weak homotopy equivalence. The space X is called the *digital circle*. Note that f does not admit a homotopy inverse, since any $g : X \to S^1$ is necessarily a constant map (as the image must be path connected and at most finitely many points).

By the *Whitehead theorem*, we have that a weak homtopy equivalence between CW complexes is a homotopy equivalence. In general, we have the following theorem.

Theorem 10.27: (*CW Approximation Theorem*)

Given any space Y, there exists a CW complex X and weak homotopy equivalence $\alpha : X \to Y$, which is called a *CW approximation* of Y. Moreover, if $f : Y_1 \to Y_2$ is a map, and $\alpha_i : X_i \to Y_i$ are CW approximations for $i = 1, 2$, then there exists a (cellular) map $g : X_1 \to X_2$, defined unique up to homotopy, such that $\alpha_2 \circ g \simeq f \circ \alpha_1$.

Note that CW approximation is applicable for any space, but since we cannot in general invert (even up to homotopy) a weak homotopy equivalence, this does not let us replace a space. However, when computing algebraic invariants like homotopy or (co)homology, it is good enough to replace any space by a CW approximation.

Remark 10.28: (*CW Type*)

A space X is said to be of *CW type* if X is homotopy equivalent to a space Y, where Y admits a CW decomposition. We have noted that the Hawaiian earring does not admit a CW decomposition, moreover it does not have CW type. Consider the *Hedgehog space*

$$X := \left\{ re^{\iota\theta} \mid 0 \leq r \leq 1, \theta \in \mathbb{Q} \right\} \subset \mathbb{C}$$

which is a dense collection of spokes. Again, X is not locally contractible at any point other than the origin, and hence, X does not admit a CW decomposition. On the other hand, X is contractible, and so, it is homotopy equivalent to a CW complex (the singleton). So, X is of CW type. It is easy to see that Whitehead theorem generalizes to CW tpye : a weak homotopy equivalence between spaces of CW type is a homotopy equivalence.