Course notes for

Algebraic Topology II (KSM4E02)

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`basic category theory — functors — chain complexes`

1.1 Categories and Functors

Category theory is the language of mathematics, it lets you to identify pattens among disparate topics. Although we shall see some definitions, they are not set in stone, and depending on the context you may need to assume extra structure.

Definition 1.1: (*Category*)

A *category* $\mathcal{C}$ consists of the following data.

1. A collection of objects, denoted $\mathrm{Ob}(\mathcal{C})$.
2. For any two objects $A, B \in \mathcal{C}$ a set $\hom_{\mathcal{C}}(A, B)$.
3. For any three objects $A, B, C \in \mathcal{C}$, a binary operation

$$\begin{aligned} \circ : \hom_{\mathcal{C}}(A, B) \times \hom_{\mathcal{C}}(B, C) &\to \hom_{\mathcal{C}}(A, C) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

   which satisfies the following.

   a. **(Associativity)** For morphisms $f : A \to B, g : B \to C, h : C \to D$, we have

   $$h \circ (g \circ f) = (h \circ g) \circ f.$$

   b. **(Identity)** For each object A, there is a morhphism $\mathrm{Id}_A : A \to A$, such that for $g : A \to B$ and any $h : C \to A$ we have

   $$g \circ \mathrm{Id}_A = g, \quad \mathrm{Id}_A \circ h = h.$$

Remark 1.2:

Definition of a category is highly dependent on the context! One can add many adjectives to specify different tpyes of categories. The above should be called a *locally small* category, as we insists that $\hom_{\mathcal{C}}(A, B)$ is a **set**. When the class of object $\mathrm{Ob}(\mathcal{C})$ is also a set, we say the category is *small*. Such set theoretic issues crop up all over category theory. Recall Russell's paradox : can you define a *set* of all sets?!

Example 1.3: (*Some categories*)

Categories crop up all over mathematics (and other fields as well).

- $\mathbf{Sets}$: Category of sets and set functions.
- $\mathbf{Top}$: Category of topological spaces and continuous functions.
- $\mathbf{Top}_*$: Category of topological spaces X with a fixed point $*_X \in X$ (called the *basepoint*), and continuous maps $f : (X, *_X) \to (Y, *_Y)$, which are basepoint preserving, i.e., $f(*_X) = *_Y$.
- $\mathbf{Grp}$: Category of groups and group homomorphisms.
- $\mathbf{Ab}$: Category of Abelian groups and group homomorphisms.
- $\boldsymbol{R} - \mathbf{Mod}$: Given a ring R, category of (left) R-modules, and R-module maps.
- $\mathbf{Cat}$: Category of all locally small categories. Note that this category itself is not locally small, as the collection of all functors between two categories need not be a set. Thus, Cat maybe called a *large* category.
- $\mathbf{Kit}$: (not a standard notaion!) Category of all *small* categories (i.e, both objects and morphisms form a set) indeed gives rise to a locally small category.
- $\boldsymbol{\Delta}$: For each $n \geq 0$, denote $[n] := \{0, 1, \cdots, n\}$. A function $f : [m] \to [n]$ is called *non-decreasing* (or *order-preserving*) if $i < j \Rightarrow f(i) \leq f(j)$. The *simplicial category* Δ consists of $[n]$ for each $n \geq 0$ as objects, and $\hom_\Delta([m], [n])$ is the set of non-decreasing functions $[m] \to [n]$.

Exercise 1.4: (*Groups as Categories*)

Let G be group. Check that we have a natural (small) category with a single object say $*$, and G as the hom set $\hom(*, *)$.

Example 1.5: (*Discrete Category*)

Given any set X, we can consider a small category whose object set is X, and given any $x, y \in X$, there is no morphism if $x \neq y$. For $x = y$, the definition forces us to consider the identity morphism $1_x : x \to x$. This is called a *discrete category*. Any small discrete category is always obtained in this way.

Definition 1.6: (*Functor*)

Given two categories $\mathcal{C}, \mathcal{D}$, a *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ is the following data.

1. For each object $c \in \mathcal{C}$, there is an object $F(c) \in \mathcal{D}$.
2. For each morphism $f : c_1 \to c_2$ in $\mathcal{C}$, there is a morphism $F(f) : F(c_1) \to F(c_2)$ in $\mathcal{D}$, that satisfies the following.
   a. F preseves the identity, i.e., $F(\mathrm{Id}_c) = \mathrm{Id}_{\{F(c)\}}$ for any object $c \in \mathcal{C}$.
   b. F preseves the composition, i.e., given morphisms $f : A \to B, g : B \to C$ in $\mathcal{C}$ we have
   $$F(g \circ f) = F(g) \circ F(f).$$

We say $F : \mathcal{C} \to \mathcal{D}$ is a *contravariant functor* if it *reverses the morphisms and compositions*, i.e., given morphism $f : c_1 \to c_2$ in $\mathcal{C}$, we have the morphism $F(f) : F(c_2) \to F(c_1)$, and the composition is preseved as $F(g \circ f) = F(f) \circ F(g)$.

Example 1.7: (*Some functors*)

- Given any category, we can always consider the identity functor.
- Given two sets X, Y and a function $f : X \to Y$, we have a functor between the two discrete categories, and conversely.
- For any mathematical object defined as a set with some additional structure, lets us define a *forgetful functor* by forgetting the extra structure. Thus, we have functors $\mathbf{Top} \to \mathbf{Sets}$, $\mathbf{Grp} \to \mathbf{Sets}$ etc. Similarly, we have forgetful functors $\mathbf{Ab} \to \mathbf{Grp}$ and $\boldsymbol{R}$-$\mathbf{Mod} \to \mathbf{Ab}$.
- Recall Δ is the simplicial category, where $\mathrm{Ob}(\Delta) = \{[n] := \{0, 1, \cdots, n\} \mid n \geq 0\}$, and $\hom_\Delta([m], [n])$ is the set of order-preserving functions. A *contravariant* functor $K : \Delta \to \mathrm{Set}$ is known as a *simplicial set*. Simplicial sets are of fundamental importance in homotopy theory.
- A covariant functor $\Delta \to \mathrm{Set}$ is known as a *cosimplicial set*, which might be an unfortunate nomenclature!

Definition 1.8: (*Opposite Category*)

Given a category $\mathcal{C}$, the *opposite category*, denoted as $\mathcal{C}^{\mathrm{op}}$, is the category which has the following data.

- $\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Obj}(\mathrm{cal}(\mathrm{C}).$
- For any $A, B \in \mathcal{C}^{\mathrm{op}}$, we have $\hom_{\mathcal{C}^{\mathrm{op}}}(A, B) := \hom_{\mathcal{C}}(B, A)$.
- For $f \in \hom_{\mathcal{C}^{\mathrm{op}}}(A, B), g \in \hom_{\mathcal{C}^{\mathrm{op}}}(B, C)$, we have

$$g \circ_{\mathcal{C}^{\mathrm{op}}} f := f \circ_{\mathcal{C}} g \in \hom_{\mathcal{C}}(C, A) = \hom_{\mathcal{C}^{\mathrm{op}}}(A, C)$$

In other words, $\mathcal{C}^{\mathrm{op}}$ is obtained from $\mathcal{C}$ by reversing the arrows (i.e., morphisms).

Exercise 1.9: (*Group Homomorphism as Functor*)

Given a group homomorphism $f : G \to H$ interpret it as a functor between the two associated one-object categories as in Exercise 1.4. Is the converse true? That is, is any (covariant) functor between these categories induced by a group homomorphism?

Given any (locally small) category, one of the most important functors are the $\hom$ functors.

Definition 1.10: ($\hom$ *functors*)

Let $\mathcal{C}$ be a category, and fix an object $X \in \mathcal{C}$. Then, the *covariant* $\hom$*-functor* is the functor

$$\begin{aligned} \hom_{\mathcal{C}}(X, \_) : \mathcal{C} &\to \mathrm{Set} \\ Y &\mapsto \hom_{\mathcal{C}}(X, Y), \end{aligned}$$

and the *contravariant* $\hom$*-functor* is the contravariant functor

$$\begin{aligned}\hom_{\mathcal{C}}(\_, X) : \mathcal{C} &\to \mathrm{Set}\\ Y &\mapsto \hom_{\mathcal{C}}(Y, X).\end{aligned}$$

Often times the $\hom$ itself may have some extra structure, in which case the range of the functors can be a different category.

Example 1.11:

Let $\mathcal{C}$ be the category of vector spaces over some field $\Bbbk$, where morphisms are $\Bbbk$-linear maps. Then, $\hom_{\mathcal{C}}(V, W)$ itself is a vector space. Thus, the $\hom$-functors for some fixed vector space V can be realized as $\hom_{\mathcal{C}}(V, \_) : \mathcal{C} \to \mathcal{C}$, and $\hom_{\mathcal{C}}(\_, V) : \mathcal{C} \to \mathcal{C}$.

1.2 Category of Chain Complexes

One of the crucial interest in homological algebra is the category of chain complexes of Abelian groups (or more generally, R-modules).

Definition 1.12: (*Chain Complex*)

A *chain complex* of Abelian groups, is a collection $\{C_n\}_{n\in\mathbb{Z}}$ of Abelian groups, and a collection $\{\partial_n : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ of group homomorphisms, called boundary maps. We denote this as

$$(C_\bullet, \partial) : \cdots \to C_n \underset{\partial_n}{\longrightarrow} C_{n-1} \to \cdots,$$

such that $\partial_n \circ \partial_{n+1} = 0$. We say $(C^\bullet, \partial)$ is a *cochain complex* if we have the maps in the opposite direction, i.e.,

$$(C^\bullet, \partial) : \cdots \leftarrow C^{n+1} \underset{\partial^n}{\longleftarrow} C^n \leftarrow \cdots,$$

Although there is no explicit rules for indexing the boundary maps, it is standard to put the index of the source object as the index for the boundary map for both chain and cochain complexes.

Definition 1.13: (*Chain map*)

A *chain map* $f_\bullet : C_\bullet \to D_\bullet$ between two chain complexes $(C_\bullet, \partial_\bullet^C)$ and $(D_\bullet, \partial_\bullet^D)$ is a collection of homomorphisms $f_n : C_n \to D_n$ such that, we have a commutative diagram as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\partial_n^C"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_n \arrow[r, "\partial_n^D"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Explicitly, we have $f_{n-1} \circ \partial_n^C = \partial_n^D \circ f_n$. Similarly, one can define *cochain maps* of cochain complexes.

Remark 1.14:

If it is clear from the context, we often drop index from the ∂_n, f_n. In particular, the definition of chain map may also be understood as $\partial \circ f = f \circ \partial$. Moreover, we sometimes use the same ∂ to denote the boundary maps of different chain complexes, so that the notation is kept light!

Definition 1.15:

A (co)chain complex $(C_\bullet, \partial)$ is called

a. *bounded below* by some $n_0 \in \mathbb{Z}$ if $C_n = 0$ for all $n < n_0$,
b. *bounded above* by some $m_0 \in \mathbb{Z}$ if $C_n = 0$ for all $n > m_0$, and
c. *bounded* if it is both bounded below and above.

Definition 1.16: (*Category of Chain Complex*)

The *category of chain complexes* $\mathrm{Ch}(\mathrm{Ab})$ is the category whose objects are chain complexes, and morphisms are the chain maps.

Remark 1.17:

Depending on the requirement, we can restrict ourselves to bounded below, bounded above, or just bounded chain complexes. Among these, we are primarily interested in the $C_\bullet$ when $C_n = 0$ for $n < 0$, which we simply write as

$$C_\bullet : \cdots \to C_2 \to C_1 \to C_0 \to 0,$$

and $C^\bullet$ when $C^n = 0$ for $n < 0$, which we simply write as

$$C^\bullet : 0 \to C^0 \to C^1 \to C^2 \to \cdots$$

Definition 1.18: (*Graded Abelian Groups*)

A *graded Abelian group* is a collection $G_\bullet = \{G_n\}_{n\in\mathbb{Z}}$ of Abelian groups. A map $f_\bullet : G_\bullet \to H_\bullet$ between two such graded groups is a collection of group homomorphisms $\{f_n : G_n \to H_n\}_{n\in\mathbb{Z}}$

Exercise 1.19: (*Graded Groups are Chain Complex*)

Interpret graded group as a chain complex. Verify that a morphism of graded groups is precisely a chain map.