Mid-semester Examination

Course: Algebraic Topology II (KSM4E02) **Instructor:** Aritra Bhowmick

Time: 2:00PM – 5:00PM, 2nd March, 2026 **Total marks:** 30

Attempt any question. You can get maximum **25 marks**.

Q1. Compute the singular homology groups of $S^p \times S^q$ for $p, q \geq 0$. [5]

Solution : Suppose S^p has the cell decomposition $\{e^0, e^p\}$ and S^q has the cell decomposition $\{f^0, f^q\}$. Then, on the product $S^p \times S^q$ the cells are $\{e^0 \times f^0, e^0 \times f^q, e^p \times f^0, e^p \times f^q\}$. The attaching map for the p and q cell are constant. Hence, the corresponding cellular boundary maps are also 0. The only nontrivial attaching map is $S^{p+q-1} \to S^p \vee S^q$. Thus, we need to compute $\partial : C^{\text{cell}}_{p+q}(S^p \times S^q) \to C^{\text{cell}}_{p+q-1}(S^p \times S^q)$.

- Suppose $p \neq 1, q \neq 1$. Then there cannot be any $(p+q-1)$-cell, and hence $\partial = 0$.
- Suppose $p = 1$. Then, an argument using the Mayer-Vietoris sequence gives $H_{p+q}(S^p \times S^q) = \mathbb{Z}$. This implies the cellular boundary map ∂ must be 0

Since every boundary map in the cellular chain complex is zero, the (cellular) homology is isomorphic to the cellular chain complex itself.

Q2. Compute the singular homology groups of $S^1 \times (S^1 \vee S^1)$. [5]

Solution : The space $X = S^1 \times (S^1 \vee S^1)$ looks like a torus lying flat on top of another torus, while touching each other along an S^1. Consider U to be the top torus, along with a small open neighborhood of the circle in the bottom torus. Similarly, let V be the bottom torus, along with a small open neighborhood of the circle in the top torus. Then, U and V both deformation retracts onto the respective torus. Also, $U \cap V$ deforms to the circle. It follows that $(X; U, V)$ is a proper triad, and we can use the Mayer-Vietoris sequence.

Let us denote $H_1(U) = \mathbb{Z}\langle a, b\rangle, H_1(V) = \mathbb{Z}\langle x, y\rangle, H_1(U \cap V) = \mathbb{Z}\langle z\rangle$, where a, x, z represents the circle in the intersection. In particular, the map $H_1(U \cap V) \to H_1(U) \oplus H_1(V)$ is given as

$$z \mapsto a + 0.b + x + 0.y = a + x.$$

Clearly this map is injetive. Since $\tilde{H}_k(U \cap V) = \tilde{H}_k(S^1) = 0$ for $k \neq 1$, it follows that $\tilde{H}_k(U \cap V) \to \tilde{H}_k(U) \oplus \tilde{H}_k(V)$ is injective for all k. But then from the reduced Mayer-Vietoris sequence, we get the short exact sequences

$$0 \to \tilde{H}_k(U \cap V) \to \tilde{H}_k(U) \oplus \tilde{H}_k(V) \to \tilde{H}_k(X) \to 0.$$

It follows that $H_1(X) = \mathbb{Z}\langle b, y, a - x\rangle = \mathbb{Z}^3$. For $k \neq 1$, we have isomorphism $\tilde{H}_k(X) = \tilde{H}_k(U) \oplus \tilde{H}_k(V)$. Putting everything together, we have

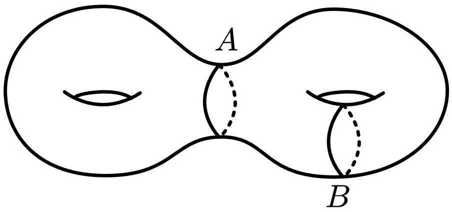
$$H_k(X) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}^3, & k = 1 \\ \mathbb{Z}^2, & k = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Q3. Compute the *relative* singular homology groups in the following cases. $[2 \times 4 = 8]$

a. $X = S^2$ and $A \subset X$ is a finite set.

b. $X = S^1 \times S^1$ and $A \subset X$ is a finite set.

c. X is the double torus, and A is the middle circle.



d. X is again the double torus, and B is the circle as above.

Solution : Since we are working with CW complexes, we have $q : X \to X/A$ induces isomorphism $H_\bullet(X, A) \to \tilde{H}_\bullet(X/A)$. So we need to figure out X/A.

Suppose $A \subset X$ has k many distinct points. We claim that

$$X/A \simeq X \vee \left(\vee_{i=1}^{k-1} S^1\right).$$

To see this, first note that $X/A \simeq Y$, where Y is obtained from X by taking a disjoint point, and joining each of the points of A by a line. Then, in Y, we can homotopy the base of these lines and identify to a single point in X. Finally, if we collapse one of the lines, we are left with $(k-1)$-many S^1 wedged to X. Finally, we can write

$$H_i(S^2, A) = \begin{cases} 0, & i = 0 \\ \mathbb{Z}^{k-1}, & i = 1 \\ \mathbb{Z}, & i = 2 \\ 0, & \text{otherwise} \end{cases} \qquad H_i(S^1 \times S^1, A) = \begin{cases} 0, & i = 0 \\ \mathbb{Z}^{k+1}, & i = 1 \\ \mathbb{Z}, & i = 2 \\ 0, & \text{otherwise} \end{cases}$$

Next, suppose X is the double torus. It is easy to see $X/A = (S^1 \times S^1) \vee (S^1 \times S^1)$, and $X/B = (S^1 \times S^1) \vee S^1$. One can immediately compute,

$$H_i(X, A) = \begin{cases} 0, & i = 0 \\ \mathbb{Z}^4, & i = 1 \\ \mathbb{Z}^2, & i = 2 \\ 0, & \text{otherwise} \end{cases} \qquad H_i(X, B) = \begin{cases} 0, & i = 0 \\ \mathbb{Z}^3, & i = 1 \\ \mathbb{Z}, & i = 2 \\ 0, & \text{otherwise} \end{cases}$$

Q4. Prove *Brouwer fixed point theorem* : any continuous map $f : D^n \to D^n$ has a fixed point, where $D^n \subset \mathbb{R}^n$ is the closed unit disc. (**Hint :** Consider the *ray* joining $f(x)$ to x.) $[4]$

Solution : Suppose $f : D^n \to D^n$ is a map without fixed points. Then, for each $x \in D^n$, there exists a unique ray from $f(x)$ to x, and intersecting the boundary $S^{n-1} = \partial D^n$ at a unique point, say, $r(x)$. One can write down an explicit formula for $r(x)$, and show that $r : D^n \to S^{n-1}$ is continuous. Note that for any $x \in S^{n-1}$ we have $r(x) = x$, since the ray from $f(x)$ to x

intersects S^{n-1} precisely at x. Thus, $r : D^n \to S^{n-1}$ is a retraction. But then $\tilde{H}_k(S^{n-1})$ is a direct summand of $\tilde{H}_k(D^n)$ for all k. Since $\tilde{H}_k(D^n) = 0$, and $\tilde{H}_{n-1}(S^{n-1}) = \mathbb{Z}$, we have a contradiction. Hence, $f : D^n \to D^n$ must have a fixed point.

Q5. Show that any fixed-point free map $f : S^n \to S^n$ is homotopic to the antipode map. (**Hint** : For $x \in S^n$, can the convex sum of $f(x)$ and $-x$ pass through the origin?) [4]

Solution : For $x \in S^n$ and $0 \leq t \leq 1$, consider the quantity $b = (1-t)f(x) + t(-x) = (1-t)f(x) - tx$. If possible, suppose $b = 0$. Then, $(1-t)f(x) = tx$. Taking norm, and noting $\|x\| = 1 = \|f(x)\|$, we have $|1-t| = |t| \Rightarrow 1-t = t \Rightarrow t = \frac{1}{2}$. But for $t = \frac{1}{1}$ we have $b = f(x) - x \neq 0$. Thus, we can define a continuous map

$$h_t(x) = \frac{(1-t)f(x) + t(-x)}{\|(1-t)f(x) + t(-x)\|}.$$

Clearly, $h_0(x) = f(x)$, and $h_1(x) = -x$ is the antipode map. Thus, h is a homotopy of f with the antipode map.

Q6. Justify that on an *even* sphere the only nontrivial group acting freely is $\mathbb{Z}/2\mathbb{Z}$. [4]

Recall : A group G acts on a space X if there is a continuous map $\varphi : G \times X \to X$ such that $\varphi(e, x) = x$, and $\varphi(g, \varphi(h \cdot x)) = \varphi(g \cdot h, x)$ holds for $g, h \in G, x \in X$. The action is *free* if for any $g \neq e$ (where $e \in G$ is the identity), we have $\varphi_g : X \to X$ given as $x \mapsto g \cdot x$ is a fixed-point free map.

Solution : Suppose G acts freely on the even sphere S^{2n}. Suppose $\varphi : G \times S^{2n} \to S^{2n}$ is the action. For any $g \in G$, we then have the homeomorphism $\varphi_g : S^{2n} \to S^{2n}$. Since homeomorphisms have degree ± 1, we have a composed map

$$\delta : G \longrightarrow \mathrm{homeo}(S^{2n}) \xrightarrow{\deg} \{\pm 1\} = \mathbb{Z}/2\mathbb{Z}.$$

Let us check that δ is a group homomorphism. Indeed, $\delta(gh^{-1}) = \deg(\varphi_{gh^{-1}}) = \deg(\varphi_g \circ \varphi_h^{-1}) = \deg(\varphi_g) \cdot \deg(\varphi_h)^{-1} = \delta(g) \cdot \delta(h)^{-1}$. Since G acts freely, for all $e_G \neq g \in G$, we have φ_g is fixed-point free, and hence, homotopic to the antipode map. Thus, $\delta(g) = (-1)^{2n+1} = -1$. This shows that $\delta : G \to \{\pm 1\}$ is an injective group map. This means $G = \{e\}$ or $G = \{\pm 1\}$. This proves the claim.