Course notes for

Algebraic Topology II (KSM4E02)

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Day 1 : 13th January, 2026

`basic category theory — functors — chain complexes`

1.1 Categories and Functors

Category theory is the language of mathematics, it lets you identify pattens among disparate topics. Although we shall see some definitions, they are not set in stone, and depending on the context you may need to assume extra structure.

Definition 1.1: (*Category*)

A *category* $\mathcal{C}$ consists of the following data.

1. A collection of objects, denoted $\mathrm{Ob}(\mathcal{C})$.
2. For any two objects $A, B \in \mathcal{C}$ a set $\hom_{\mathcal{C}}(A, B)$.
3. For any three objects $A, B, C \in \mathcal{C}$, a binary operation

$$\begin{aligned} \circ : \hom_{\mathcal{C}}(A, B) \times \hom_{\mathcal{C}}(B, C) &\to \hom_{\mathcal{C}}(A, C) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

which satisfies the following.

a. **(Associativity)** For morphisms $f : A \to B, g : B \to C, h : C \to D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

b. **(Identity)** For each object A, there is a morhphism $\mathrm{Id}_A : A \to A$, such that for $g : A \to B$ and any $h : C \to A$ we have

$$g \circ \mathrm{Id}_A = g, \quad \mathrm{Id}_A \circ h = h.$$

Remark 1.2:

Definition of a category is highly dependent on the context! One can add many adjectives to specify different tpyes of categories. The above should be called a *locally small* category, as we insists that $\hom_{\mathcal{C}}(A, B)$ is a **set**. When the class of object $\mathrm{Ob}(\mathcal{C})$ is also a set, we say the category is *small*. Such set theoretic issues crop up all over category theory. Recall Russell's paradox : can you define a *set* of all sets?!

Example 1.3: (*Some categories*)

Categories crop up all over mathematics (and other fields as well).

- $\mathbf{Sets}$: Category of sets and set functions.
- $\mathbf{Top}$: Category of topological spaces and continuous functions.
- $\mathbf{Top}_*$: Category of topological spaces X with a fixed point $*_X \in X$ (called the *basepoint*), and continuous maps $f : (X, *_X) \to (Y, *_Y)$, which are basepoint preserving, i.e., $f(*_X) = *_Y$.
- $\mathbf{Grp}$: Category of groups and group homomorphisms.
- $\mathbf{Ab}$: Category of Abelian groups and group homomorphisms.
- $\boldsymbol{R} - \mathbf{Mod}$: Given a ring R, category of (left) R-modules, and R-module maps.
- $\mathbf{Cat}$: Category of all locally small categories. Note that this category itself is not locally small, as the collection of all functors between two categories need not be a set. Thus, Cat maybe called a *large* category.
- $\mathbf{Kit}$: (not a standard notaion!) Category of all *small* categories (i.e, both objects and morphisms form a set) indeed gives rise to a locally small category.
- $\boldsymbol{\Delta}$: For each $n \geq 0$, denote $[n] := \{0, 1, \cdots, n\}$. A function $f : [m] \to [n]$ is called *non-decreasing* (or *order-preserving*) if $i < j \Rightarrow f(i) \leq f(j)$. The *simplicial category* Δ consists of $[n]$ for each $n \geq 0$ as objects, and $\hom_\Delta([m], [n])$ is the set of non-decreasing functions $[m] \to [n]$.

Exercise 1.4: (*Groups as Categories*)

Let G be group. Check that we have a natural (small) category with a single object say $*$, and G as the hom set $\hom(*, *)$.

Example 1.5: (*Discrete Category*)

Given any set X, we can consider a small category whose object set is X, and given any $x, y \in X$, there is no morphism if $x \neq y$. For $x = y$, the definition forces us to consider the identity morphism $1_x : x \to x$. This is called a *discrete category*. Any small discrete category is always obtained in this way.

Definition 1.6: (*Functor*)

Given two categories $\mathcal{C}, \mathcal{D}$, a *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ is the following data.

1. For each object $c \in \mathcal{C}$, there is an object $F(c) \in \mathcal{D}$.
2. For each morphism $f : c_1 \to c_2$ in $\mathcal{C}$, there is a morphism $F(f) : F(c_1) \to F(c_2)$ in $\mathcal{D}$, that satisfies the following.
   a. F preseves the identity, i.e., $F(\mathrm{Id}_c) = \mathrm{Id}_{\{F(c)\}}$ for any object $c \in \mathcal{C}$.
   b. F preseves the composition, i.e., given morphisms $f : A \to B, g : B \to C$ in $\mathcal{C}$ we have
   $$F(g \circ f) = F(g) \circ F(f).$$

We say $F : \mathcal{C} \to \mathcal{D}$ is a *contravariant functor* if it *reverses the morphisms and compositions*, i.e., given morphism $f : c_1 \to c_2$ in $\mathcal{C}$, we have the morphism $F(f) : F(c_2) \to F(c_1)$, and the composition is preseved as $F(g \circ f) = F(f) \circ F(g)$.

Example 1.7: (*Some functors*)

- Given any category, we can always consider the identity functor.
- Given two sets X, Y and a function $f : X \to Y$, we have a functor between the two discrete categories, and conversely.
- For any mathematical object defined as a set with some additional structure, lets us define a *forgetful functor* by forgetting the extra structure. Thus, we have functors $\mathbf{Top} \to \mathbf{Sets}$, $\mathbf{Grp} \to \mathbf{Sets}$ etc. Similarly, we have forgetful functors $\mathbf{Ab} \to \mathbf{Grp}$ and $\boldsymbol{R}\text{-}\mathbf{Mod} \to \mathbf{Ab}$.
- Recall Δ is the simplicial category, where $\mathrm{Ob}(\Delta) = \{[n] := \{0, 1, \cdots, n\} \mid n \geq 0\}$, and $\hom_\Delta([m], [n])$ is the set of order-preserving functions. A *contravariant* functor $K : \Delta \to \mathrm{Set}$ is known as a *simplicial set*. Simplicial sets are of fundamental importance in homotopy theory.
- A covariant functor $\Delta \to \mathrm{Set}$ is known as a *cosimplicial set*, which might be an unfortunate nomenclature!

Definition 1.8: (*Opposite Category*)

Given a category $\mathcal{C}$, the *opposite category*, denoted as $\mathcal{C}^{\mathrm{op}}$, is the category which has the following data.

- $\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Obj}(\mathrm{cal(C)})$.
- For any $A, B \in \mathcal{C}^{\mathrm{op}}$, we have $\hom_{\mathcal{C}^{\mathrm{op}}}(A, B) := \hom_{\mathcal{C}}(B, A)$.
- For $f \in \hom_{\mathcal{C}^{\mathrm{op}}}(A, B), g \in \hom_{\mathcal{C}^{\mathrm{op}}}(B, C)$, we have
$$g \circ_{\mathcal{C}^{\mathrm{op}}} f := f \circ_{\mathcal{C}} g \in \hom_{\mathcal{C}}(C, A) = \hom_{\mathcal{C}^{\mathrm{op}}}(A, C)$$

In other words, $\mathcal{C}^{\mathrm{op}}$ is obtained from $\mathcal{C}$ by reversing the arrows (i.e., morphisms).

Exercise 1.9: (*Group Homomorphism as Functor*)

Given a group homomorphism $f : G \to H$ interpret it as a functor between the two associated one-object categories as in Exercise 1.4. Is the converse true? That is, is any (covariant) functor between these categories induced by a group homomorphism?

Given any (locally small) category, one of the most important functors are the $\hom$ functors.

Definition 1.10: ($\hom$ *functors*)

Let $\mathcal{C}$ be a category, and fix an object $X \in \mathcal{C}$. Then, the *covariant* $\hom$*-functor* is the functor

$$\begin{aligned} \hom_{\mathcal{C}}(X, \_) : \mathcal{C} &\to \mathrm{Set} \\ Y &\mapsto \hom_{\mathcal{C}}(X, Y), \end{aligned}$$

and the *contravariant* $\hom$*-functor* is the contravariant functor

$$\begin{aligned} \hom_{\mathcal{C}}(\_, X) : \mathcal{C} &\to \mathrm{Set} \\ Y &\mapsto \hom_{\mathcal{C}}(Y, X). \end{aligned}$$

Often times the $\hom$ itself may have some extra structure, in which case the range of the functors can be a different category.

Example 1.11:

Let $\mathcal{C}$ be the category of vector spaces over some field $\Bbbk$, where morphisms are $\Bbbk$-linear maps. Then, $\hom_{\mathcal{C}}(V, W)$ itself is a vector space. Thus, the $\hom$-functors for some fixed vector space V can be realized as $\hom_{\mathcal{C}}(V, \_) : \mathcal{C} \to \mathcal{C}$, and $\hom_{\mathcal{C}}(\_, V) : \mathcal{C} \to \mathcal{C}$.

1.2 Category of Chain Complexes

One of the crucial interest in homological algebra is the category of chain complexes of Abelian groups (or more generally, R-modules).

Definition 1.12: (*Chain Complex*)

A *chain complex* of Abelian groups, is a collection $\{C_n\}_{n\in\mathbb{Z}}$ of Abelian groups, and a collection $\{\partial_n : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ of group homomorphisms, called boundary maps. We denote this as

$$(C_\bullet, \partial) : \cdots \to C_n \xrightarrow[\partial_n]{} C_{n-1} \to \cdots,$$

such that $\partial_n \circ \partial_{n+1} = 0$. We say $(C^\bullet, \partial)$ is a *cochain complex* if we have the maps in the opposite direction, i.e.,

$$(C^\bullet, \partial) : \cdots \leftarrow C^{n+1} \xleftarrow[\partial^n]{} C^n \leftarrow \cdots,$$

Although there is no explicit rules for indexing the boundary maps, it is standard to put the index of the source object as the index for the boundary map for both chain and cochain complexes.

Definition 1.13: (*Chain map*)

A *chain map* $f_\bullet : C_\bullet \to D_\bullet$ between two chain complexes $(C_\bullet, \partial_\bullet^C)$ and $(D_\bullet, \partial_\bullet^D)$ is a collection of homomorphisms $f_n : C_n \to D_n$ such that, we have a commutative diagram as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\partial_n^C"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_n \arrow[r, "\partial_n^D"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Explicitly, we have $f_{n-1} \circ \partial_n^C = \partial_n^D \circ f_n$. Similarly, one can define *cochain maps* of cochain complexes.

Remark 1.14:

If it is clear from the context, we often drop index from the ∂_n, f_n. In particular, the definition of chain map may also be understood as $\partial \circ f = f \circ \partial$. Moreover, we sometimes use the same ∂ to denote the boundary maps of different chain complexes, so that the notation is kept light!

Definition 1.15:

A (co)chain complex $(C_\bullet, \partial)$ is called

a. *bounded below* by some $n_0 \in \mathbb{Z}$ if $C_n = 0$ for all $n < n_0$,
b. *bounded above* by some $m_0 \in \mathbb{Z}$ if $C_n = 0$ for all $n > m_0$, and
c. *bounded* if it is both bounded below and above.

Definition 1.16: (*Category of Chain Complex*)

The *category of chain complexes* $\mathrm{Ch}(\mathrm{Ab})$ is the category whose objects are chain complexes, and morphisms are the chain maps.

Remark 1.17:

Depending on the requirement, we can restrict ourselves to bounded below, bounded above, or just bounded chain complexes. Among these, we are primarily interested in the $C_\bullet$ when $C_n = 0$ for $n < 0$, which we simply write as

$$C_\bullet : \cdots \to C_2 \to C_1 \to C_0 \to 0,$$

and $C^\bullet$ when $C^n = 0$ for $n < 0$, which we simply write as

$$C^\bullet : 0 \to C^0 \to C^1 \to C^2 \to \cdots$$

Definition 1.18: (*Graded Abelian Groups*)

A *graded Abelian group* is a collection $G_\bullet = \{G_n\}_{n\in\mathbb{Z}}$ of Abelian groups. A map $f_\bullet : G_\bullet \to H_\bullet$ between two such graded groups is a collection of group homomorphisms $\{f_n : G_n \to H_n\}_{n\in\mathbb{Z}}$

Exercise 1.19: (*Graded Groups are Chain Complex*)

Interpret graded group as a chain complex. Verify that a morphism of graded groups is precisely a chain map.

Day 2 : 16th January, 2026

`natural transformations – adjunction – Eilenberg-Steenrod axioms`

2.1 Natural Transformations

Definition 2.1: (*Natural transformation*)

Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta : F \Rightarrow G$ is a collection of morphisms $\eta_c : F(c) \to G(c)$ in $\mathcal{D}$, one for each $c \in \mathcal{C}$, such that given any morphism $f : X \to Y$ in $\mathcal{C}$, we have the following commutative diagram

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_Y"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

Exercise 2.2: (*Composition of Natural Transformations*)

Try to figure out two distinct ways compose natural transformations, namely, horizontally and vertically. The following diagram may help.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow"] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & \mathcal{D} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow"] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & \mathcal{E}
\end{tikzcd}$$

Try to determine the associativity of these compositions.

Definition 2.3: (*Natural Isomorphism*)

A natural transformation $\eta : F \Rightarrow G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is called a *natural isomorphism* if $\eta_c : F(c) \to G(c)$ is an isomorphism for all $c \in \mathcal{C}$.

Remark 2.4: (*Natural Transformation and Homotopy*)

You can imagine functors as maps between two spaces. Then, natural transformation can be interprated as homotopy between them! In fact this analogy can be made rigourous, and one can consider a level 3 morphism between two natural transformations, and so on and on to infinity.

Example 2.5:

Consider $\mathcal{C}$ to be the category of vector spaces over some field $\Bbbk$, with linear maps as morphisms. We have a functor $F : \mathcal{C} \to \mathcal{C}$ given by

$$F(V) = (V^*)^* = \hom_{\Bbbk}(\hom_{\Bbbk}(V, \Bbbk), \Bbbk).$$

Now, for each V, we have linear map given by evaluation

$$\begin{aligned} \eta_V : V &\to F(V) \\ \mathbf{v} &\mapsto (T \mapsto T(\mathbf{v})). \end{aligned}$$

Then, $\eta : \mathrm{Id} \Rightarrow F$ is a natural transformation (Check!).

2.2 Adjoint functors

Definition 2.6: (*Adjoint Functor*)

Let $\mathcal{C}, \mathcal{D}$ be two categories, and we have two fuctors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$. We say L is *left adjoint* to R, and R is *right adjoint* to L if there exists natural isomorphisms (i.e, set bijections)

$$\eta_{c,d} : \hom_{\mathcal{D}}(L(c), d) \to \hom_{\mathcal{C}}(c, R(d)), \quad c \in \mathcal{C}, d \in \mathcal{D}.$$

Here naturality means that given $c \in \mathcal{C}$ fixed, and any morphism $g : d_1 \to d_2$ in $\mathcal{D}$, we have a commutative diagram

$$\begin{tikzcd}
\hom_{\mathcal{D}}(L(c), d_1) \arrow[r, "\eta_{c,d_1}"] \arrow[d, "\hom_{\mathcal{D}}(L(c), g)"'] & \hom_{\mathcal{C}}(c, R(d_1)) \arrow[d, "\hom_{\mathcal{C}}(c, R(g))"] \\
\hom_{\mathcal{D}}(L(c), d_2) \arrow[r, "\eta_{c,d_2}"'] & \hom_{\mathcal{C}}(c, R(d_2))
\end{tikzcd}$$

Similarly, given $d \in \mathcal{D}$ fixed, and any morphism $f : c_1 \to c_2$ in $\mathcal{C}$, we have a commutative diagram

$$\begin{tikzcd}
\hom_{\mathcal{D}}(L(c_1), d) \arrow[r, "\eta_{c_1,d}"] & \hom_{\mathcal{C}}(c_1, R(d)) \\
\hom_{\mathcal{D}}(L(c_2), d) \arrow[r, "\eta_{c_2,d}"'] \arrow[u, "\hom_{\mathcal{D}}(L(f), d)"] & \hom_{\mathcal{C}}(c_2, R(d)) \arrow[u, "\hom_{\mathcal{C}}(f, R(d))"']
\end{tikzcd}$$

The pair of functors (L, R) is called an *adjunct pair*.

Exercise 2.7: (*Adjunction and Natural Isomorphisms*)

Let $L : \mathcal{C} \to \mathcal{D}, R : \mathcal{D} \to \mathcal{R}$ be a pair of functors. Show that (L, R) is an adjuct pair if and only if for each $c \in \mathcal{C}$ and each $d \in \mathcal{D}$ there are natural isomorphisms of functors

$$\hom_{\mathcal{D}}(L(c), \_) \Rightarrow \hom_{\mathcal{C}}(c, R(\_)), \quad \text{and} \quad \hom_{\mathcal{D}}(L(\_), d) \Rightarrow \hom_{\mathcal{C}}(\_, R(d)).$$

Exercise 2.8: (*Left and Right Adjoint of Forgetful Functor* $\mathrm{Top} \to \mathrm{Set}$)

Let $U : \mathrm{Top} \to \mathrm{Set}$ be the forgetful functor. Given a set X, we can either give it the discrete topology, or the indiscrete topology. This defines two functors $D, I : \mathrm{Set} \to \mathrm{Top}$. Show that D is left adjoint of U, and I is right adjoint of U.

Exercise 2.9: (*hom-tensor Adjunction*)

On the category Ab of Abelian groups, we have two functors for each $X \in \mathrm{Ab}$

$$\hom(X, \_) : \mathrm{Ab} \to \mathrm{Ab}, \quad \_ \otimes X : \mathrm{Ab} \to \mathrm{Ab}$$

Prove that the tensor is left adjoint to hom, i.e, for $X, Y, Z \in \mathrm{Ab}$ show that there exists natural isomorphism

$$\hom(X \otimes Y, Z) \to \hom(X, \hom(Y, Z)).$$

In fact, the isomorphism is an isomorphism of Abelian groups as well. Same statement holds true for R -modules as well.

2.3 Eilenberg-Steenrod Axioms of (co)Homology Theories

Consider the category $\mathrm{TopPair}$ whose objects are pairs of topological space (X, A) where $A \subset X$, and morphisms $f : (X, A) \to (Y, B)$ are continuous maps $f : X \to Y$, such that $f(A) \subset B$. For simplicity, we shall denote $X = (X, \emptyset)$. Given any pair (X, A), we have the following inclusion maps

$$\begin{tikzcd}
 & & (X, \emptyset) \arrow[dr, "j"] & & \\
(\emptyset, \emptyset) \arrow[r] & (A, \emptyset) \arrow[ur, "\iota"] \arrow[dr] & & (X, A) \arrow[r] & (X, X) \\
 & & (A, A) \arrow[ur] & &
\end{tikzcd}$$

For $I = [0, 1]$, we have $(X, A) \times I = (X \times I, A \times I)$. Two maps $f_0, f_1 : (X, A) \to (Y, B)$ are called homotopic if there is a map $h : (X \times A) \times I \to (Y, B)$ such that $h|_{(X,A)\times\{0\}} = f_0$ and $h|_{(X,A)\times\{1\}} = g$ holds. In particular, a homtopy of pairs restricts to a homotopy of $f_0|_A =$. Later on we shall put more restrictions on this category, one such possibility is to assume that (X, A) is a CW-pair, i.e, X is a CW complex, and A is a subcomplex.

Definition 2.10: (*Homology Theory*)

A *homology theory*, according to Eilenberg-Steenrod, is a collection, indexed by $\mathbb{Z}$, of

- functors $H_n : \mathrm{TopPair} \to \mathrm{Ab}$ (or, more generally, to category of R-mod), and
- natural transformations $\partial^n_{(X,A)} : H_n(X,A) \to H_{n-1}(A)$,

satisfying the following axioms. Without loss of generaligy, given $f : (X,A) \to (Y,B)$, we shall denote $f_* = H_n(f) : H_n(X,A) \to H_n(Y,B)$, provided the index n is understood from the context. Similarly, we shall denote $\partial = \partial^n_{(X,A)}$.

I. **Exactness Axiom** : Given any pair (X,A), there exists a long exact sequence of Abelian groups

$$\cdots \to H_{n+1}(X,A) \to H_n(A) \underset{\iota_\star}{\to} H_n(X) \underset{j_\star}{\to} H_n(X,A) \underset{\partial}{\to} H_{n-1}(A) \to \cdots$$

Here $\iota : A \hookrightarrow X$ and $j : X \hookrightarrow (X,A)$ are the inclusions.

II. **Homotopy Axiom** : Homotopic maps induce the same map in homology. That is, given two homotopic maps $f, g : (X,A) \to (Y,B)$, we have $f_\star = g_\star : H_n(X,A) \to H_n(Y,B)$.

III. **Excision Axiom** : Given $(X,A) \in \mathrm{TopPair}$ and $U \subset A$ satisfying $\overline{U} \subset \mathring{A}$, the inclusion map $\iota : (X \setminus U, A \setminus U) \hookrightarrow (X,A)$ induces isomoephism (called *excision isomorphism*) in all homology groups.

IV. **Dimension Axiom** : For the one-point space $P = \{\star\}$, we have $H_n(P) = 0$ for $n \neq 0$.

Remark 2.11: (*Extraodinary Homology Theory*)

A homology theory without the dimension axiom is called *extraodinary homology theory*. Topological K-theory is an example of extraodinary homology theory.

Remark 2.12: (*Excision*)

In their original work, Eilenberg-Steenrod assumed a weaker version of the excision axiom, namely they only considered *open* sets $U \subset X$ such that $\overline{U} \subset \mathring{A}$ holds. One of the most common homology theory (i.e, the singular homology theory) satisfies the stronger excision axiom where *any* subset $U \subset X$ with $\overline{U} \subset \mathring{A}$ can be excised out. We shall see that any (ordinary) homology theory defined on reasonable pairs (e.g. CW-paris) is isomorphic to the singular homology, and thus, satisfies the stronger excision axiom.

Exercise 2.13: (*Two Excision Axioms Can Be Equivalent*)

Consider $\mathscr{C}$ to be the category of pairs (X,A) of *compact* topological spaces, and continuous maps between them. Justify that the two excision axioms must be equivalent for this category.

Exercise 2.14:

Let $f : (X,A) \to (Y,B)$ be map. From the axiom, show that we have a commutative diagram

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & H_{n+1}(X,A) & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_{n+1}(X,A) & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & \cdots
\end{array}$$

where the vertical arrows are induced by f. That is, the homology long exact sequence is natural.

Day 3 : 20th January, 2026

`additivity in homology theory – homology of homtopy equivalent spaces – homology long exact sequence of triple`

3.1 Additivity in Homology Theory

Proposition 3.1: (*Finite Additivity*)

Suppose $X = \sqcup_{j=1}^{k} X_j$ is a finite disjoint union of spaces. Then, the inclusion maps $\iota_j : X_j \hookrightarrow X$ induces isomorphism of $H_n(X)$ onto a direct summand of $H_n(X_j)$, and $H_n(X)$ is naturally isomorphic to $\oplus_{j=1}^{n} H_n(X_j)$ for each n.

Proof : We only conider the case $X \sqcup Y$, the general case follows by induction. Note that $\varphi_X : X \hookrightarrow (X \sqcup Y, Y)$ and $\varphi_Y : Y \hookrightarrow (X \sqcup Y, X)$ induces excision isomorphism in homology. Thus, we have the commuting diagrams

$$\begin{tikzcd}
H_n(X) \arrow[r, hook, "(\iota_X)_\star"] \arrow[rr, bend right, "(\varphi_X)_\star"'] & H_n(X \sqcup Y) \arrow[r, two heads, "(j_X)_\star"] & H_n(X \sqcup Y, Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
H_n(Y) \arrow[r, hook, "(\iota_Y)_\star"] \arrow[rr, bend right, "(\varphi_Y)_\star"'] & H_n(X \sqcup Y) \arrow[r, two heads, "(j_Y)_\star"] & H_n(X \sqcup Y, X)
\end{tikzcd}$$

Then, from the long exact sequence for the pair $(X \sqcup Y, X)$, we get the split short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H_n(X) \arrow[r] & H_n(X \sqcup Y) \arrow[r] & H_n(X \sqcup Y, X) \arrow[l, bend right, "(\iota_X)_\star \circ (\varphi_X)_\star^{-1}"'] \arrow[r] & 0.
\end{tikzcd}$$

Thus, $(\iota_X)_\star$ maps $H_n(X)$ isomorphically onto a direct summand of $H_n(X \sqcup Y)$. Similarly, $(\iota_Y)_\star$ maps $H_n(Y)$ isomorphically onto a direct summand of $H_n(X \sqcup Y)$ as well. Finally, by the exicision isomorphism, we have $H_n(X \sqcup Y) \cong H_n(X) \oplus H_n(X \sqcup Y, X) \cong H_n(X) \oplus H_n(Y)$. $\square$

Exercise 3.2: (*Finite Union of Pairs*)

Let (X_j, A_j) be pairs of spaces for $1 \leq i \leq k$. Denote $(X, A) = (\sqcup X_i, \sqcup A_i)$. Show that $\iota_j : (X_j, A_j) \hookrightarrow (X, A)$ induces an isomorphism of $H_n(X_j, A_j)$ onto a direct summand of $H_n(X, A)$, and moreover, $H_n(X, A)$ is naturally isomorphic to the direct sum $\oplus_{j=1}^{k} H_n(X_j, A_j)$.

Hint : Use Proposition 3.1 and the naturality of the long exact sequence (Exercise 2.14).

The finite additivity of homology theory cannot be improved to arbitrary sum, and there are examples of homology theories which does not split for arbitrary union of spaces. On the other hand, we shall see later that singular homology abides by this. Hence, Milnor added one extra axiom.

Definition 3.3: (*Additive Homology Theory*)

A homology theory is called an *additive homology theory* if given a collection of spaces $\{X_\alpha\}_{\alpha \in I}$, the inclusion maps $\iota_\alpha : X_\alpha \hookrightarrow X := \sqcup X_\alpha$ induces an isomorphism of $H_n(X_\alpha)$ onto a direct summand of $H_n(X)$, and moreover, $H_n(X)$ is naturally isomorphic to $\oplus H_n(X_\alpha)$.

3.2 Homology of Homotopy Equivalent Spaces

Recall, $f : X \to Y$ is a homotopy equivalence if there is a map $g : Y \to X$ such that

$$g \circ f \simeq \mathrm{Id}_X, \quad f \circ g \simeq \mathrm{Id}_Y.$$

One can similarly define homotopy equivalence $f : (X, A) \to (Y, B)$ for pairs, which essentially requires that that the homotopy restricts to a homotopy of the restricted map.

Proposition 3.4: (*Homotopy Equivalence induces Homology Isomorphism*)

Let $f : X \to Y$ be a homotopy equivalence. Then, $f_\star : H_n(X) \to H_n(Y)$ is an isomorphism for all n. Similarly, a homotopy equivalence $(X, A) \to (Y, B)$ also induces isomorphism at the homology groups.

Proof : Let $g : Y \to X$ be a homotopy inverse. Now, from the functoriality and the homotopy invariance, we have

$$\mathrm{Id} = H_n(\mathrm{Id}_X) = H_n(g \circ f) = H_n(g) \circ H_n(f) = g_\star \circ f_\star,$$

and similarly, we have $f_\star \circ g_\star = \mathrm{Id}$. Hence, $f_\star : H_n(X) \to H_n(Y)$ is an isomorphism, with inverse $g_\star$. Similar argument holds for $(X, A) \simeq (Y, B)$ as well. $\square$

Corollary 3.5: (*Homeomorphism induces Homology Isomorphism*)

Let $f : X \to Y$ be a homeomorphism of spaces. Then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism. Similarly, homeomorphism $(X, A) \cong (Y, B)$ induces homology isomorphism as well.

3.3 Homology Long Exact Sequence of Triples

Let us consider a triple (X, A, B) of spaces where $B \subset A \subset X$. Then, we have inclusions

$$\iota : (A, B) \hookrightarrow (X, B), \quad j : (X, B) \hookrightarrow (X, A).$$

A map $f : (X, A, B) \to (Y, C, D)$ of triples is a continuous map $f : X \to Y$ such that $f|_A : A \to C$ and $f|_B : B \to D$ holds. A triple leads to a natural long exact sequence of homology groups. We shall need the following.

Lemma 3.6: (*Relative Homology of Space w.r.t. itself*)

For any space X, we have $H_n(X, X) = 0$.

Proof : By the exicision isomorphism, it follows that $H_n(X, X) \cong H_n(\emptyset, \emptyset) = H_n(\emptyset)$. Now, for the pair $(\emptyset, \emptyset)$, we have long exact sequence $\cdots \to H_n(\emptyset) \underset{\iota_\star}{\to} H_n(\emptyset) \underset{j_\star}{\to} H_n(\emptyset) \to \cdots$. Since $\iota = \mathrm{Id} = j$, we have

$i_\star = \mathrm{Id} = j_\star$. But by exactness, we have $j_\star \circ \iota_\star = 0$. Hence, we have $\iota_* = 0 = j_\star$, and consequently, $H_n(\emptyset) = 0$. Thus, $H_n(X, X) = 0$ for any space X. □

Exercise 3.7:

Give a proof of $H_n(X, X) = 0$ without using excision.

Hint : Use the long exact sequence of the pair (X, X).

Theorem 3.8: (*Homology Long Exact Sequence of Triples*)

Let (X, A, B) be a triple with $B \subset A \subset X$. Then, there exists a natural long exact sequence

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A, B) \xrightarrow{\iota_\star} H_n(X, B) \xrightarrow{j_\star} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B) \cdots,$$

which is natural with respect to maps of triples. The boundary map is given as the composition

$$\partial : H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{j_\star} H_{n-1}(A, B),$$

where the first boundary is from the long exact sequence of the pair (X, A), and the second map is induced by the inclusion $A \hookrightarrow (A, B)$.

Proof : The proof boils down to checking exactness at each point, using the respective long exact sequences associated to the pairs (A, B), (X, B), and (X, A). Let us color code the arrows as follows.

$$\cdots \longrightarrow H_{n+1}(A, B) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(A, B) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_{n+1}(X, B) \longrightarrow H_n(X) \longrightarrow H_n(B) \longrightarrow H_n(X, B) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_{n+1}(X, A) \longrightarrow H_n(X) \longrightarrow H_n(A) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

Now, we have the following commutative diagram.

$$\begin{tikzcd}
H_n(A) \arrow[r] \arrow[d] & H_n(X) \arrow[d] \arrow[dr, bend left] & & \\
H_n(A, B) \arrow[r, "\iota_\star"] \arrow[dr, bend right] & H_n(X, B) \arrow[r, "j_\star"] \arrow[d] & H_n(X, A) \arrow[d] \arrow[dr, bend left, "\partial"] & \\
& H_{n-1}(B) \arrow[r] \arrow[dr, bend right] & H_{n-1}(A) \arrow[r] \arrow[d] & H_{n-1}(A, B) \arrow[d] \\
& & H_{n-1}(X) \arrow[r] & H_{n-1}(X, B)
\end{tikzcd}$$

The horizontal arrows are induced by the inclusions $(A, B) \hookrightarrow (X, B) \hookrightarrow (X, A)$, and the commutativity of each squeare follows from the naturality of the long exact sequence of pairs. Let us now check the exactness at each position.

1. $\ker j_\star \supset \operatorname{im} \iota_\star$: We need to show $j_\star \circ \iota_\star = 0$. We have a commuting diagram of spaces

$$\begin{tikzcd}
(A,B) \arrow[r, "\iota"] \arrow[dr, bend right] & (X,B) \arrow[r, "j"] & (X,A) \\
& (A,A) \arrow[ur, bend right] &
\end{tikzcd}$$

Since $H_n(A,A) = 0$ (Lemma 3.6), by functoriality, we have $j_\star \circ \iota_\star = 0$.

2. $\mathbf{ker}\, j_\star \subset \mathbf{im}\, \iota_\star$: We look at the following diragram

$$\begin{tikzcd}
\overset{w}{H_n(A)} \arrow[r, "\iota_\star"] \arrow[d, "j_\star"'] & \overset{z}{H_n(X)} \arrow[d, "j_\star"'] & \\
\overset{y}{H_n(A,B)} \arrow[r, "\iota_\star"] \arrow[dr, bend right, "\partial"'] & \overset{x}{H_n(X,B)} \arrow[r, "j_\star"] \arrow[d, "\partial"'] & H_n(X,A) \arrow[d, "\partial"] \\
& H_{n-1}(B) \arrow[r, "\iota_\star"'] & H_{n-1}(A)
\end{tikzcd}$$

Suppose $j_\star(x) = 0$ for some $x \in H_n(X,B)$. Then,

$$\iota_\star \partial(x) = \partial j_\star x = 0 \Rightarrow \partial(x) \in \ker(\iota_\star) = \mathrm{im}(\partial).$$

So, there exists some $y \in H_n(A,B)$ such that

$$\partial x = \partial y = \partial \iota_\star(y) \Rightarrow \partial(x - \iota_\star(y)) = 0 \Rightarrow x - \iota_\star(y) \in \ker \partial = \mathrm{im}\, j_\star.$$

So, there exists some $z \in H_n(X)$ such that

$$j_\star(z) = x - \iota_\star(y) \Rightarrow j_\star(z) = j_\star(x - \iota_\star(y)) = j_\star(x) - j_\star \iota_\star(x) = 0 \Rightarrow z \in \ker j_\star = \mathrm{im}\, \iota_\star.$$

So, there exists some $w \in H_n(A)$ such that $\iota_\star(w) = z$. Define $y_1 = y + j_\star(w) \in H_n(A,B)$. Then,

$$\iota_\star(y_1) = \iota_\star(y) + j_\star \iota_\star(w) = \iota_\star(y) + j_\star(z) = \iota_\star(y) + x - \iota_\star(y) = x.$$

This proves the claim.

3. $\mathbf{ker}\, \partial \supset \mathbf{im}\, j_\star$: We only need to show $\partial \circ j_\star = 0$. We have the following diagram.

$$\begin{tikzcd}
H_n(X,B) \arrow[r, "j_\star"] \arrow[d] & H_n(X,A) \arrow[d] \arrow[dd, bend left=60, "\partial"] \\
H_{n-1}(B) \arrow[r] \arrow[dr, bend right, "0"'] & H_{n-1}(A) \arrow[d] \\
& H_{n-1}(A,B)
\end{tikzcd}$$

The 0 map is a consequence of the long exact sequence of the pair (A,B). Then, chasing the diagram, it follows that $\partial \circ j_\star = 0$.

4. $\mathbf{ker}\, \partial \subset \mathbf{im}\, j_\star$: We look at the following diagram.

$$\begin{tikzcd}
\overset{w}{H_n(X)} \arrow[d, "j_\star"'] \arrow[dr, bend left, "j_\star"] & & \\
\overset{z}{H_n(X,B)} \arrow[r, "j_\star"] \arrow[d, "\partial"'] & \overset{x}{H_n(X,A)} \arrow[d, "\partial"] \arrow[dr, bend left, "\partial"] & \\
\overset{y}{H_{n-1}(B)} \arrow[r, "\iota_\star"] \arrow[dr, bend right, "\iota_\star"'] & H_{n-1}(A) \arrow[r, "j_\star"] \arrow[d, "\iota_\star"] & H_{n-1}(A,B) \\
& H_{n-1}(X) &
\end{tikzcd}$$

Suppose $\partial(x) = 0$ for some $x \in H_n(X,A)$. Now,

$$j_\star(\partial(x)) = 0 \Rightarrow \partial(x) \in \ker(j_\star) = \operatorname{im}(\iota_\star).$$

So, there exists some $y \in H_{n-1}(B)$ such that $\iota_\star(y) = \partial(x)$. Since $B \hookrightarrow A \hookrightarrow X$, we get

$$\iota_\star(y) = \iota_\star(\iota_\star(y)) = \iota_\star(\partial(x)) = 0 \Rightarrow y \in \ker(\iota_\star) = \operatorname{im}(\partial).$$

So, there exists some $z \in H_n(X,B)$ such that $\partial(z) = y$. Now,

$$\partial(x - j_\star(z)) = \partial(x) - \iota_\star\partial(z) = \partial(x) - \iota_\star(y) = 0 \Rightarrow x - j_\star(z) \in \ker(\partial) = \operatorname{im}(j_\star).$$

So, there exists some $w \in H_n(X)$ such that $j_\star(w) = x - j_\star(z)$. Define, $z_1 = z + j_\star(w) \in H_n(X,B)$. Then, we have

$$j_\star(z_1) = j_\star(z) + j_\star(w) = j_\star(z) + x - j_\star(z) = x.$$

This proves the claim.

5. $\mathbf{ker}\,\boldsymbol{\iota_\star} \supset \mathbf{im}\,\boldsymbol{\partial}$: We only need to show $\iota_\star \circ \partial = 0$. We have the following diagram.

$$\begin{tikzcd}
H_n(X,A) \arrow[d] \arrow[dr, bend left, "0"] \arrow[dd, bend right=60, "\partial"'] & \\
H_{n-1}(A) \arrow[r] \arrow[d] & H_{n-1}(X) \arrow[d] \\
H_{n-1}(A,B) \arrow[r, "\iota_\star"'] & H_{n-1}(X,B)
\end{tikzcd}$$

The 0 map is a consequence of the long exact sequence of the pair (X,A). Then, chasing the diagram, it follows that $\iota_\star \circ \partial = 0$.

6. $\mathbf{ker}\,\boldsymbol{\iota_\star} \subset \mathbf{im}\,\boldsymbol{\partial}$: We look at the following diagram.

$$\begin{tikzcd}
 & \overset{z}{H_{n-1}(B)} \arrow[d, "\iota_\star"'] \arrow[dr, bend left, "\iota_\star"] & \\
\overset{w}{H_n(X,A)} \arrow[r, "\partial"] \arrow[dr, bend right, "\partial"'] & \overset{y}{H_{n-1}(A)} \arrow[r, "\iota_\star"] \arrow[d, "j_\star"'] & H_{n-1}(X) \arrow[d, "j_\star"'] \\
 & \overset{x}{H_{n-1}(A,B)} \arrow[r, "\iota_\star"] \arrow[dr, bend right, "\partial"'] & H_{n-1}(X,B) \arrow[d, "\partial"'] \\
 & & H_{n-2}(B)
\end{tikzcd}$$

Suppose for some $x \in H_{n-1}(A,B)$ we have $\iota_\star(x) = 0$. Then,

$$\partial(x) = \partial(\iota_\star(x)) = 0 \Rightarrow x \in \ker(\partial) = \operatorname{im}(j_\star).$$

So, there exists some $y \in H_{n-1}(A)$ such that $j_\star(y) = x$. Now,

$$j_\star(\iota_\star(y)) = \iota_\star(j_\star(y)) = \iota_\star(x) = 0 \Rightarrow \iota_\star(y) \in \ker(j_\star) = \operatorname{im}(\iota_\star).$$

So, there exists some $z \in H_{n-1}(B)$ such that $\iota_\star(z) = \iota_\star(y)$. As $B \hookrightarrow A \hookrightarrow X$, we have

$$\iota_\star(y - \iota_\star(z)) = \iota_\star(y) - \iota_\star(z) = 0 \Rightarrow y - \iota_\star(z) \in \ker(\iota_\star) = \operatorname{im}(\partial).$$

So, there exists some $w \in H_n(X,A)$ such that $\partial(w) = y - \iota_\star(z)$. We then have,

$$\partial(w) = j_\star(\partial(w)) = j_\star(y - \iota_\star(z)) = j_\star(y) - 0 = x.$$

This proves the claim.

Hence, we have proved that the sequence is exact at all points. Since all the maps involved are natural, one can *easily* show that the sequence is natural with respect to maps of triple as well (Check!). $\square$

Day 4 : 23rd January, 2026

`reduced homology – triads – long exact sequence of proper triads`

4.1 Reduced Homology

Given a space X, we have the unique map $f : X \to \{\star\}$. If X is nonempty, fixing a point $x_0 \in X$, we have a continuous map $r : X \to P := \{x_0\}$. Then, P is a retract of X. Hence, by functoriality, we see $H_n(P)$ is a direct summand of $H_n(X)$ for all n. Let us write $H_n(X) = \tilde{H}_n(X) \oplus H_n(P)$. By the dimension axiom, for all $n \neq 0$, it follows that $\tilde{H}_n(X) = H_n(X)$.

Definition 4.1: (*Reduced Homology*)

Given a space X, its *reduced homology* groups are defined as $\tilde{H}_n(X) := \ker(H_n(f) : H_n(X) \to H_n(\star))$, for $f : X \to \{\star\}$ the unique constant map. If X is nonempty, we have $H_n(X) \cong \tilde{H}_n(X) \oplus H_n(\star)$.

Corollary 4.2: (*Reduced Homology of a Point*)

We have $\tilde{H}_n(X) = H_n(X)$ for all $n \neq 0$, and $\tilde{H}_n(\star) = 0$ for all n.

Lemma 4.3:

Let $f : X \to Y$ be a map. Then, we have an induced map $f_\star : \tilde{H}_0(X) \to \tilde{H}_0(Y)$.

Proof : We have a diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \tilde{H}_0(X) \arrow[r, hook] \arrow[d, dashed] & H_0(X) \arrow[r, two heads] \arrow[d, "H_0(f)"'] & H_0(\star) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \tilde{H}_0(Y) \arrow[r, hook] & H_0(Y) \arrow[r, two heads] & H_0(\star) \arrow[r] & 0
\end{tikzcd}$$

Note that for any $x \in \tilde{H}_0(X)$, we have $H_0(f)(x) \in \ker(H_0(Y) \to H_0(\star)) = \tilde{H}_0(Y)$. Hence, $H_0(f)$ restricts to a map $\tilde{H}_0(X) \to \tilde{H}_0(Y)$. $\square$

Theorem 4.4: (*Long Exact Sequence of Reduced Homology Groups*)

Given a pair (X, A), we have the long exact sequence

$$\cdots \to H_{n+1}(X, A) \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X, A) \to \cdots$$

Proof : We only need to consider the case $n = 0$, i.e, we need to prove the exactness of the part

$$\cdots \to H_1(X, A) \to \tilde{H}_0(A) \to \tilde{H}_0(X) \to H_0(X, A) \to \cdots$$

Let us consider the long exact sequence for the pair $(\star, \star)$. Naturality gives the following commutative diagram of long exact sequences.

$$\begin{tikzcd}
 & & \tilde{H}_0(A) \arrow[r] & \tilde{H}_0(X) \arrow[dr, bend left] & & \\
\cdots \arrow[r] & H_1(X,A) \arrow[r] \arrow[d] \arrow[ur, bend left] & H_0(A) \arrow[r] \arrow[d] & H_0(X) \arrow[r] \arrow[d] & H_0(X,A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \underbrace{H_1(\star,\star)}_{0} \arrow[r] & H_0(\star) \arrow[r] & H_0(\star) \arrow[r] & \underbrace{H_0(\star,\star)}_{0} \arrow[r] & \cdots
\end{tikzcd}$$

Since $\tilde{H}_0(A)$ and $\tilde{H}_0(X)$ is defined as the kernel of the vertical maps, we get the induced blue arrows, and mroreover, the exactness for the reduced homology groups follows from the commutativity of the two rows (Check!). □

Definition 4.5: (*Homologically Trivial Space*)

A space X is called *homologically trivial* if $H_n(X) = 0$ for all $n \neq 0$, and $\tilde{H}_0(X) = 0$ (or in other words, $\tilde{H}_n(X) = 0$ for all n). For $A \neq \emptyset$, the pair (X, A) is called homologically trivial if $H_n(X, A) = 0$ for all n.

Exercise 4.6:

Suppose $A \neq \emptyset$, and (X, A) is homologically trivial. Show that $\iota : A \hookrightarrow X$ induces isomorphism of both unreduced and reduced homology groups. Conversely, if $\iota_\star$ is an isomorphism (for either unreduced or the reduced homology groups), then show that (X, A) is homologically trivial.

4.2 Homology Exact Sequence of Triads

Definition 4.7: (*Proper Triad*)

A *triad* is a triple $(X; X_1, X_2)$, where $X_1, X_2 \subset X$ are subspaces. We say $(X; X_1, X_2)$ is a *proper triad* if the inclusion maps

$$\kappa_1 : (X_2, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_1), \quad \kappa_2 : (X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2)$$

induce isomorphisms of homology groups.

Note that $(X; X_1, X_2)$ and $(X; X_2, X_1)$ are distinct triads, but one of them is proper if and only if the other one is as well. A triad maybe proper with resepect to one homology theory, but fails to be proper with respect to another.

Exercise 4.8:

Suppose $X_1, X_2 \subset X$ are closed subsets, such that $X = X_1 \cup X_2$, and $\overline{X_1 \setminus (X_1 \cap X_2)} \cap \overline{X_2 \setminus (X_1 \cap X_2)} = \emptyset$. Then, the triad $(X; X_1, X_2)$ is a proper triad for any homology theory.

Hint : Check that the inclusion maps κ_1, κ_2 are excision maps.

Exercise 4.9: (*Triad and Triple*)

Suppose (X, A, B) is a triple, i.e, $B \subset A \subset X$. Then, show that $(X; A, B)$ is a proper triad.

Hint : Recall Lemma 3.6.

Lemma 4.10: (*Direct Sum Decomposition*)

Consider the commutative diagram of groups.

$$\begin{tikzcd}
C_1 & & C_2 \\
& B \arrow[ul, "j_1"'] \arrow[ur, "j_2"] & \\
A_2 \arrow[uu, "k_1"] \arrow[ur, "\iota_2"'] & & A_1 \arrow[uu, "k_2"'] \arrow[ul, "\iota_1"]
\end{tikzcd}$$

Suppose, $\ker(j_\alpha) = \operatorname{im}(\iota_\alpha)$ for $\alpha = 1, 2$. Then, the following are equivalent.

1. k_1, k_2 are isomorphisms
2. $0 \to A_i \to B \to C_i \to$ is exact for $i = 1, 2$. Moreover, $B = \operatorname{im}(\iota_1) \oplus \operatorname{im}(\iota_2)$ is a direct sum.

Proof : Suppose k_1, k_2 are isomorphisms. Commutativity implies that j_1, j_2 are epic, and ι_1, ι_2 are monic. Thus, $0 \to A_i \to B \to C_i \to$ is exact for $i = 1, 2$. Note that $\ker(j_1) \cap \ker(j_2) = \ker(j_1) \cap \operatorname{im}(\iota_2) = 0$, since otherwise k_1 will fail to be iso. Let us prove the direct sum decomposition. For any $b \in B$, consider

$$a_2 = (k_1)^{-1} j_1(b) \in A_2, \quad a_1 = (k_2)^{-1} j_2(b) \in A_1.$$

Set $b' = \iota_1(a_1) + \iota_2(a_2)$. Now, $j_1(b') = j_1(b)$ and $j_2(b') = j_2(b)$. Thus, $b - b' \in \ker(j_1) \cap \ker(j_2) = 0$, which means $b = b'$. Thus, every $b \in B$ can be written as a sum from $\operatorname{im}(\iota_1) + \operatorname{im}(\iota_2)$. On the other hand, $\operatorname{im}(\iota_1) \cap \operatorname{im}(\iota_2) = \ker(j_1) \cap \ker(j_2) = 0$. Hence, we have direct sum decomposition. $\square$

Exercise 4.11: (*Finite Additivity*)

Prove Proposition 3.1 for relative homology directly using Lemma 4.10.

Hint : Given (X_i, A_i), consider $(X = X_1 \sqcup X_2, A = A_1 \sqcup A_2)$. We have a diagram of spaces.

$$\begin{tikzcd}
& (X, X_2 \sqcup A) & & (X, X_1 \sqcup A_2) & \\
& & (X, A) \arrow[ul] \arrow[ur] & & \\
(X_1, A_1) \arrow[r] & (X_1 \sqcup A_2, A) \arrow[uu] \arrow[ur] & & (X_2 \sqcup A, A) \arrow[uu] \arrow[ul] & (X_2, A_2) \arrow[l]
\end{tikzcd}$$

Theorem 4.12:

A triad $(X; X_1, X_2)$ is proper if and only if $\iota_\alpha : (X_\alpha, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_1 \cap X_2)$ induces a monomorphism, and gives a direct sum decomposition of $H_\star(X_1 \cup X_2, X_1 \cap X_2)$.

Proof : At the space level, we have the following diagram.

$$\begin{tikzcd}
(X_1 \cup X_2, X_1) & & (X_1 \cup X_2, X_2) \\
& (X_1 \cup X_2, X_1 \cap X_2) \arrow[ul, "j_1"'] \arrow[ur, "j_2"] & \\
(X_2, X_1 \cap X_2) \arrow[uu, "\kappa_1"] \arrow[ur, "\iota_2"'] & & (X_1, X_1 \cap X_2) \arrow[uu, "\kappa_2"'] \arrow[ul, "\iota_1"]
\end{tikzcd}$$

We have the triples $(X_1 \cup X_2, X_1, X_1 \cap X_2)$ and $(X_1 \cup X_2, X_2, X_1 \cap X_2)$. From Theorem 3.8, we we have $\ker (j_\alpha)_\star = \operatorname{im} (\iota_\alpha)_\star$ for $\alpha = 1, 2$. We conclude the proof from Lemma 4.10. □

Definition 4.13: (*Boundary Operator of Triad*)

Given a proper triad $(X; X_1, X_2)$, the *boundary operator* is defined as the composition

$$\partial : H_n(X, X_1 \cup X_2) \xrightarrow{\partial} H_{n-1}(X_1 \cup X_2) \longrightarrow H_{n-1}(X_1 \cup X_2, X_2) \xrightarrow{(\kappa_2)_\star^{-1}} H_n(X_1, X_1 \cap X_2).$$

It should be noted that this boundary map is the boundary map of the triple $(X, X_1 \cup X_2, X_1)$, followed by (the inverse of) an excision isomorphism.

Theorem 4.14: (*Homology Long Exact Sequence of Proper Triad*)

Given a proper triad $(X; X_1, X_2)$ we have a long exact sequence of homology groups

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X, X_1 \cup X_2) \arrow[d, "\partial"'] & & \\
& H_n(X_1, X_1 \cap X_2) \arrow[r] & H_n(X, X_2) \arrow[r] & H_n(X, X_1 \cup X_2) \arrow[d, "\partial"'] & \\
& & & H_{n-1}(X_1, X_1 \cap X_2) \arrow[r] & \cdots
\end{tikzcd}$$

Moreove, the sequence is natural with respect to map of triads.

Proof : We have the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X, X_2) \arrow[r] & H_{n+1}(X, X_1 \cup X_2) \arrow[r, "\partial"] \arrow[dr, bend right, "\partial"'] & H_n(X_1 \cup X_2, X_2) \arrow[r] & H_n(X, X_2) \arrow[r] & \cdots \\
& & & H_n(X_1, X_1 \cap X_2) \arrow[u, "\cong"', "(\kappa_2)_\star"] \arrow[ur, bend right] & &
\end{tikzcd}$$

The top row is exact, being the homology long exact sequence of the triple $(X, X_1 \cup X_2, X_2)$. Hence, the blue sequence is also exact by commutativity of the triangles. Naturality can also be proved *easily*, as the top row is natural and so is the boundary map (Check!). $\square$

Day 5 : 27th January, 2026

`Mayer-Vietoris sequence — suspension isomorphism — homology of spheres — relative Mayer-Vietoris sequence — singular chains — singular homology`

5.1 Mayer-Vietoris Sequence of a Proper Triad

Before describing the sequence, let us observe the following hexagonal lemma.

Lemma 5.1: (*Hexagonal Lemma*)

Suppose, we have a diagram of groups, where each triangle commutes.

$$\begin{tikzcd}
 & A_0 \arrow[dl, "l_1"'] \arrow[dr, "l_2"] \arrow[dd, "\iota_0"] & \\
C_1 & & C_2 \\
 & B \arrow[ul, "j_1"] \arrow[ur, "j_2"'] \arrow[dd, "j_0"] & \\
A_2 \arrow[uu, "k_1"] \arrow[ur, "\iota_2"] \arrow[dr, "h_1"'] & & A_1 \arrow[uu, "k_2"'] \arrow[ul, "\iota_1"'] \arrow[dl, "h_2"] \\
 & C_0 &
\end{tikzcd}$$

Assume that $\ker(\iota_\alpha) = \operatorname{im}(j_\alpha)$ for $\alpha = 1, 2$, $j_0 \circ \iota_0 = 0$, and k_1, k_2 are isomorphisms. Then, the left and right sides of the hexagon differs by a side, i.e, $h_1 \circ k_1^{-1} \circ l_2 = -h_2 \circ k_2^{-1} \circ l_2$.

Proof : We can apply Lemma 4.10 to the blue part of the diagram. In particular, for any $a \in A_0$, we can uniquely write

$$\iota_0(a) = \iota_1(k_2^{-1} j_2 \iota_0(a)) + \iota_2(k_1^{-1} j_1 \iota_0(a)).$$

Applying j_0, and using $j_0 \circ \iota_0 = 0$, we have

$$0 = j_0 \iota_0(a) = h_2 k_2^{-1} j_2 \iota_0(a) + h_1 k_1^{-1} j_1 \iota_0(a) = h_2 k_2^{-1} l_2(a) + h_1 k_1^{-1} l_1(a).$$

As $a \in A_0$ is arbitrary, we have the claim. □

Let us consider a proper triad $(X; X_1, X_2)$ with $X = X_1 \cup X_2$, and set $A = X_1 \cap X_2$. At the space level, we have the following commuting diagram.

$$\begin{tikzcd}
 & X_1 \arrow[dr, "m_1"] \arrow[rrrr, bend left, "n_1"] & & (X, X_2) & & (X_1, A) \arrow[ll, "k_2"'] \arrow[dl, "\iota_1"] \\
A \arrow[ur, "h_1"] \arrow[rr, "\iota"] \arrow[dr, "h_2"'] & & X \arrow[ur, "l_2"] \arrow[rr, "j"] \arrow[dr, "l_1"'] & & (X, A) \arrow[ul, "j_2"'] \arrow[dl, "j_1"] & \\
 & X_2 \arrow[ur, "m_2"'] \arrow[rrrr, bend right, "n_2"'] & & (X, X_1) & & (X_2, A) \arrow[ll, "k_1"] \arrow[ul, "\iota_2"']
\end{tikzcd}$$

Passing to homology, we have the following commuting diagram.

$$\begin{tikzcd}
 & H_n(X_1) \arrow[dr, "m_{1\star}"] \arrow[rrrr, bend left, green, "n_{1\star}"] & & H_n(X, X_2) & & H_n(X_1, A) \arrow[ll, "k_{2\star}", "\cong"'] \arrow[dl, hook, "\iota_{1\star}"] \arrow[dr, green, "\partial_2"] & \\
H_n(A) \arrow[ur, green, "h_{1\star}"] \arrow[rr, blue, "\iota_\star"] \arrow[dr, red, "h_{2\star}"'] & & H_n(X) \arrow[ur, "l_{2\star}"] \arrow[rr, blue, "j_\star"] \arrow[dr, "l_{1\star}"'] & & H_n(X, A) \arrow[ul, "j_{2\star}"'] \arrow[dl, "j_{1\star}"] \arrow[rr, blue, "\partial"] & & H_{n-1}(A) \\
 & H_n(X_2) \arrow[ur, "m_{2\star}"'] \arrow[rrrr, bend right, red, "n_{2\star}"'] & & H_n(X, X_1) & & H_n(X_2, A) \arrow[ll, "k_{1\star}"', "\cong"] \arrow[ul, hook, "\iota_{2\star}"'] \arrow[ur, red, "\partial_1"'] &
\end{tikzcd}$$

As the triad is proper, we have $k_{1\star}, k_{2\star}$ are isomorphisms, which gives the diagonal short exact sequences by Lemma 4.10. The commutativity involving the boundary maps follows from the naturality of the homology long exact sequence. The colored arrows are part of long exact sequence of (X_1, A), (X_2, A), and (X, A). We define a sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{\psi} H_n(X_1) \oplus H_n(X_2) \xrightarrow{\varphi} H_n(X) \xrightarrow{\Delta} H_{n-1}(A) \longrightarrow \cdots$$

where the maps are as follows:

$$\begin{aligned}
\psi(u) &= (h_{1\star}(u), -h_{2\star}(u)), && u \in H_n(A) \\
\varphi(v_1, v_2) &= m_{1\star}(v_1) + m_{2\star}(v_2), && v_1 \in H_n(X_1), v_2 \in H_n(X_2) \\
\Delta(w) &= -\partial_1 k_{1\star}^{-1} l_{1\star}(w) = \partial_2 k_{2\star}^{-1} l_{2\star}(w), && w \in H_n(X).
\end{aligned}$$

The definition of Δ is a consequence of Lemma 5.1.

Theorem 5.2: (*Mayer-Vietoris Sequence for Proper Triad*)

Given a proper triad $(X; X_1, X_2)$ with $X = X_1 \cup X_2$ and $A = X_1 \cap X_2$, we have a long exact sequence,

$$\cdots \longrightarrow H_n(A) \xrightarrow{\psi} H_n(X_1) \oplus H_n(X_2) \xrightarrow{\varphi} H_n(X) \xrightarrow{\Delta} H_{n-1}(A) \longrightarrow \cdots$$

known as the *Mayer-Vietoris sequence*, which is natural with respect to morphisms of triads.

Proof : As usual, the proof requires explicitly checking the exactness at each point.

1. $\mathbf{ker}\,\varphi \supset \mathbf{im}\,\psi$ **:** For any $u \in H_n(A)$, we have
$$\varphi\psi(u) = \varphi(h_{1\star}(u), -h_{2\star}(u)) = m_{1\star}h_{1\star}(u) - m_{2\star}h_{2\star}(u) = \iota_\star(u) - \iota_\star(u) = 0.$$
Thus, $\varphi \circ \psi = 0 \Rightarrow \ker\varphi \supset \operatorname{im}\psi$.
2. $\mathbf{ker}\,\varphi \subset \mathbf{im}\,\psi$ **:** Suppose $\varphi(v) = 0$ for some $v = (v_1, v_2) \in H_n(X_1) \oplus H_n(X_2)$. Then,
$$0 = j_\star\varphi(v) = j_\star m_{1\star}(v_1) + j_\star m_{2\star}(v_2) = \iota_{1\star}n_{1\star}(v_1) + \iota_{2\star}n_{2\star}(v_2)$$
Now, by Lemma 4.10, we have $\iota_{1\star}, \iota_{2\star}$ are monic, and $\operatorname{im}(\iota_{1\star}) \cap \operatorname{im}(\iota_{2\star}) = 0$. Hence, $n_{1\star}(v_1) = 0 = n_{2\star}(v_2)$. By exactness of the long exact sequence of (X_1, A) and (X_2, A) respectively, there exists $u_1, u_2 \in H_n(A)$ such that $v_1 = h_{1\star}(u_1), v_2 = h_{2\star}(u_2)$. In other words,
$$0 = \varphi(v) = m_{1\star}h_{1\star}(u_1) + m_{2\star}h_{2\star}(u_2) = \iota_\star(u_1) + \iota_\star(u_2) = \iota_\star(u_1 + u_2).$$
Now, $u_1 + u_2 \in \ker(\iota_\star) = \operatorname{im}\partial$. Thus, there is some $w \in H_{n+1}(X, A)$ such that $\partial(w) = u_1 + u_2$. Again by Lemma 4.10, there are (unique) $w_1 \in H_{n+1}(X_1, A), w_2 \in H_{n+1}(X_2, A)$ such that $w = \iota_{1\star}(w_1) + \iota_{2\star}(w_2)$. Then,
$$u_1 + u_2 = \partial(w) = \partial\iota_{1\star}(w_1) + \partial\iota_{2\star}(w_2) = \partial_2(w_1) + \partial_1(w_2).$$
Set, $u = u_1 - \partial_2(w_1) = -(u_2 - \partial_1(w_2))$. We then have,
$$h_{1\star}(u) = h_{1\star}(u_1) - \cancel{h_{1\star}\partial_2(w_1)} = v_1, h_{2\star}(u) = -(h_{2\star}(u_2) - \cancel{h_{2\star}\partial_1(w_2)}) = -v_2.$$
Hence, $\psi(u) = (h_{1\star}(u), -h_{2\star}(u)) = (v_1, v_2) = v$. This proves the claim.
3. $\mathbf{ker}\,\mathbf{\Delta} \supset \mathbf{im}\,\varphi$ **:** For $v = (v_1, v_2) \in H_n(X_1) \oplus H_2(X_2)$, we have
$$\Delta\varphi(v) = \Delta(m_{1\star}(v_1) + m_{2\star}(v_2)) = -\partial_1 k_{1\star}^{-1} l_{1\star} m_{1\star}(v_1) + \partial_2 k_{2\star}^{-1} l_{2\star} m_{2\star}(v_2) = 0.$$
Thus, $\Delta \circ \varphi = 0 \Rightarrow \ker\Delta \supset \operatorname{im}\varphi$.
4. $\mathbf{ker}\,\mathbf{\Delta} \subset \mathbf{im}\,\varphi$ **:** Suppose for some $w \in H_n(X)$, we have $\Delta(w) = 0$. Thus, $\partial_1 k_{1\star}^{-1} l_{1\star}(w) = 0 = \partial_2 k_{2\star}^{-1} l_{2\star}(w)$. By exactness, for $\alpha = 1, 2$, there is $v_\alpha \in H_n(X_\alpha)$ such that $n_{\alpha\star}(v_\alpha) = k_{\alpha\star}^{-1} l_{\alpha\star}(w)$. By Lemma 4.10, we can (uniquely) write
$$\begin{aligned} j_\star(w) &= i_{1\star}k_{2\star}^{-1}j_{2\star}j_\star(w) + i_{2\star}k_{1\star}^{-1}j_{1\star}j_\star(w) \\ &= i_{1\star}k_{2\star}^{-1}l_{2\star}(w) + \iota_{2\star}k_{1\star}^{-1}l_{1\star}(w) \\ &= \iota_{1\star}n_{1\star}(v_1) + \iota_{2\star}n_{2\star}(v_2) \\ &= j_\star m_{1\star}(v_1) + j_\star m_{2\star}(v_2). \end{aligned}$$
Thus, $w - m_{1\star}(v_1) - m_{2\star}(v_2) \in \ker(j_\star) = \operatorname{im}(\iota_\star)$. Hence, there exists some $u \in H_n(A)$ such that $\iota_\star(u) = w - m_{1\star}(v_1) - m_{2\star}(v_2)$. Set, $v_1' = v_1 + h_{1\star}(u), v_2' = v_2$. Then
$$\varphi(v_1', v_2') = m_{1\star}(v_1 + h_{1\star}(u)) + m_{2\star}(v_2)$$

$$= m_{1\star}(v_1) + (w - m_{1\star}(v_1) - m_{2\star}(v_2)) + m_{2\star}(v_2) = w.$$

This proves the claim.

5. $\mathbf{ker}\,\psi \supset \mathbf{im}\,\boldsymbol{\Delta}$ **:** For $w \in H_n(X)$, we have

$$\psi\Delta(w) = (h_{1\star}\partial_2 k_{2\star}^{-1})l_{2\star}(w), h_{2\star}\partial_1 k_{1\star}^{-1} l_{1\star}(w) = (0,0) = 0.$$

Thus, $\psi \circ \Delta = 0 \Rightarrow \ker\psi \supset \operatorname{im}\Delta$.

6. $\mathbf{ker}\,\psi \subset \mathbf{im}\,\boldsymbol{\Delta}$ **:** Suppose for some $u \in H_n(A)$ we have $\psi(u) = 0$. Thus, $h_{1\star}(u) = 0 = h_{2\star}(u)$. By exactness, for $\alpha = 1, 2$ there are $x_\alpha \in H_{n+1}(X_\alpha, A)$ such that $\partial_1(x_2) = u$ and $\partial_2(x_1) = -u$. Then,

$$\partial\iota_{1\star}(x_1) + \partial\iota_{2\star}(x_2) = \partial_2(x_1) + \partial_1(x_2) = -u + u = 0 \Rightarrow \iota_{1\star}(x_1) + \iota_{2\star}(x_2) \in \ker(\partial) = \operatorname{im}(j_\star).$$

Thus, there exists $w \in H_{n+1}(X)$ such that $j_\star(w) = -\iota_{1\star}(x_1) - \iota_{2\star}(x_2)$. Then,

$$\begin{aligned}\Delta(w) &= -\partial_1 k_{1\star}^{-1} l_{1\star}(w) = -\partial_1 k_{1\star}^{-1} j_{1\star} j_\star(w) \\ &= \partial_1 k_{1\star}^{-1} j_{1\star}(\iota_{1\star}(x_1) + \iota_{2\star}(x_2)) \\ &= 0 + \partial_1 k_{1\star}^{-1} j_{1\star}\iota_{2\star}(x_2) \\ &= \partial_1 k_{1\star}^{-1} k_{1\star}(x_2) = \partial_1(x_2) = u.\end{aligned}$$

This proves the claim.

Thus, the Mayer-Vietoris sequence is exact. One can *easily* prove the naturality of the sequence with respect to maps of triads (Check!). □

5.2 The Suspension Isomorphism

Given a space, recall the *(unreduced) cone* on X is

$$CX = \frac{X \times [0,1]}{X \times \{0\}},$$

and the *(unreduced) suspension* of X is

$$\Sigma X = \frac{CX}{X \times \{1\}}.$$

It is well known that $CX \simeq \star$, i.e, the cone is always contractible. Consequently we have the following.

Proposition 5.3: (*Cone on a Space Is Homologically Trivial*)

Given a space X, we have CX is homologically trivial, i.e, $\tilde{H}_n(CX) = 0$ for all n.

Let us now prove a fundamental property of (ordinary) homology theory.

Theorem 5.4: (*Suspension Isomorphism*)

Given a space X, there exists a natural isomorphism $\tilde{H}_n(\Sigma X) \cong \tilde{H}_{n-1}(X)$.

Proof : Consider $A := q\left(X \times \left[0, \frac{3}{4}\right]\right), B := q\left(X \times \left[\frac{1}{4}, 1\right]\right) \subset \Sigma X$, where $q : X \times [0,1] \to \Sigma X$ is the quotient map. It is easy to see that $(\Sigma X; A, B)$ is a proper triad. Now,

$$\Sigma X = A \cup B, A \cong CX \simeq \star, B \cong CX \simeq \star, A \cap B \cong X \times \left[\frac{1}{4}, \frac{3}{4}\right] \simeq X.$$

From the reduced version of the Mayer-Vietoris sequence, we immediately see that $\Delta : \tilde{H}_n(\Sigma X) \to \tilde{H}_{n-1}(A \cap B)$ is an isomorphism. Since $A \cap B$ deformation ratracts onto X, we have the claim. $\square$

Corollary 5.5: (*Reduced Homology of Sphere*)

The reduced homology of the n-sphere is $\tilde{H}_k(S^n) = \begin{cases} 0, & k \neq n \\ H_0(\star), & k = n \end{cases}$.

Proof : We apply Theorem 5.4. Since $S^n = \Sigma S^{n-1}$, by induction, we only need compute the homology of S^0, which is just two points. By Proposition 3.1, we have $H_n(S^0) = H_n(\star) \oplus H_n(\star)$. Then, by the dimension axiom, it follows that $\tilde{H}_k(S^0) = \begin{cases} 0, & k \neq 0 \\ H_0(\star), & k = 0 \end{cases}$. $\square$

5.3 Relative Mayer-Vietoris Sequence

Let us now consider an arbitrary proper triad $(X; X_1, X_2)$. We have the following diagram.

$$\begin{tikzcd}
 & & H_n(X, X_1 \cap X_2) \arrow[dl, "h_{2\star}"'] \arrow[dr, "h_{1\star}"] \arrow[dd, "j_\star"] & & \\
 & H_n(X, X_2) \arrow[dr, "m_{2\star}"'] \arrow[dddd, bend right=60, "\delta_2"'] & & H_n(X, X_1) \arrow[dl, "m_{1\star}"] \arrow[dddd, bend left=60, "\delta_1"] & \\
 & & H_n(X, X_1 \cup X_2) \arrow[dl, "\partial_1"'] \arrow[dr, "\partial_2"] \arrow[dd, "\partial"] & & \\
 & H_{n-1}(X_1 \cup X_2, X_1) & & H_{n-1}(X_1 \cup X_2, X_2) & \\
 & & H_{n-1}(X_1 \cup X_2, X_1 \cap X_2) \arrow[ul, "d_{1\star}"'] \arrow[ur, "d_{2\star}"] \arrow[dd, "\iota_\star"] & & \\
 & H_{n-1}(X_2, X_1 \cap X_2) \arrow[uu, "\kappa_{1\star}", "\cong"'] \arrow[ur, "\iota_{2\star}"] \arrow[dr, "n_{1\star}"'] & & H_{n-1}(X_1, X_1 \cap X_2) \arrow[uu, "\cong", "\kappa_{2\star}"'] \arrow[ul, "\iota_{1\star}"'] \arrow[dl, "n_{2\star}"] & \\
 & & H_{n-1}(X, X_1 \cap X_2) & &
\end{tikzcd}$$

The colored arrows are part of the corresponding long exact sequences of different triples. The isomorphims $\kappa_{1\star}, \kappa_{2\star}$ follows from the proper triad $(X_1 \cup X_2; X_1, X_2)$. In particular, we can apply Lemma 5.1. Let us define a sequence

$$\cdots \longrightarrow H_n(X, X_1 \cap X_2) \xrightarrow{\psi} H_n(X, X_1) \oplus H_n(X, X_2) \xrightarrow{\varphi} H_n(X, X_1 \cup X_2) \xrightarrow{\Delta} H_{n-1}(X, X_1 \cap X_2) \longrightarrow \cdots$$

where the maps are as follows:

$$\begin{aligned} \psi(u) &= (h_{1\star}(u), -h_{2\star}(u)), && u \in H_n(X, X_1 \cap X_2) \\ \varphi(v_1, v_2) &= m_{1\star}(v_1) + m_{2\star}(v_2), && v_1 \in H_n(X, X_1), v_2 \in H_n(X, X_2) \\ \Delta(w) &= -n_{1\star} k_{1\star}^{-1} \partial_1(w) = n_{2\star} k_{2\star}^{-1} \partial_2(w), && w \in H_n(X, X_1 \cup X_2). \end{aligned}$$

Theorem 5.6: (*Relative Mayer-Vietoris Sequence*)

Given a proper triad $(X; X_1, X_2)$, there exists a long exact sequence

$$\cdots \longrightarrow H_n(X, X_1 \cap X_2) \xrightarrow{\psi} H_n(X, X_1) \oplus H_n(X, X_2) \xrightarrow{\varphi} H_n(X, X_1 \cup X_2) \xrightarrow{\Delta} H_{n-1}(X, X_1 \cap X_2) \longrightarrow \cdots$$

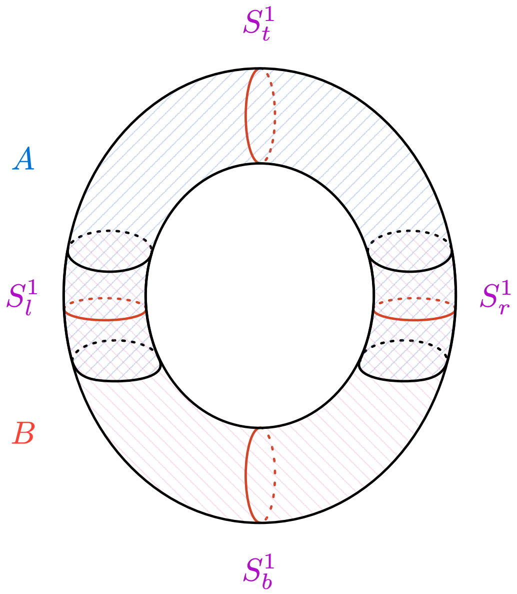
known as the *relative Mayer-Vietoris sequence*, which is moreover natural with respect to morphism of triads.

Day 6 : 6th February, 2026

`homology of` T^2 `– singular homology – singular homology of point – categorical digression : kernel, cokernel, abelian category – snake lemma`

6.1 Homology of Torus $T^2 = S^1 \times S^1$ (with $\mathbb{Z}$ coefficients)

Suppose $H_\star$ is a homology theory with $H_0(\star) = \mathbb{Z}$. Let us now compute the homology groups of the torus $T^2 = S^1 \times S^1$ using the Mayer-Vietoris sequence (Theorem 5.2). Consider the following decomposition $T^2 = A \cup B$.



Observe that $A \cong B \simeq S^1$, and $A \cap B \simeq S^1 \sqcup S^1$. Then, by Corollary 4.2, we have $H_1(S^1) = \mathbb{Z} = H_0(S^1)$, and all other homology of S^1 is zero. By Proposition 3.1, we have $H_\star(A \cap B) = H_\star(S^1) \oplus H_\star(S^1) = \mathbb{Z} \oplus \mathbb{Z}$ for $\star = 0, 1$, and 0 otherwise. It is easy to obeserve that $(A, A \cap B) \hookrightarrow (T^2, B)$ and $(B, A \cap B) \hookrightarrow (T^2, A)$ satisfy the hypothesis for excision, and thus, $(T^2;, A, B)$ is a proper triad.

Let us consider the diagrams

$$\begin{tikzcd}
S^1_l \arrow[r, hook] \arrow[d, equal] & A \cap B \arrow[d, hook] & S^1_r \arrow[l, hook'] \arrow[d, equal] \\
S^1_t \arrow[r, hook] & A & S^1_t \arrow[l, hook']
\end{tikzcd}
\qquad
\begin{tikzcd}
S^1_l \arrow[r, hook] \arrow[d, equal] & A \cap B \arrow[d, hook] & S^1_r \arrow[l, hook'] \arrow[d, equal] \\
S^1_b \arrow[r, hook] & B & S^1_b \arrow[l, hook']
\end{tikzcd}$$

which commute up to homtopy. Consequently, we can identify the map $\psi : H_\star(A \cap B) \to H_\star(A) \oplus H_\star(B)$ from the Mayer-Vietoris sequence as

$$\psi(x, y) = (x + y, -x - y).$$

Note that ψ is an injective map. Let us consider the exact sequence

$$\cdots \to \underbrace{H_2(A) \oplus H_2(B)}_{0} \to H_2(T^2) \to \underbrace{H_1(A \cap B)}_{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\psi} \underbrace{H_1(A) \oplus H_1(B)}_{\mathbb{Z}\oplus\mathbb{Z}} \to \cdots.$$

As $\psi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is injective, we have

$$H_2(T^2) = \ker(\psi) = \{(x, y) \mid x = -y\} \cong \mathbb{Z}.$$

Next, we compute $H_1(T^2)$. Note that $\varphi : H_\star(A) \oplus H_\star(B) \to H_\star(T^2)$ is given by

$$\varphi(x,y) = x - y.$$

Now, using the *reduced* Mayer-Vietories sequence, we have

$$\cdots \to \underbrace{\tilde{H}_1(A) \oplus \tilde{H}_1(B)}_{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\varphi} \tilde{H}_1(T^2) \to \underbrace{\tilde{H}_0(A \cap B)}_{\mathbb{Z}} \xrightarrow{\psi} \underbrace{\tilde{H}_0(A) \oplus \tilde{H}_0(B)}_{0} \to \cdots$$

We have $\operatorname{im} \varphi = \mathbb{Z}$, and thus, injectivity of ψ gives the short exact sequence

$$0 \to \underbrace{\operatorname{im}(\varphi)}_{\mathbb{Z}} \to H_1(T^2) \to \mathbb{Z} \to 0.$$

This implies $H_1(T^2) = \mathbb{Z} \oplus$. Finally, again the reduced Mayer-Vietoris sequence gives the exact sequence,

$$\cdots \to \underbrace{\tilde{H}_0(A) \oplus \tilde{H}_0(B)}_{0} \to \tilde{H}_0(T^2) \to \underbrace{\tilde{H}_{-1}(A \cap B)}_{0} \to \cdots$$

Hence, $H_0(T^2) = \tilde{H}_0(T^2) \oplus H_0(\star) = 0 \oplus \mathbb{Z} = \mathbb{Z}$. It is easy to see, $H_k(T^2) = 0$ for $k \geq 3$ from the Mayer-Vietoris sequence. Hence, we have computed

$$H_k(T^2) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z} \oplus \mathbb{Z}, & k = 1 \\ \mathbb{Z}, & k = 2 \\ 0, & k > 2. \end{cases}$$

6.2 Singular Homology Theory

As a first step in defining the singular homology, let us begin with n-simplex.

Definition 6.1: (n*-simplex*)

The standard (or geometric) n*-simplex* is defined as

$$\Delta^n := \left\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum t_i = 1\right\}$$

Denoting $e_i := (0, \ldots, 1, \ldots, 0)$ to be the i^{th} standard unit vector in $\mathbb{R}^{n+1}$, we have Δ^n is the convex hull of $\{e_0, \ldots, e_n\}$

In the language of category theory, the collection $\{\Delta_n\}$ is a *cosimplicial space*. Explicitly, recall from Example 1.3, the simplicial category $\mathbf{\Delta}$. We have a covariant functor $\Delta : \mathbf{\Delta} \to \mathbf{Top}$ given by $\Delta(n) = \Delta^n$. To describe the morphisms, consider a (weakly) order preserving map $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$. Then, we have

$$\begin{aligned} \Delta(\alpha) : \Delta^m &\longrightarrow \Delta^n \\ \sum_{i=0}^m t_i e_i &\mapsto \sum_{i=0}^m t_i e_{\alpha(i)}. \end{aligned}$$

In $\mathbf{\Delta}$, there are some special maps. For $0 \leq i \leq n$, we have the *face maps* $\delta_i^n : [n-1] \to [n]$ given by

$$
\begin{tikzcd}
{[n-1]} \arrow[d, "\delta_i^n"'] & : & 0 \arrow[d] & 1 \arrow[d] & \cdots & i-1 \arrow[d] & i \arrow[dr] & i+1 \arrow[dr] & & \cdots & n-1 \arrow[dr] & \\
{[n]} & & 0 & 1 & & i-1 & i & i+1 & i+2 & & & n
\end{tikzcd}
$$

That is, δ_i^n is the map that *misses* i. On the other hand, for $0 \leq i \leq n$, we have the *degeneracy maps* $\sigma_i^n : [n+1] \to [n]$ given by

$$
\begin{tikzcd}
{[n+1]} \arrow[d, "\sigma_i^n"'] & : & 0 \arrow[d] & 1 \arrow[d] & \cdots & i \arrow[d] & i+1 \arrow[dl] & i+2 \arrow[dl] & \cdots & n+1 \arrow[dl] \\
{[n]} & & 0 & 1 & & i & i+1 & & n &
\end{tikzcd}
$$

That is σ_i^n is the map that repeats i twice. It is a well-known fact that any morphism of $\mathbf{\Delta}$ can be written as finite composition of these δ_i^n and σ_j^m. These maps satisfy some identities, known as the *simplicial identities*:

$$
\begin{aligned}
\delta_j^{n+1} \circ \delta_i^n &= \delta_i^{n+1} \circ \delta_{j-1}^n, && i < j, \\
\sigma_j^{n+1} \circ \sigma_i^n &= \sigma_i^{n+1} \circ \sigma_{j+1}^n, && i \leq j, \\
\sigma_j^{n-1} \circ \delta_i^n &= \begin{cases} \delta_i^{n-1} \circ \sigma_{j-1}^n, & i < j, \\ \mathrm{Id}_{[n-1]}, & i = j, j+1, \\ \delta_{i-1}^{n-1} \circ \sigma_j^n, & i > j+1. \end{cases}
\end{aligned}
$$

Denote the face and degeneracy maps

$$
\begin{aligned}
d_i^n := \Delta(\delta_i^n) : \Delta^{n-1} &\longrightarrow \Delta^n, & s_i^n := \Delta(\sigma_i^n) : \Delta^{n+1} &\longrightarrow \Delta^n \\
\sum t_j e_j &\longmapsto \sum t_j e_{\delta_i^n(j)}, & \sum t_j e_j &\longmapsto \sum t_j e_{\sigma_i^n(j)}.
\end{aligned}
$$

Definition 6.2: (*Singular Chain Complex*)

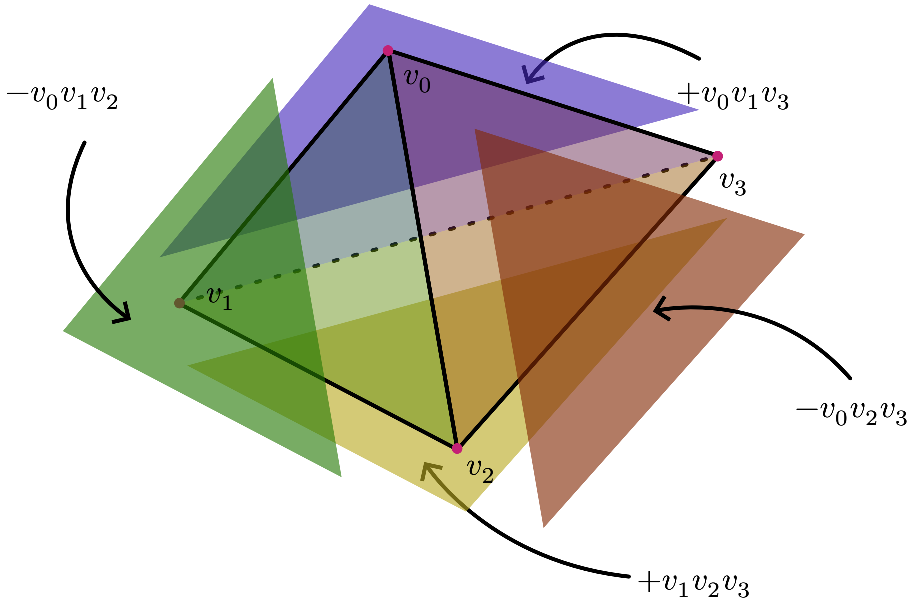
Given a space X, a *singular n-simplex* is a continuous map $\sigma : \Delta^n \to X$. The *singular chian complex* $S_\bullet(X)$ of X consists of the following data.

- The abelian group $S_n(X)$ freely generated by all singular n-simplicies. For $n < 0$, we have $S_n(X) = 0$.
- The boundary maps $\partial_n : S_n(X) \to S_{n-1}(X)$ defined on a generator $\sigma : \Delta^n \to X$ via

$$
\partial_n(\sigma) := \sum_{i=1}^{n} \sigma \circ d_i^n,
$$

  and then extended $\mathbb{Z}$-linearly.

Any element of $S_n(X)$ is called a *singular n-chain*, an element of $\ker(\partial_n) \subset S_n(X)$ is called a *singular n-cycle*, an element of $\operatorname{im}(\partial_{n+1}) \subset S_n(X)$ is called a *singular n-boundary*.



Visual representaion of the boundary map : $\partial_2(\Delta^3)$ is the signed sum of the faces

Exercise 6.3: (*Singular Chain Complex is a Functor*)

Check that $S_\bullet : \mathrm{Top} \to \mathrm{Ch}$ is a functor, where Ch is the category of chain complexes and chain maps.

Hint : Given a map $f : X \to Y$, define $S_n(f)(\sigma) = f \circ \sigma$ for a singular n-simplex $\sigma : \Delta^n \to X$, and extend linearly. Check that $S_\bullet(f) : S_\bullet(X) \to S_\bullet(Y)$ is a chain map. Then, check that $S_\bullet(g \circ f) = S_\bullet(g) \circ S_\bullet(f)$ and $S_\bullet(\mathrm{Id}_X) = \mathrm{Id}_{S_\bullet(X)}$.

Proposition 6.4: (*Boundary in Singular Chain Complex*)

$\partial_n \circ \partial_{n+1} = 0$, and thus, $(S_\bullet(X), \partial)$ is indeed a chain complex.

Proof : We need to check on generators only. Consider $\sigma : \Delta^{n+1} \to X$. Then we compute

$$\begin{aligned}
\partial_n\partial_{n+1}(\sigma) = \partial_n\left(\sum_{j=0}^{n+1}(-1)^j\sigma\circ d_j^{n+1}\right) &= \sum_{j=0}^{n+1}(-1)^j\partial_n\left(\sigma\circ d_j^{n+1}\right) = \sum_{j=0}^{n+1}\sum_{i=0}^{n}(-1)^{i+j}\left(\sigma\circ d_j^{n+1}\circ d_i^n\right)\\
&= \sum_{i<j}(-1)^{i+j}\left(\sigma d_j^{n+1}d_i^n\right) + \sum_{j\le i}(-1)^{i+j}\left(\sigma d_j^{n+1}d_i^n\right)\\
&= \sum_{i<j}(-1)^{i+j}\left(\sigma d_i^{n+1}d_{j-1}^n\right) + \sum_{j\le i}(-1)^{i+j}\left(\sigma d_j^{n+1}d_i^n\right)\\
&= \sum_{i\le j'}(-1)^{i+j'+1}\left(\sigma d_i^{n+1}d_{j'}^n\right) + \sum_{j\le i}(-1)^{i+j}\left(\sigma d_j^{n+1}d_i^n\right)\\
&= -\sum_{i\le j}(-1)^{i+j}\left(\sigma d_i^{n+1}d_j^n\right) + \sum_{i\le j}(-1)^{i+j}\left(\sigma d_i^{n+1}d_j^n\right)\\
&= 0.
\end{aligned}$$

Hence, $\partial_n \circ \partial_{n+1} = 0$. Consequently, $(S_\bullet(X), \partial)$ is a chain complex. $\square$

Definition 6.5: (*Homology of a Chain Complex*)

Given any chain complex $C_\bullet = (C_n, \partial)$, the *$n^{\text{th}}$-homology group* is defined as the quotient

$$H_n(C_\bullet) = \frac{\ker(\partial_n : C_n \to C_{n-1})}{\operatorname{im}(\partial_{n+1} : C_{n+1} \to C_n)}.$$

Remark 6.6: (*Cochain Complex and Cohomology*)

We shall also consider the following case

$$C^\bullet : \cdots \to C^n \xrightarrow{\partial^n} C^{n+1} \to \cdots$$

such that $\partial^{n+1} \circ \partial^n = 0$. We say $C^\bullet$ is a *cochain complex*, and the *n^{th}-cohomology group* of $C^\bullet$ is defined as the quotient

$$H^n(C^\bullet) = \frac{\ker(\partial^n : C^n \to C^{n+1})}{\operatorname{im}(\partial^{n-1} : C^{n-1} \to C^n)}.$$

Observe that given a cochain complex $(C^\bullet, \partial^\bullet)$, we have a chain complex $(D_\bullet, \partial_\bullet)$ defined by $D_n := C^{-n}$, and $\partial_n = \partial^{-n}$.

Exercise 6.7: (*Homology is an Additive Functor*)

Given a chain map $f_\bullet : C_\bullet \to D_\bullet$, define

$$H_n(f_\bullet)([x]) = [f_n(x)], \quad [x] \in H_n(C).$$

Verify that this is well-defined, and $H_n : \mathrm{Ch} \to \mathrm{Ab}$ is a functor. Similarly, H^n is a functor as well from the category of cochain complexes to abelian groups. Moreover, verify that H_n is an *additive* functor, i.e, given $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$, we have $H_n(af_\bullet + bg_\bullet) = aH_n(f_\bullet) + bH_n(g_\bullet)$, for scalars a, b (i.e, $a, b \in R = \mathbb{Z}$)

Definition 6.8: (*Singular Homology*)

Given a space X, the *singular n-homology* is the n^{th}-homology group of the singular chain complex $S_\bullet(X)$, and it is dentoed as $H_n(X)$.

Remark 6.9: (*Homology is a Functor*)

It follows from Exercise 6.3 and Exercise 6.7 that $H_n : \mathrm{Top} \to \mathrm{Ab}$ is a functor as it is a composition of two functors.

Let us compute an example!

Proposition 6.10: (*Singular Homology of a Singleton*)

$H_n(\star) = \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0. \end{cases}$

<u>**Proof**</u> **:** Observe that for each n, there exists a unique map $\sigma_n : \Delta^n \to \star$, namely the constant map. Thus, we have $S_n(X) = \mathbb{Z}$ for $n \geq 0$. Let us figure out the boundary maps. For $n > 0$, we have

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i (\sigma_n \circ d_i^n) = \sum_{i=0}^{n} (-1)^i (\sigma_{n-1}) = \begin{cases} \sigma_{n-1}, & n \text{ is even} \\ 0, & n \text{ is odd.} \end{cases}$$

Thus, the singular n-complex $S_\bullet(\star)$ is given as

$$\cdots \to \underbrace{S_4}_{\mathbb{Z}} \xrightarrow{\text{Id}} \underbrace{S_3}_{\mathbb{Z}} \xrightarrow{0} \underbrace{S_2}_{\mathbb{Z}} \xrightarrow{\text{Id}} \underbrace{S_1}_{\mathbb{Z}} \xrightarrow{0} \underbrace{S_0}_{\mathbb{Z}} \to 0.$$

Immediately we get $H_n(\star) = \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0. \end{cases}$ $\square$

The above shows that singular homology satisfies the dimension axiom from the Eilenberg-Steenrod axioms.

6.3 A Categorical Digression : Product, Coproduct, Kernel, Cokernel

In this section, we re-interpret some notions from algebra in category theoretic language. If it seems too abstract, you can skip it completely!

Definition 6.11: (*Product and Initial Object*)

Let $\mathcal{C}$ be a category, and $\{A_i\}_{\{i \in I\}}$ be a (possibly empty!) collection of objects of $\mathcal{C}$. The *product* is then defined via the following universal property.

> The product is an object $X \in \mathcal{C}$ and a collection of maps $\pi_i : X \to A_i$, such that given any other object $Y \in \mathcal{C}$ and any other collection of maps $f_i : Y \to A_i$, there exists a **unique** map $f : Y \to X$ such that $f_i = \pi_i \circ f$. We have the following diagram.

$$\begin{tikzcd} Y \arrow[d, dashed, "\exists! f"'] \arrow[dr, bend left, "f_i"] & \\ X \arrow[r, "\pi_i"'] & A_i \end{tikzcd}$$

An *initial object* in $\mathcal{C}$ is a product of an empty collection.

Since product is defined via an universal property, if it exists, it is unique up to unique isomorphism. We have an alternative, perhaps more useful definition of the initial object.

Exercise 6.12: (*Initial Object*)

Show that the initial object of a category $\mathcal{C}$ (if exists) can be equivalently defined as an object X such that given any object $Y \in \mathcal{C}$, there exists a unique morphism $X \to Y$.

Example 6.13: (*Examples of Product and Initial Objects*)

Here a few examples, that one should verify!

- Set : product is the Cartesian product of sets, and initial object is the emptyset.
- Group : product is the direct product, and initial object is the group with one element.
- Top : product is the Cartesian product of the underlying sets equipped with the product topology, and initial object is the emptyset.
- $\mathrm{Top}_\star$: product is the *smash product*, i.e, $\prod(X_i, x_i) = \frac{\prod X_i}{\bigvee X_i}$, and initial object is the singleton.

The dual notion to product is the coproduct.

Definition 6.14: (*Coproduct and Final Object*)

Let $\mathcal{C}$ be a category, and $\{A_i\}_{\{i \in I\}}$ be a (possibly empty!) collection of objects of $\mathcal{C}$. The *coproduct* is then defined via the following universal property.

> The coproduct is an object $X \in \mathcal{C}$ and a collection of maps $\iota_j : A_j \to X$, such that given any other object $Y \in \mathcal{C}$ and any other collection of maps $f_i : A_j \to Y$, there exists a **unique** map $f : X \to Y$ such that $f_i = f \circ \iota_j$. We have the following diagram.

$$\begin{tikzcd}
A_j \arrow[r, "\iota_j"] \arrow[d, "f_j"'] & X \arrow[dl, dashed, bend left, "\exists! f"] \\
Y &
\end{tikzcd}$$

A *final object* in $\mathcal{C}$ is a coproduct of an empty collection.

Again, coproduct (if exists) is unique up to unique isomorphism. Moreover, we have the following useful characterization of final objects.

Exercise 6.15: (*Final Object*)

Show that the final object of a category $\mathcal{C}$ (if exists) can be equivalently defined as an object X such that given any object $Y \in \mathcal{C}$, there exists a unique morphism $Y \to X$.

Example 6.16: (*Examples of Coproduct and Final Objects*)

- Set : coproduct is the disjoint union of sets, and final object is the singleton set.
- Ab : coproduct is the direct sum of Abelian groups, and final object is the group with one element.
- Group : coproduct is the *free product* of groups, and final object is the group with one element.
- Top : coproduct is the disjoint union of the underlying sets equipped with the disjoint union topology, and final object is the singleton.
- $\mathrm{Top}_\star$: coproduct is the wedge product of spaces, and final object is the singleton.

We now define the notion of (pre)additive categories, our motivation is the category Ab of Abelian groups.

Definition 6.17: (*Pre-additive Category*)

A category $\mathcal{C}$ is called an *pre-additive category* if for any $X, Y \in \mathcal{C}$ we have $\hom_{\mathcal{C}}(X, Y)$ is an Abelian group, and moreover, the composition

$$\hom_{\mathcal{C}}(X, Y) \times \hom_{\mathcal{C}}(Y, Z) \to \hom_{\mathcal{C}}(X, Z)$$

is bilinear. A functor $F : \mathcal{C} \to \mathcal{D}$ between pre-additive categories is called an *additive functor* if $F : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(F(X), F(Y))$ is a group homomorphism for all $X, Y \in \mathcal{C}$.

Proposition 6.18: (*Zero Object*)

In a pre-additive category $\mathcal{C}$, an object $\star$ is an initial object if and only if it is a final object, whence it is called a *zero object*

Proof : Suppose $\star \in \mathcal{C}$ is an initial object. Given any $X \in \mathcal{C}$, there exists a unique map $e : \star \to X$. Now, $\hom_{\mathcal{C}}(X, \star)$ is an Abelian group, and hence, we have the zero map $0_X : X \to \star$. If possible, suppose $f : X \to \star$ is another map. As $\mathcal{C}$ is a category, we have $\mathrm{Id}_\star : \star \to \star$, and as $\mathcal{C}$ is pre-additive, we have $0 : \star \to \star$. Then, by uniqueness, we have $\mathrm{Id}_\star = 0$. But then,

$$f = f \circ \mathrm{Id}_\star = f \circ 0 = 0_X,$$

where the last equality follows from the bilinearity of the composition. Thus, there exists a unique map $X \to \star$, which makes $\star$ into a final object. Similar argument works for the other direction as well. $\square$

In fact, one can generalize the above. In a pre-additive category, any finite product is a coproduct, and coversely, any finite coproduct is a product, whence they are called *biproduct*. Let $A, B \in \mathcal{C}$ be two objects in a pre-additive category. Suppose the product $A \oplus B \in \mathcal{C}$ exists. We have maps

$$A \xtwoheadleftarrow{\pi_A} A \oplus B \xtwoheadrightarrow{\pi_B} B.$$

From universal property of product, we get unique maps in the following diagrams.

$$\begin{tikzcd}
A \arrow[dr, dashed, "\iota_A"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "\mathrm{Id}_A"'] & & \\
& A \oplus B \arrow[r, "\pi_B"'] \arrow[d, "\pi_A"] & B \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[dr, dashed, "\iota_B"] \arrow[drr, bend left, "\mathrm{Id}_B"] \arrow[ddr, bend right, "0"'] & & \\
& A \oplus B \arrow[r, "\pi_B"'] \arrow[d, "\pi_A"] & B \\
& A &
\end{tikzcd}$$

Observe that,

$$\pi_A \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \mathrm{Id}_A \circ \pi_A + 0 \circ \pi_B = \pi_A + 0 = \pi_A,$$

and

$$\pi_B \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = 0 \circ \pi_A + \mathrm{Id}_B \circ \pi_B = 0 + \pi_B = \pi_B.$$

Then, from the universal property of the product, we have

$$\iota_A \circ \pi_A + \iota_B \circ \pi_B = \mathrm{Id}_{A \oplus B}.$$

Next, we check that

$$A \overset{\iota_A}{\hookrightarrow} A \oplus B \overset{\iota_B}{\hookleftarrow} B$$

is the coproduct. Indeed, for any $f : A \to C$ and $g : B \to C$, consider the map

$$h = f \circ \pi_A + g \circ \pi_B : A \oplus B \to C.$$

Then,

$$h \circ \iota_A = f \circ \pi_A \circ \iota_A + g \circ \pi_B \circ \iota_A = f \circ \mathrm{Id}_A + g \circ 0 = f + 0 = f,$$

and similarly,

$$h \circ \iota_B = f \circ \pi_A \circ \iota_B + g \circ \pi_B \circ \iota_B = f \circ 0 + g \circ \mathrm{Id}_B = 0 + g = g.$$

Now, consider $\theta : A \oplus B \to C$ be any map satisfying $\theta \circ \iota_A = f, \theta \circ \iota_B = g$. Then,

$$h = f \circ \pi_A + g \circ \pi_B = \theta \circ \iota_A \circ \pi_A + \theta \circ \iota_B \circ \pi_B = \theta \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \theta \circ \mathrm{Id}_{A \oplus B} = \theta.$$

Hence, $h : A \oplus B$ is the unique such map. This justifies that $A \overset{\iota_A}{\hookrightarrow} A \oplus B \overset{\iota_B}{\hookleftarrow} B$ is indeed a coproduct. Similarly, we can show that any finite coproduct is also a product.

Conversely, suppose we have objects A, B, C and maps $\iota_A : A \to C, \iota_B : B \to C, \pi_A : C \to A, \pi_B : C \to B$, satisfying

$$\pi_A \circ \iota_A = \mathrm{Id}_A, \pi_A \circ \iota_B = 0, \pi_B \circ \iota_A = 0, \pi_B \circ \iota_B = \mathrm{Id}_B, \iota_A \circ \pi_A + \iota_B \circ \pi_B = \mathrm{Id}_C.$$

Then, it follows that $A \overset{\pi_A}{\twoheadleftarrow} C \overset{\pi_B}{\twoheadrightarrow} B$ is a product, and $A \overset{\iota_A}{\hookrightarrow} C \overset{\iota_B}{\hookleftarrow} B$ is a coproduct.

Definition 6.19: (*Additive Category*)

A pre-additive category is called an *additive category* if it admits all finite products (hence all finite coproducts). In particular, the initial object exists (which is also the final object), and is called the *zero object* of the category. Given any two objects $X, Y \in \mathcal{C}$, we have the unique *zero map* $0 : X \to Y$ between them.

Proposition 6.20:

An additive functor (Definition 6.17) between additive categories preserve finite products. In particular, zero object is mapped to the zero object.

Proof : Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories. Firstly, let us show that $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$. Note that $\mathrm{Id}_{0_{\mathcal{C}}} = 0$. Since F is an additive functor, we have $\mathrm{Id}_{F(0_{\mathcal{C}})} = F\big(\mathrm{Id}_{0_{\mathcal{C}}}\big) = F(0) = 0$. That is, the identity map of $F(0_{\mathcal{C}})$ is the zero map. But then $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$.

Next, let $A, B \in \mathcal{C}$. We need to show that $F(A \oplus B) = F(A) \oplus F(B)$. Recall that $A \oplus B$ is both the product and coproduct. Thus, we have maps $\iota_A : A \to A \oplus B, \iota_B : B \to A \oplus B$ and $\pi_A : A \oplus B \to A, \pi_B : A \oplus B \to B$, which moreover satisfy

$$\pi_A \circ \iota_A = \mathrm{Id}_A, \pi_A \circ \iota_B = 0, \pi_B \circ \iota_A = 0, \pi_B \circ \iota_B = \mathrm{Id}_B, \iota_A \circ \pi_A + \iota_B \circ \pi_B = \mathrm{Id}_{A \oplus B}.$$

But then applying F we get the same relations for the object $F(A \oplus B)$. Hence, $F(A \oplus B)$ is the biproduct of $F(A)$ and $F(B)$. Inductively, it follows taht F takes any finite biproduct to a biproduct. $\square$

Example 6.21: (*Examples of Additive Category*)

Category of Abelian groups (and more generally, category of R-modules for a commutative ring R) is an additve category. The category Ch of chain complexes of Abelian groups (or even modules) is also an additive category.

Exercise 6.22:

Let R be a commutative ring with 1. Let $\mathcal{C}$ be the one-object category with hom-set R. Verify that $\mathcal{C}$ is a pre-additive category, which is not additive.

The goal is to now define (co)kernels, and relate them to monic and epic maps.

Definition 6.23: (*Kernel and Cokernel*)

Let $\mathcal{C}$ be an additive category, and $f : X \to Y$ is a map.

- The *kernel* of f is a map $\iota : \ker(f) \to X$ with $f \circ \iota = 0$, satisfying the universal property :

$$\begin{tikzcd} \ker(f) \arrow[r, "\iota"] & X \arrow[r, "f"] & Y \\ Z \arrow[u, dashed, "\exists!\tilde{g}"] \arrow[ur, "g"'] \arrow[urr, bend right, "0"'] & & \end{tikzcd}$$

given any map $g : Z \to X$ with $f \circ g = 0$, there exists a unique map $\tilde{g} : Z \to \ker(f)$ such that $\iota \circ \tilde{g} = g$.

- The *cokernel* of f is a map $\pi : X \to \mathrm{coker}(f)$ with $\pi \circ f = 0$, satisfying the universal property :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[drr, bend right, "0"'] & Y \arrow[r, "\pi"] \arrow[dr, "h"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists!\tilde{h}"] \\ & & W \end{tikzcd}$$

given any map $h : X \to W$ with $h \circ f = 0$, there exists a unique map $\tilde{h} : \mathrm{coker}(f) \to W$ such that $\tilde{h} \circ \pi = h$.

In the category of Abelian groups (or modules), the kernel of a map is indeed the 0-set. For a homomorphism $f : X \to Y$ of Abelian groups, we have $\mathrm{coker}(f) = \frac{B}{\mathrm{im}(f)}$.

Remark 6.24: (*Categorical Image and Coimage*)

Since we have defined (co)kernel as a map, we can now define the (co)kernel of them as well. This leads to the following definition : given a map $f : X \to Y$, the *image* is defined as $\ker(\mathrm{coker}(f))$ and the *coimage* is defined as $\mathrm{coker}(\ker(f))$. One can show that there is a natural map $\mathrm{coim}(f) \to \mathrm{im}(f)$, and the *first isomorphism theorem* states that this map is an isomorphism. In the category of Abelian groups, the categorical image matches with the set-theoretic image (which is already a subgroup), but in the category of groups one needs to take the *normal completion* of the set-theoretic image (which is only a subgroup).

Let us now define injective and surjective maps, and relate them kernel and cokernel.

Definition 6.25: (*Monic and Epic Maps*)

Let $f : X \to Y$ be a map in a category $\mathcal{C}$.

- f is called *monic* (or *monomorphism* or *injective*) if given any two maps $g_1, g_2 : Z \to X$ with $f \circ g_1 = f \circ g_2$, we have $g_1 = g_2$.
- f is called *epic* (or *epimorphism* or *surjective*) if given any two maps $h_1, h_2 : Y \to W$ with $h_1 \circ f = h_2 \circ f$, we have $h_1 = h_2$.

In the category Set, the above definition boils down to the usual definition of injective and surjective set maps. The categorical definition has the advantage that we do not need to the objects to be a set. Thus, for example, we can readily define monic/epic chain maps.

Caution 6.26: (*Monic, Epic and Iso*)

Suppose, $f : X \to Y$ admits a right inverse, i.e, there is a map $s : Y \to X$ such that $s \circ f = \mathrm{Id}_X$. Then, f is necessarily monic. But the converse may not be true! Consider the map $f : \mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2: clearly f is monic, but admits no right inverse. Similarly, any map admitting a left inverse is epic, but not conversely. By definition, an isomorphism admits both left and right inverse, and hence, is both monic and epic. In the category of commutative rings, the inclusion $\mathbb{Z} \to \mathbb{Q}$ is both monic and epic, but it is **not** an isomorphism!

Definition 6.27: (*Abelian Category*)

A category $\mathcal{C}$ is called an *Abelian category* if the following holds.

- $\mathcal{C}$ is an additive category (i.e, each $\hom$ is an Abelian groups, composition is bilinear, and all finite (co)products exists including the zero object).
- Any map in $\mathcal{C}$ has kernel and cokernel (this is sometimes called a *pre-Abelian category*).
- Every monic map is a kernel of some map, and every epic map is a cokernel of some map (this is called having *nomal monic* and *conormal epic* maps).

The category of modules over a commutative ring is an Abelian category, and so is the category of chain complexes of such modules! In the literature, following Grothendieck, additional axioms (which deals with arbitrary (co)products) are added to the definition of an Abelian category. In an Abelian category, a map which is both monic and epic is necessarily an iso.

Exercise 6.28: (*Monic and Epic maps in an Abelian Category*)

Let $f : X \to Y$ be a map in an Abelian category $\mathcal{C}$. Verify the following.

- f is monic if and only if $\ker(f) = 0$.
- $\iota : \ker(f) \to X$ is monic, and thus, $\ker(\ker(f)) = 0$.
- f is epic if and only if $\operatorname{coker}(f) = 0$.
- $\pi : Y \to \operatorname{coker}(f)$ is epic, and thus, $\operatorname{coker}(\operatorname{coker}(f)) = 0$.
- f is an iso $\Leftrightarrow$ f is both monic and epic $\Leftrightarrow$ $\ker(f) = 0 = \operatorname{coker}(f)$.

Note that by $\ker(f) = 0$ (and similarly, $\operatorname{coker}(f) = 0$), one should understand that the object $\ker(f)$ is the zero object, and the map $\ker(f) \to X$ is (necessarily) the 0 map.

6.4 Snake Lemma

After all the abstract nonsense, let us summarize the takeaway of the previous section! Suppose $f : X \to Y$ is a morphism of modules. Then, there exists an exact sequence

$$0 \to \ker(f) \hookrightarrow X \xrightarrow{f} Y \twoheadrightarrow \operatorname{coker}(f) \to 0.$$

Given a chain map $f_\bullet : C_\bullet \to D_\bullet$, kernel, cokernel, and image of $f_\bullet$ can be computed degree-wise, with naturally induced boundary maps. In particular, we have

$$0 \to P_\bullet \xrightarrow{f_\bullet} Q_\bullet \xrightarrow{g_\bullet} R_\bullet \to 0$$

is a short exact sequence of chain maps precisely when

$$0 \to P_n \xrightarrow{f_n} Q_n \xrightarrow{g_n} R_n \to 0$$

is a short exact sequence for each n. Note that any module A (and in particular 0), can be realized as a chain complex (*concentrated at degree* 0) by putting A at the 0^{th} place, and putting 0 everywhere else (and necessarily with 0 as boundary maps). Moreover, given any chain map $f_\bullet : C_\bullet \to D_\bullet$, we have an exact sequence

$$0 \to \ker(f)_\bullet \to C_\bullet \xrightarrow{f_\bullet} D_\bullet \to \operatorname{coker}(f)_\bullet \to 0$$

of chain complexes and maps.

We now state one of the most important results in homological algebra!

Lemma 6.29: (*Snake Lemma!*)

Consider the following commutative diagram

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' &
\end{tikzcd}$$

where the two rows are exact. Then, there exists an exact sequence

$$\ker(f) \to \ker(a) \to \ker(b) \to \ker(c) \xrightarrow{\partial} \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c) \to \operatorname{coker}(g'),$$

where ∂ is called the *boundary map*. Moreover, the sequence is natural.

Proof : We have the following diagram

$$\begin{tikzcd}
 & \ker(a) \arrow[r] \arrow[d] & \ker(b) \arrow[r] \arrow[d] & \ker(c) \arrow[d] \arrow[ddll, "\partial"] & \\
\ker(f) \arrow[r] \arrow[ur, bend left] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] \arrow[d] & C' \arrow[r] \arrow[d] & \operatorname{coker}(g') \\
 & \operatorname{coker}(a) \arrow[r] & \operatorname{coker}(b) \arrow[r] & \operatorname{coker}(c) \arrow[ur, bend right] &
\end{tikzcd}$$

The blue arrows can be induced naturally. Note that the colored arrows look like a snake! Let us define the map ∂. Say, $x \in \ker(c) \subset C$. Since g is surjective, we have some $y \in B$ such that $x = g(y)$. By commutativity, $g'b(y) = cg(y) = c(x) = 0 \Rightarrow b(y) \in \ker(g') = \operatorname{im}(f')$. Thus, there is some $z \in A'$ such that $f'(z) = b(y)$. Let us define $\partial(x) = [z] = z + \operatorname{im}(a)$. We need to show that ∂ is well-defined. Suppose, we have some $y' \in B$ such that $g(y') = c$. Then, $g'b(y') = cg(y') = 0 \Rightarrow b(y') \in \ker(g') = \operatorname{im}(f')$ and so, $b(y') = f'(z')$ for some $z' \in A'$. We have, $g(y - y') = g(y) - g(y') = x - x = 0 \Rightarrow y - y' \in \ker(g) = \operatorname{im}(f)$, and so $y - y' = f(w)$ for some $w \in A$. Now, $f'(z - z') = b(y - y') = bf(w) = f'a(w)$. As f' is injective, we have $z - z' = a(w)$. But then $[z] = [z']$ in $\operatorname{coker}(a)$. This proves that ∂ is well-defined. A similar argument shows that ∂ is a homomorphism. The exactness of the sequence is left as an exercise!

As for naturaility, we consider a commutative diagram of the form

$$
\begin{tikzcd}
 & & & A_2 \arrow[rr, "f"] \arrow[dd, "a_2"] & & B_2 \arrow[rr, "g"] \arrow[dd, "b_2"] & & C_2 \arrow[rr] \arrow[dd, "c_2"] & & 0 \\
 & & A_1 \arrow[ur, red] \arrow[rr, "f"] \arrow[dd, "a_1"] & & B_1 \arrow[ur, red] \arrow[rr, "g"] \arrow[dd, "b_1"] & & C_1 \arrow[ur, red] \arrow[rr] \arrow[dd, "c_1"] & & 0 & \\
 & 0 \arrow[rr] & & A_2' \arrow[rr, "f'"] & & B_2' \arrow[rr, "g'"] & & C_2' & & \\
0 \arrow[rr] & & A_1' \arrow[ur, red] \arrow[rr, "f'"'] & & B_1' \arrow[ur, red] \arrow[rr, "g'"'] & & C_1' \arrow[ur, red] & & &
\end{tikzcd}
$$

where all the rows are exact. Then, the red arrwos determine a commutative diagram of the corresponding 8-term exact sequences. Again, the proof is a diagram chasing argument, and left as an exercise! □

Day 7 : 8^{th} February, 2026

`long exact sequence from short exact sequence of chain complexes – relative singular homology – long exact sequence of singular homology – chain homotopy invariance – singular homology of contractible space`

7.1 Long Exact Sequence in Homology

So far we have been working with only Abelian groups, which are nothing but $\mathbb{Z}$-modules. More generally, we can fix a commutative ring R, and work with R-modules. Generalizing even further, we can consider any arbitrary *Abelian category* $\mathcal{A}$, where 0-maps and taking quotients makes sense (see Definition 6.27). Then, we have the chain complex of objects from $\mathcal{A}$, denoted as $\mathrm{Ch}(\mathcal{A})$. The definition of n^{th}-homology group makes sense for $\mathrm{Ch}(\mathcal{A})$ as well. Homological algebra is the study of the (co)homology of (co)chain complex over some Abelian category.

Theorem 7.1: (*Long Exact Sequence of Homology of Chain Complex*)

Given a short exact sequence $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \to 0$ of chain complexes, there exists a long exact sequence of homology groups :

$$\cdots \to H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{\iota_\star} H_n(B) \xrightarrow{j_\star} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots,$$

which is natural with respect to maps of short exact sequences of chain complexes.

Proof : We consider part of the diagram

$$\begin{tikzcd}
 & \ker \partial_n^A \arrow[d, hook] & \ker \partial_n^B \arrow[d, hook] & \ker \partial_n^C \arrow[d, hook] & \\
0 \arrow[r] & A_n \arrow[r, "\iota_n"] \arrow[d, "\partial_n^A"'] & B_n \arrow[r, "j_n"] \arrow[d, "\partial_n^B"'] & C_n \arrow[r] \arrow[d, "\partial_n^C"'] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\iota_{n-1}"] \arrow[d, two heads] & B_{n-1} \arrow[r, "j_{n-1}"] \arrow[d, two heads] & C_{n-1} \arrow[r] \arrow[d, two heads] & 0 \\
 & \operatorname{coker} \partial_n^A & \operatorname{coker} \partial_n^B & \operatorname{coker} \partial_n^C &
\end{tikzcd}$$

Since both the rows are exact, applying the snake lemma (Lemma 6.29), we have the exact sequence

$$0 \to \ker \partial_n^A \to \ker \partial_n^B \to \ker \partial_n^C \to \operatorname{coker} \partial_n^A \to \operatorname{coker} \partial_n^B \to \operatorname{coker} \partial_n^C \to 0.$$

This holds for each n. In particular, we consider the following diagram

$$\begin{tikzcd}
 & \operatorname{coker} \partial_{n+1}^A \arrow[r] \arrow[d, "\delta_n^A"'] & \operatorname{coker} \partial_{n+1}^B \arrow[r] \arrow[d, "\delta_n^B"'] & \operatorname{coker} \partial_{n+1}^C \arrow[r] \arrow[d, "\delta_n^C"'] & 0 \\
0 \arrow[r] & \ker \partial_{n-1}^A \arrow[r] & \ker \partial_{n-1}^B \arrow[r] & \ker \partial_{n-1}^C &
\end{tikzcd}$$

By above, the rows are exact. The maps $\delta_n^A, \delta_n^B, \delta_n^C$ are induced from $\partial_n^A, \partial_n^B, \partial_n^C$ respectively. Thus, the diagram is commutative. Observe that $\ker \delta_n^A = H_n(A)$ and $\operatorname{coker} \delta_n^A = H_{n-1}(A)$, and similarly for δ_n^B, δ_n^C. Indeed, we have the diagram which might be of help

$$\begin{tikzcd}
A_{n+1} \arrow[r, "\partial^A_{n+1}"] & A_n \arrow[r, "\partial^A_n"] \arrow[d, two heads] & A_{n-1} \arrow[r, "\partial^A_{n-1}"] & A_{n-2} \\
H_n(A) \arrow[r, hook] & \text{coker } \partial^A_{n+1} \arrow[r, "\delta^A_n"'] \arrow[ur, "\widetilde{\partial}^A_n"] & \ker \partial^A_{n-1} \arrow[u, hook] \arrow[r, two heads] & H_{n-1}(A)
\end{tikzcd}$$

Hence, again applying the snake lemma, we have an exact sequence

$$H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C).$$

Chasing the diagram carefully, shows that the connecting maps are induced by $\iota_\bullet$ and $j_\bullet$. Pasting these exact sequences together, we get the long exact sequence in homology. Since snake lemma produces *natural* exact sequences, again a diagram chase shows that the long exact sequence is natural. □

Remark 7.2: (*The Boundary Map*)

Let us describe the boundary map explicitly. Consider part of the diagram

$$\begin{tikzcd}
& B_n \arrow[rr, two heads, "j_n"] \arrow[dd, "\partial^B_n"'] & & C_n \\
& & y \arrow[r, mapsto] \arrow[dd] & x \arrow[dd] \\
A_{n-1} \arrow[r, hook, "\iota_n"] & B_{n-1} & & \\
z \arrow[rr, mapsto] & & \partial^B_n(y) \arrow[r, mapsto] & 0
\end{tikzcd}$$

Suppose $\alpha \in H_n(C)$ is represented as $\alpha = [x]$ for some $x \in C_n$, with $\partial(x) = 0$. Then, it follows that $\partial(\alpha) = [z]$. The well-definedness is a consequence of the snake lemma. Thus, we can 'define' $\partial([x]) = [\iota_n^{-1}\partial^B_n j_n^{-1}(x)]$.

7.2 Relative Singular Homology and Long Exact Sequence

Let us now extend our definition of singular homology to pairs of spaces (X, A) with $A \subset X$. Given such a pair (X, A), observe that any singular n-simplex $\sigma : \Delta^n \to A$ is naturally an n-simplex in X via the inclusion map $\iota : A \hookrightarrow X$. Thus, we have an *injective* map $S_\bullet(\iota) : S_\bullet(A) \hookrightarrow S_\bullet(X)$. Let us define the cokernel

$$S_n(X, A) := \frac{S_n(X)}{S_n(A)}.$$

Observe that $S_n(X, A)$ is an Abelian group freely generated by singular n-simplexes $\sigma : \Delta^n \to X$ such that $\mathrm{im}(\sigma)$ is not *completely* contained in A. We have the well-defined *relative* boundary map

$$\begin{aligned}
\partial^{X,A}_n : S_n(X, A) &\longrightarrow S_{n-1}(X, A) \\
\left[\sum a_\sigma \sigma\right] &\longmapsto \sum a_\sigma [\partial^X_n(\sigma)].
\end{aligned}$$

In particular, if we have an n-chain in X whose boundary is compleletly contained in A, then its relative boundary is zero. It follows that $S_\bullet(X, A)$ is a chain complex, which is called the *relative singular chain complex*.

Exercise 7.3: (*Relative Singular Chain Complex is Functorial*)

Verify that $S_\bullet : \mathrm{TopPair} \to \mathrm{Ch}$ is a functor, and we can identify $S_n(X) = S_n(X, \emptyset)$ (which justifies the same notation). Moreover, $j : X = (X, \emptyset) \to (X, A)$ induces the quotient map $S_\bullet(X) \to S_\bullet(X, A)$.

The homology groups

$$H_n(X, A) \coloneqq H_n(S_n(X, A))$$

are called the *relative singular homology groups* of the pair (X, A). By Exercise 6.7 and Exercise 7.3, it follows that the relative singular homology groups are functors as well. In particular, given $f : (X, A) \to (Y, B)$, we have $H_n(f) = H_n(S_\bullet(f))$.

We have a short exact sequence of chain complexes

$$0 \to S_\bullet(A) \xhookrightarrow{\iota_\bullet} S_\bullet(X) \xrightarrow{j_\bullet} S_\bullet(X, A) \to 0,$$

where $\iota : A \hookrightarrow X$ and $j : X \hookrightarrow (X, A)$ are the space level maps.

Theorem 7.4: (*Long Exact Sequence of Singular Homology*)

There exists a natural long exact sequence

$$\cdots \to H_n(A) \xrightarrow{\iota_\star} H_n(X) \xrightarrow{j_\star} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots \to H_0(A) \xrightarrow{\iota_\star} H_0(X) \xrightarrow{j_\star} H_0(X, A) \to 0.$$

Proof : The proof is immediate from Theorem 7.1. $\square$

Theorem 7.4 justifies that singular homology satisfies the exactness axiom.

7.3 Chain Homotopy Invariance

Let us recall the notion of *chain maps* $f_\bullet : C_\bullet \to D_\bullet$ (Definition 1.13), which is given a by a collection of maps $f_n : C_n \to D_n$ such that $f_{n-1} \circ \partial_n^C = \partial_n^D \circ f_n$ holds.

Given a chain complex $C_\bullet$, the *shift (or translation)* of $C_\bullet$ by degree k is the chain complex $C[k]_\bullet$ defined as $C[k]_n = C_{n+k}$ and $\partial_n^{C[k]} = (-1)^k \partial_{n+k}^C$. We shall see later why the sign $(-1)^k$ appears in the definition. A *degree k chain map* $C_\bullet \to D_\bullet$ is a chain map $C_\bullet \to D[k]_\bullet$. In particular, a chain map $C_\bullet \to D_\bullet$ has degree 0.

Exercise 7.5: (*Shift Functor*)

Let $\Sigma^k : \mathrm{Ch} \to \mathrm{Ch}$ be the functor that shifts a chain complex by degree k. Check that $\Sigma^k \circ \Sigma^l = \Sigma^{k+l}$ and $\Sigma^0 = \mathrm{Id}$. In particular, Σ^k is an *isomorphism* with inverse Σ^{-k}.

Definition 7.6: (*Chain Map and Chain Homotopy*)

Given two chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$, a *chain homotopy* between them is a degree 1 map $h_\bullet : C_\bullet \to D_{\bullet+1}$ such that $f - g = \partial \circ h + h \circ \partial$ holds. That is, we have the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, bend right, "f_{n+1}"'] \arrow[d, bend left, "g_{n+1}"] & C_n \arrow[r, "\partial^C_n"] \arrow[dl, "h_n"'] \arrow[d, bend right, "f_n"'] \arrow[d, bend left, "g_n"] & C_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"'] \arrow[d, bend right, "f_{n-1}"'] \arrow[d, bend left, "g_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"'] & D_n \arrow[r, "\partial^D_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

where, we have $f_n - g_n = h_{n-1} \circ \partial^C_n + \partial^D_{n+1} \circ h_n$

Exercise 7.7: (*Chain Homotopy is an Equivalence Relation*)

Check that the chain homotopy is an equivalence relation on the collection of all chain maps $C_\bullet \to D_\bullet$.

Proposition 7.8: (*Homology of Chain Homotopic Maps*)

Suppose $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are homotopic chain maps. Then, they induce the same map between homology groups.

Proof : Suppose $f - g = \partial h + h\partial$ for a chain homotopy $h : C_\bullet \to D_{\bullet+1}$. Since homology is an additive functor (Exercise 6.7), we only need to show that $H_n(\partial h + h\partial) = 0$. For a class $[\alpha] \in H_n(C)$, we have $\partial(\alpha) = 0$. Recall, $[\alpha] = \alpha + \operatorname{im} \partial$ Then, it follows

$$H_n(\partial h + h\partial)[\alpha] = [(\partial h + h\partial)(\alpha)] = [\partial h(\alpha)] = 0.$$

Thus, $H_n(f) = H_n(g)$ follows for all n. $\square$

7.4 Singular Homology of Contractible Space

Before proving the homotopy invariance for singular homology, let us start with a contractible space, say, X. Fix a point $x_0 \in X$. Let us define a chain map $\varepsilon_\bullet : S_\bullet(X) \to S_\bullet(X)$ by setting $\varepsilon_n = 0$ for $n \neq 0$, and

$$\varepsilon_0\Big(\sum a_\sigma \sigma\Big) = \Big(\sum a_\sigma\Big)\sigma_0,$$

where $\sigma_0 : \Delta^0 \to X$ is the 0-simplex that maps to x_0.

Proposition 7.9:

$\varepsilon_\bullet : S_\bullet(X) \to S_\bullet(X)$ is a chain map.

Proof : The only nontrivial part is the commutativity of the diagram

$$\begin{tikzcd} S_1(X) \arrow[r, "d_1"] \arrow[d, "\varepsilon_1 = 0"'] & S_0(X) \arrow[d, "\varepsilon_0"] \\ S_1(X) \arrow[r, "d_1"'] & S_0(X) \end{tikzcd}$$

Suppose $\sigma : \Delta^1 \to X$ is a singular 1-simplex. Then,

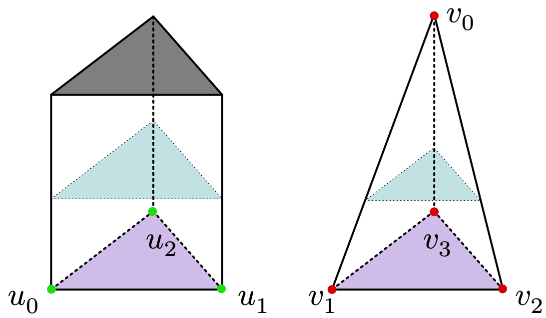
$$\varepsilon_0 d_1(\sigma) = \varepsilon_0(\sigma \circ d_0 - \sigma \circ d_1) = (1-1)\sigma_0 = 0.$$

Thus, $\varepsilon_0 d_1 = 0 = d_1 0$. This proves that $\varepsilon_\bullet$ is a chain map. □

Now, suppose $h : X \times [0,1] \to X$ is a hometopy from Id_X to the constant map $c_{x_0} : X \to X$ which maps everything to a point $x_0 \in X$. Using h, we define a chain homotopy between $\varepsilon_\bullet$ and the identity chain map. Firstly, consider the map

$$\begin{aligned} q_n : \Delta^n \times [0,1] &\longrightarrow \Delta^{n+1} \\ ((t_0, ..., t_n), s) &\longmapsto (s, (1-s)t_0, ..., (1-s)t_n) \end{aligned}$$

Since $s + \sum_{i=0}^{n} (1-s)t_i = s + (1-s) = 1$, the map is well-defined, and clearly, it is continuous. As q_n is a surjective map between compact T^2 spaces, we have q_n is a quotient map.



The map $q_2 : \Delta^2 \times [0,1] \to \Delta^3$

In fact, it follows that q_n identifies $\Delta^n \times \{1\}$ to the vertex $v_0 = (1, 0, ..., 0) \in \Delta^{n+1}$, and keeps every other point as it is. Note that q_0 is the identity map $[0,1] \to [0,1]$. Now, given a singular n-simplex $\sigma : \Delta^n \to X$, consider the diagram

$$\begin{tikzcd} \Delta^n \times [0,1] \arrow[r, "\sigma \times \mathrm{Id}"] \arrow[d, "q_n"] & X \times [0,1] \arrow[r, "h"] & X \\ \Delta^{n+1} \arrow[urr, dashed, bend right, "s_n(\sigma)"'] & & \end{tikzcd}$$

Since $h(X \times \{1\}) = \{x_0\}$, it follows from the universal property of a quotient map that there exists a unique continuous map $s_n(\sigma) : \Delta^{n+1} \to X$ such that

$$s_n(\sigma) \circ q_n = h \circ \left(\sigma \times \mathrm{Id}_{[0,1]}\right).$$

Extending linearly, we have a map $s_n : S_n(X) \to S_{n+1}(X)$.

Now, observe that for $0 < i \leq n+1$ we have the diagram

$$
\begin{tikzcd}
\Delta^{n-1}\times[0,1] \arrow[r, "d^n_{i-1}\times \mathrm{Id}"] \arrow[d, "q_{n-1}"'] & \Delta^n\times[0,1] \arrow[r, "\sigma\times \mathrm{Id}"] \arrow[d, "q_n"] & X\times[0,1] \arrow[r, "h"] & X \\
\Delta^n \arrow[r, "d_i^{n+1}"'] \arrow[rrru, dashed, bend right=60, "s_{n-1}(\sigma\circ d^n_{i-1})"'] & \Delta^{n+1} \arrow[rru, dashed, "s_n(\sigma)"'] & &
\end{tikzcd}
$$

Hence, $s_n(\sigma)d_i^{n+1} = s_{n-1}(\sigma d^n_{i-1})$ holds for $0 < i \leq n+1$. Clearly, $s_n(\sigma)d_0^{n+1} = \sigma$.

Proposition 7.10:

$s_\bullet : S_\bullet(X) \to S_{\bullet+1}(X)$ is a chain homotopy betwen Id and $\varepsilon_\bullet$.

Proof : For a 0-simplex $\sigma : \Delta^0 \to X$, we have

$$\partial(s\sigma) = (s\sigma)d_0 - (s\sigma)d_1 = \sigma - \sigma_0,$$

since $h(\sigma(0),1) = x_0$. For a singular n-simplex $\sigma : \Delta^n \to X$ with $n \geq 1$, we compute

$$
\begin{aligned}
\partial(s\sigma) &= \sum_{i=0}^{n+1}(-1)^i(s\sigma)d_i^{n+1} = (s\sigma)d_0^{n+1} - \sum_{i=1}^{n+1}(-1)^{i-1}(s\sigma)d_i^{n+1}\\
&= \sigma - \sum_{i=1}^{n+1}(-1)^{i-1}s(\sigma d^n_{i-1})\\
&= \sigma - s\left(\sum_{i=0}^{n}(-1)^i\sigma d_i^n\right), \text{by replacing } i \leftrightarrow i-1, \text{and by linearity of } s\\
&= \sigma - s(\partial\sigma)\\
\Rightarrow \sigma - \underbrace{\varepsilon_n(\sigma)}_{0} &= \partial(s\sigma) + s(\partial\sigma).
\end{aligned}
$$

Thus, we have $\mathrm{Id} - \varepsilon = \partial s + s\partial$. In other words, $s_\bullet$ is a chain homotopy between Id and $\varepsilon_\bullet$ $\square$

Theorem 7.11: (*Singular Homology of Contractible Space*)

Let X be a contracitble space. Then, $H_n(X) = 0$ for all $n \neq 0$.

Proof : Suppose $h : X \times [0,1] \to X$ is a homotopy that takes Id_X to the constant map c_{x_0} for some point $x_0 \in X$. Then, as above, construct the chain map $\varepsilon_\bullet : S_\bullet(X) \to S_\bullet(X)$. Since by Proposition 7.10 Id is chain homotopic to ε, an application of Proposition 7.8 it follows that they give the same map in homology. But homology is an additive functor. Hence, $\mathrm{Id} = H_n(\mathrm{Id}) = H_n(\varepsilon) = 0$ for $n \neq 0$. The only way identity map can be the zero map if the object is zero. Thus, $H_n(X) = 0$ for $n \neq 0$. $\square$

Exercise 7.12: (0^{th}-*Homology and Path Component*)

Show that $H_0(X)$ is freely generated by the path components of X. In particular, $H_0(X) = \mathbb{Z}$ for a path connected space.

Day 8 : 10th February, 2026

homotopy invariance of singular homology — barycentric subdivision

8.1 Homotopy Invariance of Singular Homology

Our goal is to show that homotopic maps induce the same homomorphism of singular homology groups. Given a space X, consider the maps

$$\begin{aligned}\xi^X : X &\to X \times [0,1] \\ x &\mapsto (x, 0).\end{aligned} \qquad \begin{aligned}\eta^X : X &\to X \times [0,1] \\ x &\mapsto (x, 1).\end{aligned}$$

They are clearly natural in X, and hence, induce natural chain maps

$$\xi_\bullet^X, \eta_\bullet^X : S_\bullet(X) \to S_\bullet(X \times [0,1]).$$

In particular, $\xi_\bullet, \eta_\bullet : S_\bullet(\_) \Rightarrow S_\bullet(\_ \times [0,1])$ are natural transformations.

Proposition 8.1:

The chain maps $\xi_\bullet, \eta_\bullet$ are naturally chain homotopic.

Proof : We need to construct maps $s_n^X : S_n(X) \to S_{n+1}(X \times [0,1])$, such that it is a chain homotopy:

$$\eta_n^X - \xi_n^X = \partial s_n^X + s_{n-1}^X \partial.$$

Moreover, we need it to be natural: for any $f : X \to Y$, we require

$$S_{n+1}(f \times \mathrm{Id}) \circ s_n^X = s_n^Y \circ S_n(f).$$

We construct s_n^X inductively.

- For a 0-simplex $\sigma : \Delta^0 \to X$, let us define $s_0^X(\sigma) : \Delta^1 = \Delta^0 \times [0,1] \to X \times [0,1]$ by

  $$s_0^X(\sigma)(t) = (\sigma(0), t).$$

  It follows that

  $$\partial\big(s_0^X(\sigma)\big) = s_0^X(\sigma)d_0 - s_0^X(\sigma)d_1 = \eta_0^X\sigma - \xi_0^X\sigma.$$

  Naturality is apparent from the definition.

- Inductively assume that we have constructed s_k^X for $k < n$, for all space X. We construct s_n^X. The identity map $\mathrm{Id} : \Delta^n \to \Delta^n$ is a singular n-simplex, denote $\iota_n \in S_n(\Delta^n)$ to be the corresponding element. In order to define s_n^X, we require $s_n^{\Delta^n}$ to satisfy

  $$\partial s_n^{\Delta^n}(\iota_n) = \eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial\iota_n),$$

  where the right-hand-side is defined by induction. We have the following diagram

$$
\begin{tikzcd}
S_n(\Delta^n) \arrow[r, "\partial"] \arrow[d, shift right, "\eta_n^{\Delta^n}"'] \arrow[d, shift left, "\xi_n^{\Delta^n}"] & S_{n-1}(\Delta^n) \arrow[r, "\partial"] \arrow[dl, "s_{n-1}^{\Delta^n}"'] \arrow[d, shift right, "\eta_{n-1}^{\Delta^n}"'] \arrow[d, shift left, "\xi_{n-1}^{\Delta^n}"] & S_{n-2}(\Delta^n) \arrow[dl, "s_{n-2}^{\Delta^n}"] \\
S_n(\Delta^n \times I) \arrow[r, "\partial"'] & S_{n-1}(\Delta^n \times I) &
\end{tikzcd}
$$

Observe that

$$
\begin{aligned}
&\partial\big(\eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial\iota_n)\big) \\
&= \eta_{n-1}^{\Delta^n}(\partial\iota_n) - \xi_{n-1}^{\Delta^n}(\partial\iota_n) - \partial\big(s_{n-1}^{\Delta^n}(\partial\iota_n)\big), \quad \text{since } \xi, \eta \text{ are chain maps} \\
&= \eta_{n-1}^{\Delta^n}(\partial\iota_n) - \xi_{n-1}^{\Delta^n}(\partial\iota_n) - \big(\eta_{n-1}^{\Delta^n}(\partial\iota_n) - \xi_{n-1}^{\Delta^n}(\partial\iota_n) - s_{n-2}^{X}(\partial\partial\iota_n)\big), \\
&\qquad\qquad\qquad\qquad \text{as } s_{n-1}^{\Delta^n}, s_{n-2}^{\Delta^n} \text{ are part of the chain homotopy} \\
&= 0, \quad \text{as } \partial\partial\iota_n = 0.
\end{aligned}
$$

In other words, the RHS is an n-cycle. Since $\Delta^n \times [0,1]$ is contractible, by Theorem 7.11, the RHS is a boundary. In particular, we can *choose* some singular $(n+1)$-chain $a \in S_{n+1}(\Delta^n \times I)$ such that

$$\partial a = \eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial\iota_n).$$

Set, $s_n^{\Delta^n}(\iota_n) = a$. Then, for any space X and any singular n-simplex $\sigma : \Delta^n \to X$, set

$$s_n^X(\sigma) = S_{n+1}(\sigma \times \mathrm{Id})(a).$$

Let us verify the required conditions. We have the diagram

$$
\begin{tikzcd}[column sep=small]
 & & & S_n(\Delta^n) \arrow[rr, "\partial"] \arrow[dl, "S_n(\sigma)"'] \arrow[ddll, "s_n^{\Delta^n}"'] \arrow[dd, shift right, "\eta_n^{\Delta^n}"'] \arrow[dd, shift left, "\xi_n^{\Delta^n}"] & & S_{n-1}(\Delta^n) \arrow[dl, "S_{n-1}(\sigma)"'] \arrow[ddll, "s_n^{\Delta^n}"'] \arrow[dd, shift right, "\eta_{n-1}^{\Delta^n}"'] \arrow[dd, shift left, "\xi_{n-1}^{\Delta^n}"] \\
 & & \overset{\sigma}{S_n(X)} \arrow[rr, "\partial"] \arrow[ddll, "s_n^X"'] \arrow[dd, shift right, "\eta_n^X"'] \arrow[dd, shift left, "\xi_n^X"] & & S_{n-1}(X) \arrow[ddll, "s_{n-1}^X"'] \arrow[dd, shift right, "\eta_n^X"'] \arrow[dd, shift left, "\xi_{n-1}^X"] & \\
 & \overset{a}{S_{n+1}(\Delta^n \times I)} \arrow[rr, "\partial"] \arrow[dl, "S_{n+1}(\sigma \times \mathrm{Id})"'] & & S_n(\Delta^n \times I) \arrow[rr, "\partial"] \arrow[dl, "S_n(\sigma \times \mathrm{Id})"'] & & S_{n-1}(\Delta^n \times I) \arrow[dl, "S_{n-1}(\sigma \times \mathrm{Id})"'] \\
S_{n+1}(X \times I) \arrow[rr, "\partial"] & & S_n(X \times I) \arrow[rr, "\partial"] & & S_{n-1}(X \times I) &
\end{tikzcd}
$$

We compute

$$
\begin{aligned}
\partial s_n^X(\sigma) &= \partial S_{n+1}(\sigma \times \mathrm{Id})(a) = S_n(\sigma \times \mathrm{Id})(\partial a) \\
&= S_n(\sigma \times \mathrm{Id})\big(\eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial\iota_n)\big) \\
&= \eta_n^X S_n(\sigma)(\iota_n) - \xi_n^X S_n(\sigma)(\iota_n) - s_{n-1}^X S_{n-1}(\sigma)(\partial\iota_n), \\
&\qquad\qquad \text{as } \xi^X, \eta^X \text{ are natural transformations, and } s_{n-1} \text{ is natural} \\
&= \eta_n^X S_n(\sigma)(\iota_n) - \xi_n^X S_n(\sigma)(\iota_n) - s_{n-1}^X \partial S_n(\sigma)(\iota_n) \\
&= \eta_n^X \sigma - \xi_n^X \sigma - s_{n-1}^X \partial\sigma.
\end{aligned}
$$

Thus, s_n^X satisfies the chain homotopy condition. Next, consider a continuous map $f : X \to Y$. We have the diagram

$$\begin{tikzcd}
 & \overset{\sigma}{S_n(X)} \arrow[rr, "S_n(f)"] \arrow[dl, "s_n^X"'] & & \overset{f\circ\sigma}{S_n(Y)} \arrow[dl, "s_n^Y"'] \\
S_{n+1}(X\times I) \arrow[rr, "S_n(f\times \mathrm{Id})"'] & & S_{n+1}(Y\times I) & \\
 & \underset{a}{S_{n+1}(\Delta^n\times I)} \arrow[ul, bend left, "S_{n+1}(\sigma\times\mathrm{Id})"] \arrow[ur, bend right, "S_{n+1}((f\circ\sigma)\times \mathrm{Id})"'] & &
\end{tikzcd}$$

We compute

$$\begin{aligned}
S_{n+1}(f\times \mathrm{Id})s_n^X(\sigma) &= S_{n+1}(f\times \mathrm{Id})S_{n+1}(\sigma\times \mathrm{Id})(a) \\
&= S_{n+1}((f\circ\sigma)\times \mathrm{Id})(a) \\
&= s_n^Y(f\circ\sigma) \\
&= s_n^Y S_n(f)(\sigma).
\end{aligned}$$

This proves naturality of s_n

Hence, inductively, we have a natural chain homotopy $s_\bullet : \xi_\bullet \simeq \eta_\bullet$. □

Note that the chain homotopy obtained in Proposition 8.1 is not unique since we made a choice at each induction step, but the homotopy is still natural.

Theorem 8.2: (*Singular Homology is Homotopy Invariant*)

Suppose $f, g : (X, A) \to (Y, B)$ are homotopic maps. Then, $H_n(f) = H_n(g) : H_n(X, A) \to H_n(Y, B)$ for all n.

Proof : Suppose $h : (X, A) \times I \to (Y, B)$ is a homtopy $f \simeq g$. We have the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\eta^A", "\xi^A"'] \arrow[d, hook, "\iota"'] & A\times[0,1] \arrow[r, "h|_{A\times[0,1]}"] \arrow[d, hook, "\iota\times\mathrm{Id}"'] & B \arrow[d, hook] \\
X \arrow[r, "\eta^X", "\xi^X"'] & X\times[0,1] \arrow[r, "h"'] & Y
\end{tikzcd}$$

where, $\xi^X(x) = (x, 0), \eta^X(x) = (x, 1), \xi^A(a) = (a, 0), \eta^A(a) = (a, 1)$. By Proposition 8.1, we have a natural chain homotopy $s_\bullet^X : S_\bullet(\eta^X) \simeq S_\bullet(\xi^X)$ and $s_\bullet^A : S_\bullet(\eta^A) \simeq S_\bullet(\xi^A)$. Naturality implies

$$S_{n+1}(\iota\times \mathrm{Id}) \circ s_n^A = s_n^X \circ S_n(\iota).$$

Hence, we can define $s_n^{X,A} : S_n(X, A) \to S_{n+1}(X \times [0, 1], A \times [0, 1])$, which is clearly a chain homotopy $S_\bullet(\eta^{X,A}) \simeq S_\bullet(\xi^{X,A})$. Then,

$$S_n(g) - S_n(f) = S_n(h\circ\eta^{X,A}) - S_n(h\circ\xi^{X,A}) = S_n(h)(S_n(\eta^{X,A}) - S_n(\xi^{X,A}))$$

$$= S_n(h) \circ \left(\partial s_n^{X,A} + s_{n-1}^{X,A}\partial\right) = \partial\left(S_{n+1}(h)s_n^{X,A}\right) + \left(S_n(h)s_{n-1}^{X,A}\right)\partial.$$

Thus, $\zeta_n = S_{n+1}(h)s_n^{X,A} : S_n(X) \to S_{n+1}(Y)$ is a chain homtopy $S_\bullet(g) \simeq S_\bullet(f)$. Hence, by Proposition 7.8, we have $H_n(f) = H_n(g)$. □

8.2 Barycentric Subdivision

Our next goal is to show that singular homology satisfy the exicion axiom. For that we first need to construct *barycentric subdivision* of simplicies.

Let $D \subset \mathbb{R}^n$ be a convex set. Given points $v_0, ..., v_p \in D$, the *affine singular p-simplex* is defined as

$$\begin{aligned} \sigma = s\left[v_0, ..., v_p\right] : \Delta^p &\to D \\ \sum_{i=0}^{p} \lambda_i e_i &\mapsto \sum_{i=0}^{p} \lambda_i v_i, \end{aligned}$$

where e_i are the standard unit vectors of $\mathbb{R}^{p+1}$. Note that convexitty of D implies that σ is well-defined. The *barycenter* of σ is defined as

$$\sigma^\beta := \frac{1}{p+1}\sum_{i=0}^{p} v_i.$$

In particular, we shall be interested in the identity map $\iota_p : \Delta^p \to \Delta^p$ and the corresponding barycenter ι_p^β.

Now, D being convex, is contractible. In fact, for each $v \in D$, we have a contracting homotopy

$$\begin{aligned} H_v : D \times [0,1] &\to D \\ (u, t) &\mapsto (1-t)u + tv. \end{aligned}$$

Using the cone construction from earlier, we then get a chain homotopy $S_\bullet(D) \to S_{\bullet+1}(D)$ (Proposition 7.10). Let us denote this chain homotopy as

$$\begin{aligned} S_p(D) &\to S_{p+1}(D) \\ \sigma &\mapsto v \cdot \sigma. \end{aligned}$$

In particular, the affine simplex $\sigma = \left[v_0, ..., v_p\right]$ is mapped to $v \cdot \sigma := \left[v, v_0, ..., v_p\right]$. Recall from Proposition 7.10 that

$$\partial(v \cdot \sigma) = \begin{cases} \sigma - v \cdot \partial\sigma, & p > 0, \\ \sigma - \varepsilon(\sigma)v, & p = 0, \end{cases}$$

where $\varepsilon : S_0(D) \to \mathbb{Z}$ is given by $\varepsilon(\sum n_\sigma \sigma) = \sum n_\sigma$.

Let us now inductively define an operator $\mathcal{B}_p = \mathcal{B}_p(X) : S_p(X) \to S_p(X)$ for any space X, called the *barycentric subdivision*. For any $\sigma : \Delta^p \to X$ define,

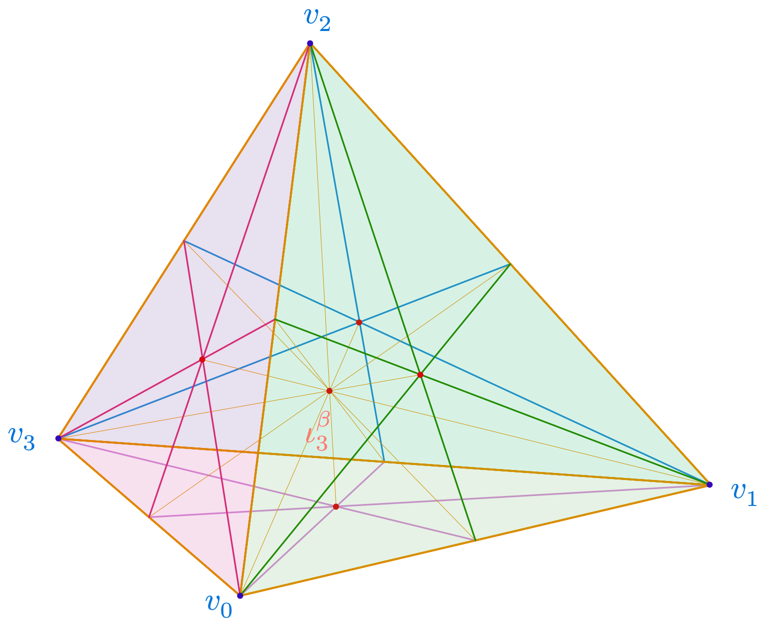
$$\mathcal{B}_p^X(\sigma) = S_p(\sigma)\mathcal{B}_p^{\Delta^p}(\iota_p),$$

where

$$\mathcal{B}_p^{\Delta^p}(\iota_p) = \begin{cases} i_0, & p = 0 \\ \iota_p^\beta \cdot \mathcal{B}_{p-1}^{\Delta^p}(\partial\iota_p), & p > 0. \end{cases}$$

Clearly this is well-defined.

Let us visualize the barycentric subdivision of ι_3, the identity simplex of Δ^3.



Barycentric sub-division of Δ^3

On each fact, we have the corresponding barycenter. For the 1-simplex $[0, 1]$, the barycenter is the just midpoint. The sub-dvision operator is taking the *cone* over the simplices of the barycentric sub-division of the lower-dimensional face. You can see that there are 24 (check!) new Δ^3 after subdividing it once.

Day 9 : 13th February, 2026

chain homotopy of barycentric subdivision – iterated barycentric subdivision – excision in singular homology – additivity in singular homology – Hurewicz homomorphism

9.1 Barycentric Subdivision and Chain Homtopy

Intuitively, barycentric subdivision divides a given chain into smaller pieces with matching boundary, so that it should not change the boundary, and have no effect on the homology. We show that this operator is in fact a natrual chain map, which is naturally chain homotopic to the identity.

Proposition 9.1: (*Barycentric Subdivision is a Natural Chain Map*)

The barycentric subdivision operator $\mathcal{B}^X_\bullet : S_\bullet(X) \to S_\bullet(X)$ is a chain map, which is natural in X. Moreover, it is naturally chain homotopic to the identity map.

Proof : Given any $f : X \to Y$, we have

$$S_n(f)\mathcal{B}^X_n(\sigma) = S_n(f)S_n(\sigma)\mathcal{B}^{\Delta^p}_n(\iota_n) = S_n(f \circ \sigma)\mathcal{B}^{\Delta^n}_n(\iota_n) = \mathcal{B}^Y_n(f \circ \sigma) = \mathcal{B}^Y_n S_n(f)(\sigma),$$

which proves the naturality. Next, we check

$$\partial\mathcal{B}^{\Delta^p}_p(\iota_p) = \partial\big(\iota^\beta_p \cdot \mathcal{B}^{\Delta^p}_{p-1}\big)(\partial\iota_p) = \mathcal{B}^{\Delta^p}_{p-1}(\partial\iota_p) - \iota^\beta_p \cdot \partial\partial\iota_p = \mathcal{B}^{\Delta^p}_{p-1}(\partial\iota_p), \quad p > 0,$$

and $\partial\mathcal{B}^{\Delta^0}_0(\iota_0) = \partial(\iota_0) = 0$. Then, for any $\sigma : \Delta^p \to X$ we have

$$\begin{aligned}\mathcal{B}^X_{p-1}\partial\sigma &= \mathcal{B}^X_{p-1}\partial S_p(\sigma)\iota_p = \mathcal{B}^X_{p-1}S_{p-1}(\sigma)(\partial\iota_p) = S_{p-1}(\sigma)\mathcal{B}^{\Delta_p}_{p-1}(\partial\iota_p) \\ &= S_{p-1}(\sigma)\partial\mathcal{B}^{\Delta_p}_{p-1}(\iota_p) = \partial S_p(\sigma)\mathcal{B}(\iota_p) = \partial\mathcal{B}(\sigma).\end{aligned}$$

This proves that $\mathcal{B}^X_\bullet$ is a chain map.

Next, we construct a natural chain homotopy $\mathrm{Id} \simeq \mathcal{B}^X_\bullet$. We define $\mathcal{T}^X_n : S_n(X) \to S_{n+1}(X)$ inductively. For any $\sigma : \Delta^n \to X$, set $\mathcal{T}^X_n(\sigma) = S_{n+1}(\sigma)\mathcal{T}^{\Delta^n}_n(\iota_n)$, where we have

$$\mathcal{T}^{\Delta^n}_n(\iota_n) = \iota^\beta_n \cdot \big(\iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n)\big).$$

Assume inductively that $\mathcal{T}^X_p$ is natural, and part of the chain homotopy for $p < n$. Then we compute

$$\begin{aligned}\partial\mathcal{T}^{\Delta^n}_n(\iota_n) &= \partial\big(\iota^\beta_n \cdot \big(\iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n)\big)\big) \\ &= \iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n) - \iota^\beta_n \cdot \big(\partial\iota_n - \partial\mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n)\big) \\ &= \iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n) - \iota^\beta_n \cdot \big(\mathcal{B}^{\Delta^n}_{n-1}(\partial\iota_n) + \mathcal{T}^{\Delta^n}_{n-2}(\partial\partial\iota_n)\big), \quad \text{by induction} \\ &= \iota_n - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n) - \iota^\beta_n \cdot \mathcal{B}^{\Delta^n}_{n-1}(\partial\iota_n), \quad \text{as } \partial\partial\iota_n = 0 \\ &= \iota_n - \mathcal{B}^{\Delta^n}_n(\iota_n) - \mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n), \quad \text{by definition of } \mathcal{B}^{\Delta^n}_n(\iota_n).\end{aligned}$$

Now, for any $\sigma : \Delta^n \to X$ we have

$$\begin{aligned}\partial\mathcal{T}^X_n(\sigma) &= \partial S_{n+1}(\sigma)\mathcal{T}^{\Delta^n}_n(\iota_n) = S_n(\sigma)\partial\mathcal{T}^{\Delta^n}_n(\iota_n) \\ &= S_n(\sigma)(\iota_n) - S_n(\sigma)\mathcal{B}^{\Delta^n}_n(\iota_n) - S_n(\sigma)\mathcal{T}^{\Delta^n}_{n-1}(\partial\iota_n) \\ &= \sigma - \mathcal{B}^X_n(\sigma) - \mathcal{T}^X_{n-1}S_{n-1}(\sigma)(\partial\iota_n), \text{by definition and by naturality}\end{aligned}$$

$$= \sigma - \mathcal{B}_n^X(\sigma) - \mathcal{T}_{n-1}^X \partial S_n(\sigma)(\iota_n)$$
$$= \sigma - \mathcal{B}_n^X(\sigma) - \mathcal{T}_{n-1}^X \partial\sigma.$$

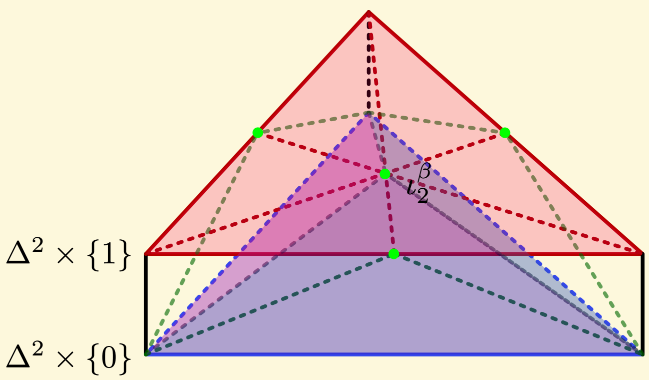
Thus, $\mathcal{T}_n^X$ is part of the required chain homotopy. Also, for any $f : X \to Y$, we have

$$S_{n+1}(f)\mathcal{T}_n^X(\sigma) = S_{n+1}(f)S_{n+1}(\sigma)\mathcal{T}_n^{\Delta^n}(\iota_n) = S_{n+1}(f\sigma)\mathcal{T}_n^{\Delta^n}(\iota_n) = \mathcal{T}_n^Y(f\sigma) = \mathcal{T}_n^Y S_n(f)(\sigma),$$

which proves the naturality. Hence, by induction, we see that $\mathcal{T}_\bullet^X : \mathrm{Id} - \mathcal{B}_\bullet^X$ is a chain homotopy, which is natural in X. □

Remark 9.2:

Here is the geometric intution behidn the chain homotopy constructed in Proposition 9.1. Given the n -simplex $\iota_n : \Delta^n \to \Delta^n$, the homotopy can be thought of as a homtopy $\Delta^n \times [0,1] \to \Delta^n$, with ι_n on $\Delta^n \times \{0\}$ and $\mathcal{B}_n^{\Delta^n}(\iota_n)$ on $\Delta^n \times \{1\}$.



The chain homotopy for $\iota_2 : \Delta^2 \to \Delta^2$ with $\mathcal{B}_2^{\Delta^2}(\iota_2)$.

We can do this by coning over all the simplices of $\Delta^n \times \{0\} \cup \partial\Delta^n \times [0,1]$ with the barycenter of $\Delta^n \times \{1\}$ as the coning point.

9.2 Iterated Barycentric Subdivision

Given a space X, consider $\mathcal{U}$ to be a collection of subsets such that $X = \cup_{U\in\mathcal{U}} \mathring{U}$, i.e, interiors of the sets from $\mathcal{U}$ form an open cover of X. A singular n-simplex $\sigma : \Delta^n \to X$ is called *$\mathcal{U}$-small* if $\sigma(\Delta^n) \subset U$ for some $U \in \mathcal{U}$. Note that the simplicies appearing in the boundary of an $\mathcal{U}$-small simplex is again $\mathcal{U}$-small. Hence, we can define $S_\bullet^{\mathcal{U}}(X) \subset S_\bullet(X)$ to be the chain complex generated by $\mathcal{U}$-small singular simplices, with (the restriction of) the usual boundary map. The corresponding homology is denoted as $H_\bullet^{\mathcal{U}}(X)$. Our goal is to show that $H_\bullet^{\mathcal{U}}(X)$ is isomorphic to $H_\bullet(X)$. We achieve this by representing a homology class by a chain of $\mathcal{U}$-small simplices, by iterating the barycentric subdivision finitely many times.

Lemma 9.3: (*Diameter of Affine Singular Simplex*)

Let $v_0, ..., v_p \in \mathbb{R}^n$. Then, $\mathcal{B}_p\left[v_0, ..., v_p\right]$ is a linear combination of affine simplices, each with diameter at most $\frac{p}{p+1}\ \mathrm{diam}\left[v_0, ..., v_p\right] = \frac{p}{p+1} \max_{i,j}\left\|v_i - v_j\right\|$, where we consider the Euclidean norm.

Proof : First, let us observe that $\mathrm{diam}\ \left[v_0, ..., v_p\right] = \max_{i,j}\left\|v_i - v_j\right\|$. Let $x, y \in \left[v_0, ..., v_p\right]$, and say $x = \sum \lambda_i v_i$ with $\sum \lambda_i = 1$ and $\lambda_i \geq 0$. Then,

$$\|x - y\| = \left\|\sum \lambda_i v_i - \sum \lambda_i y\right\| = \left\|\sum \lambda_i (v_i - y)\right\| \leq \sum \lambda_i \|v_i - y\| \leq \max_i \|v_i - y\|$$

In particular, taking $y = v_j$, we have $\left\|x - v_j\right\| \leq \max_i \left\|v_i - v_j\right\|$. But then for arbitrary x, y we have $\|x - y\| \leq \max_i \|v_i - y\| \leq \max_{i,j}\left\|v_i - v_j\right\|$. Thus, $\mathrm{diam}\left[v_0, ..., v_p\right] \leq \max_{i,j}\left\|v_i - v_j\right\|$. Clearly the diameter is attained as well, which shows the equality.

Next, we inductively show that diameter of each simplex in $\mathcal{B}_p\left[v_0, ..., v_p\right]$ is at most $\frac{p}{p+1} \max_{i,j}\left\|v_i - v_j\right\|$. It is trivial for $p = 0$. We have

$$\mathcal{B}_p\left[v_0, ..., v_p\right] = S_p(\sigma)\left(\iota_p^\beta \cdot \mathcal{B}_{p-1}(\partial \iota_p)\right) = \sum (-1)^i \sigma^\beta \cdot \mathcal{B}_{p-1}\left[v_0, ..., \hat{v_\imath}, ..., v_p\right],$$

where $\sigma = \left[v_0, ..., v_p\right]$ is the affine singular simplex. By induction, the simplices in $\mathcal{B}_{p-1}\left[v_0, ..., \hat{v_\imath}, ..., v_p\right]$ has diameter at most

$$\frac{p-1}{p}\ \mathrm{diam}\left[v_0, ..., \hat{v_\imath}, ..., v_p\right] \leq \frac{p-1}{p}\ \mathrm{diam}\ \left[v_0, ..., v_p\right] < \frac{p}{p+1}\ \mathrm{diam}\ \left[v_0, ..., v_p\right]$$

. The simplices in $\mathcal{B}_p\left[v_0, ..., v_p\right]$ has vertices σ^β and the vertices from $\mathcal{B}_{p-1}\left[v_0, ..., \hat{v_\imath}, ... v_p\right]$. Let us compute $\sup\left\{\left\|\sigma^\beta - x\right\| \mid x \in \left[v_0, ..., v_p\right]\right\}$. Say $x = \sum \lambda_i v_i \in \left[v_0, ..., v_p\right]$. Then, $\left\|\sigma^\beta - x\right\| \leq \max_i \left\|\sigma^\beta - v_i\right\|$. Now,

$$\begin{aligned}\left\|\sigma^\beta - v_j\right\| &= \left\|\frac{1}{p+1}\left(\sum v_i\right) - v_j\right\| = \frac{1}{p+1}\left\|\sum (v_i - v_j)\right\| \\ &\leq \frac{1}{p+1} \sum \left\|v_i - v_j\right\| \leq \frac{p}{p+1} \max_{i,j}\left\|v_i - v_j\right\| \\ &= \frac{p}{p+1}\ \mathrm{diam}\left[v_0, ..., v_p\right].\end{aligned}$$

The claim then follows by induction. □

Lemma 9.4: (*Iterated Barycentric Subdivision*)

The inclusion of chain complex $S_\bullet^{\mathcal{U}}(X) \to S_\bullet(X)$ induces an isomorphism in homology.

Proof : Suppose $\sigma : \Delta^n \to X$ is a singular n-simplex. Then, consider the collection $\mathcal{U}_\sigma = \left\{\sigma^{-1}\left(\mathring{U}\right) \mid U \in \mathcal{U}\right\}$, which is an open cover of the compact metric space Δ^n. Then, there exists a Lebesgue number, say, $\delta > 0$ for this covering. Suppose $k = k(\sigma)$ is a natural number such that $\left(\frac{n}{n+1}\right)^k \mathrm{diam}[e_0, ..., e_n] < \delta$. Then, it follows from Lemma 9.3 that each of the simplices appearing in $\mathcal{B}^{\circ k}[e_0, ..., e_n]$ is contained in one of the sets from $\mathcal{U}_\sigma$. But then $\mathcal{B}^{\circ k}(\sigma)$ is $\mathcal{U}$-small by naturality.

Next, using Proposition 9.1, let us show that $\mathcal{B}_\bullet^{\circ k} \simeq \mathrm{Id}$. Indeed, consider

$$\mathcal{T}^k := \sum_{i=0}^{k-1} \mathcal{T}\mathcal{B}^{\circ i}.$$

Then, we have

$$\partial\mathcal{T}^k + \mathcal{T}^k\partial = \sum \partial\mathcal{T}\mathcal{B}^{\circ i} + \mathcal{T}\mathcal{B}^{\circ i}\partial = \sum(\partial\mathcal{T} + \mathcal{T}\partial)\mathcal{B}^{\circ i} = \sum(\mathrm{Id} - \mathcal{B})\mathcal{B}^{\circ i} = \mathrm{Id} - \mathcal{B}^{\circ k}.$$

Thus, $\mathcal{T}^k : \mathrm{Id} \simeq \mathcal{B}^{\circ k}$ is a chain homotopy, which is natural by construction.

Finally, let us show that the induced map $\Theta : H_\bullet^{\mathcal{U}}(X) \to H_\bullet(X)$ is an isomorphism.

- **Injectivity** : Suppose $\Theta(\alpha) = 0$ for some $\alpha \in H_n^{\mathcal{U}}(X)$. Say, $\alpha = [a]$ for some n-cycle $a \in S_n^{\mathcal{U}}(X)$. Then, $a = \partial b$ for some $b \in S_{n+1}(X)$. By using Lemma 9.3 to each of the simplices appearing in b, we have some k such that $\mathcal{B}^{\circ k}(b)$ is $\mathcal{U}$-small. Then, $b - \mathcal{B}^{\circ k}(b) = \partial\mathcal{T}^k(b) + \mathcal{T}^k\partial b = \partial\mathcal{T}^k(b) + \mathcal{T}^k(a) \Rightarrow \partial b - \partial\mathcal{B}^{\circ k}(b) = 0 + \partial\mathcal{T}^k(a) \Rightarrow a = \partial b = \partial\big(\mathcal{T}^k(a) + \mathcal{B}^{\circ k}(b)\big)$. Now, by construction, $\mathcal{T}$ takes $\mathcal{U}$-small chain to $\mathcal{U}$-small ones, and hence, $\mathcal{T}^k(a) \in S_{n+1}^{\mathcal{U}}(X)$. Clearly $\mathcal{B}^{\circ k}(b) \in S_{n+1}^{\mathcal{U}}(X)$. Thus, a is an $\mathcal{U}$-small boundary. In other words, $\alpha = [a] = 0$ in $H_n^{\mathcal{U}}(X)$. This shows, Θ is injective.
- **Surjectivity** : Next, suppose $a \in S_n(X)$ is a cycle representing a class in $H_n(X)$. Then, there exists some k such that $\mathcal{B}^{\circ k}(a)$ is an $\mathcal{U}$-small n-chain. As $\mathcal{B}^{\circ k}$ is a chain map, it follows that $\mathcal{B}^{\circ k}(a)$ is a cycle. Also, $a - \mathcal{B}^{\circ k}(a) = \partial\mathcal{T}^k(a) + \mathcal{T}^k(\partial a) = \partial\mathcal{T}^k(a)$ implies that a is homologous to the cycle $\mathcal{B}^{\circ k}(a)$, which represents a class in $H_n^{\mathcal{U}}(X)$. Clearly, $\theta[\mathcal{B}^{\circ k}(a)] = [a]$, proving the surjectivity.

Thus, $\Theta : H_\bullet^{\mathcal{U}}(X) \to H_\bullet(X)$ is an isomorphism. □

Let us also show the same as above for a pair (X, A). Denote $\mathcal{U} \cap A := \{U \cap A \mid U \in \mathcal{U}\}$. Clearly, the interiors of sets of $\mathcal{U} \cap A$ covers A. Let us defien

$$S_\bullet^{\mathcal{U}}(X, A) := \frac{S_\bullet^{\mathcal{U}}(X)}{S_\bullet^{\mathcal{U}\cap A}(A)},$$

which is clearly a chain complex. Denote the homology as $H_\bullet^{\mathcal{U}}(X, A)$

Lemma 9.5: (*Barycentrinc Subdivision in Relative Homology*)

The inclusion map $\theta : S_\bullet^{\mathcal{U}}(X, A) \to S_\bullet(X, A)$ induces an isomorphism $\Theta : H_\bullet^{\mathcal{U}}(X, A) \to H_\bullet(X, A)$.

Proof : We have a commutative diagram of short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet^{\mathcal{U}\cap A}(A) \arrow[r] \arrow[d, "\theta_A"'] & S_\bullet^{\mathcal{U}}(X) \arrow[r] \arrow[d, "\theta_X"'] & S_\bullet^{\mathcal{U}}(X, A) \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & S_\bullet(A) \arrow[r] & S_\bullet(X) \arrow[r] & S_\bullet(X, A) \arrow[r] & 0
\end{tikzcd}$$

By Theorem 7.1, we have the commutative diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_n^{\mathcal{U}\cap A}(A) \arrow[r] \arrow[d, "H_n(\theta_A)"'] & H_n^{\mathcal{U}}(X) \arrow[r] \arrow[d, "H_n(\theta_X)"'] & H_n^{\mathcal{U}}(X, A) \arrow[r] \arrow[d, "H_n(\theta)"'] & H_{n-1}^{\mathcal{U}\cap A}(A) \arrow[r] \arrow[d, "H_{n-1}(\theta_A)"'] & H_{n-1}^{\mathcal{U}}(X) \arrow[r] \arrow[d, "H_{n-1}(\theta_X)"'] & \cdots \\
\cdots \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

By Lemma 9.4, 4-out-of-5 vertical arrows are isomorphisms. Hence, by the 5-lemma, it follows that $H_n(\theta)$ is an isomorphism as well. ☐

9.3 Excision in Singular Homology

We are now in a position to prove the excision theorem in singular homology.

Theorem 9.6: (*Excision*)

Given a pair (X, A) and a subset $B \subset A$, the inclusion

$$\iota : (X \setminus B, A \setminus B) \hookrightarrow (X, A)$$

induces an isomorphism in homology, provided $\overline{B} \subset \mathring{A}$ holds.

Proof : Let us get an equivalent statement first. Set

$$U := A, \quad V := X \setminus B.$$

Then, $\overline{B} \subset B \Rightarrow X \setminus \overline{B} \subset X \setminus B = V \Rightarrow X \setminus \overline{B} \subset \mathring{V}$. Since $\overline{B} \subset \mathring{A} = \mathring{U}$, it follows that the interiors of U, V cover X. Thus, we can now apply Lemma 9.4 for the collection $\mathcal{U} := \{U, V\}$.

Observe that $S_\bullet^{\mathcal{U}}(X) = S_\bullet(U) + S_\bullet(V)$, as an $\mathcal{U}$-small simplex is either contained in U or in V (and possibly in both). Also, $S_\bullet(U) \cap S_\bullet(V) = S_\bullet(U \cap V)$. Note that

$$\frac{S_n(V)}{S_n(U \cap V)} = \frac{S_n(V)}{S_n(U) \cap S_n(V)} \simeq \frac{S_n(U) + S_n(V)}{S_n(U)} = \frac{S_n^{\mathcal{U}}(X)}{S_n(U)}.$$

The isomorphism in the middle is the second isomorphism theorem. In particular, it is induced by the natural inclusion $S_n(V) \subset S_n(U) + S_n(V)$. Observe that in Lemma 9.4, the barycentric subdivision and the chain homotopy (and hence, their iterates), takes chains in $S_\bullet(A)$ to itself. In particular, passing to the quotient, we see that $\frac{S_\bullet^{\mathcal{U}}(X)}{S_\bullet(U)} \to \frac{S_\bullet(X)}{S_\bullet(U)}$ induces isomorphism in homology. Alternatively, observe that $S_\bullet^{\mathcal{U} \cap A}(A) = S_\bullet(A)$, and so, we can use Lemma 9.5 for $S_\bullet^{\mathcal{U}}(X, A) = \frac{S_n^\bullet(X)}{S_\bullet(U)}$. Hence, we have isomorphism $H_\bullet(V, U \cap V) \to H_\bullet(X, U)$.

Translating back to our original notaion, we have $V = X \setminus B, U \cap V = A \cap (X \setminus B) = A \setminus B$. Hence, we have the exicision isomorphism $H_\bullet(X \setminus B, A \setminus B) \to H_\bullet(X, A)$. ☐

9.4 Additivity of Singular Homology

As proved in Proposition 3.1, we immediately have that singular homology is finitely additve. But it is more than that!

Theorem 9.7: (*Homology of Path Components*)

Given a space X, we have $H_n(X) = \bigoplus_{P \in \pi_0(X)} H_n(P)$, where $\pi_0(X)$ is the set of path components of X. In particular, singular homology is additive (Definition 3.3).

Proof : Since Δ^n is path connected, and n-simplex is contained in a path component. Thus, we have a natural decomposion $S_n(X) = \bigoplus_{P \in \pi_0(X)} S_n(P)$. Clearly the boundary map restricts to each summand. Then, kernel and image splits as well. We have, $H_n(X) = \bigoplus H_n(P)$.

Given a disjoint union $X = \sqcup X_\alpha$, each X_α is union of path components. In particular, the above argument works and gives $H_\bullet(X) = \bigoplus H_\bullet(X_\alpha)$. This shows that singular homology is additive. $\square$

Thus, when computing the singular homology, we might as well assume that the space is path connected.

9.5 Hurewicz Homomorphism

Since $\Delta^1 = \{(t_0, t_1) \in \mathbb{R}^2 \mid t_0 + t_1 = 1, t_0 \geq 0, t_1 \geq 0\}$ is essentially the interval, a singular 1-simplex is nothing but a path in the space X. Given $\sigma : \Delta^1 \to X$, let us consider the path

$$\begin{aligned} P(\sigma) : [0,1] &\to X \\ t &\mapsto \sigma(1-t, t). \end{aligned}$$

Conversely, any path $\gamma : [0,1] \to X$ can be thought of as a 1-simplex $S(\gamma)(t_0, t_1) = \gamma(t_1)$. Clearly, $S \circ P = \mathrm{Id} = P \circ S$. The standard $\frac{1}{2}$-concatenation of paths then gives a concatenation of 1-simplices as $\sigma, \tau : \Delta^1 \to X$, with $\sigma(0,1) = \tau(1,0)$ via $\sigma \star \tau = S(P(\sigma) \star P(\tau))$. Explicitly, we have

$$(\sigma \star \tau)(t_0, t_1) = \begin{cases} \sigma(2t_0 - 1, 2t_1), & 0 \leq t_1 \leq \frac{1}{2} \\ \tau(2t_0, 2t_1 - 1), & \frac{1}{2} \leq t_1 \leq 1. \end{cases}$$

We define a 2-simplex $\omega : \Delta^2 \to X$ by

$$\omega(t_0, t_1, t_2) = (\sigma \star \tau)\left(t_0 + \frac{t_1}{2}, \frac{t_1}{2} + t_2\right).$$

Then, one can check that

$$\partial \omega = \omega|_{t_0=0} - \omega|_{t_1=0} + \omega|_{t_2=0} = \tau - \sigma \star \tau + \sigma.$$

Indeed, for $t_0 = 0$, we have $t_1 + t_2 = 1 \Rightarrow \frac{t_1}{2} + t_2 = 1 - \frac{t_1}{2}$. Clearly, $1 - \frac{t_1}{2} \leq \frac{1}{2} \Rightarrow t_1 \geq 1$ is not possible. Thus,

$$\omega|_{t_0=0} = (\sigma \star \tau)\left(\frac{t_1}{2}, 1 - \frac{t_1}{2}\right) = \tau(t_1, 1 - t_1) = \tau(t_1, t_2).$$

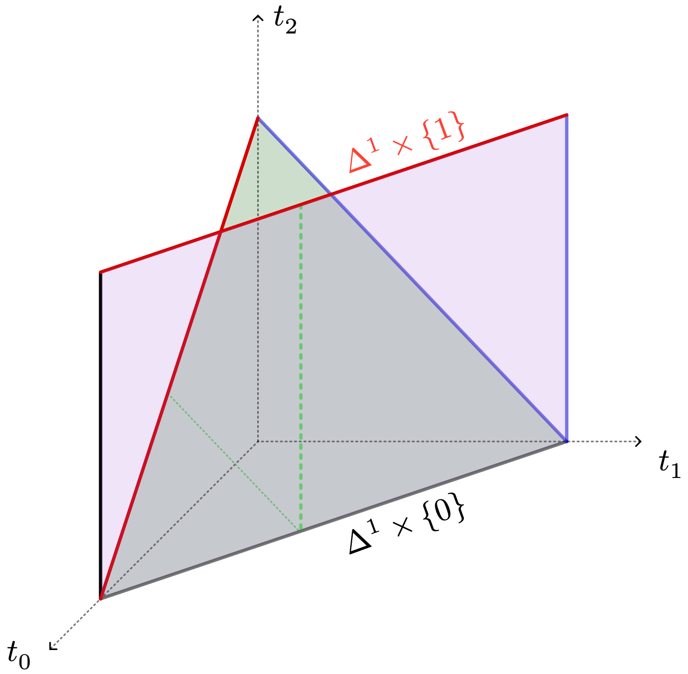
Similarly, $\omega|_{t_2=0} = \sigma$. That $\omega|_{t_1=0} = \sigma \star \tau$ is evident from the equation. But then we have

$$[\sigma \star \tau] = [\sigma] + [\tau] \in \frac{S_1(X)}{B_1(X)},$$

where $B_1(X) = \mathrm{im}(\partial : S_2(X) \to S_1(X))$.

Now, consider the quotient map

$$\begin{aligned} q : \Delta^1 \times [0,1] &\to \Delta^2 \\ (t_0, t_1, s) &\mapsto (t_0, (1-s)t_1, st_1). \end{aligned}$$



The identification $\Delta^1 \times [0,1] \to \Delta^2$.

Suppose $h : \Delta^1 \times [0,1] \to X$ is an end-point preserving homotopy between two paths. It follows that h passes to the quotient, and gives a 2-simplex, say, α. Moreover,

$$\partial\alpha = c - h_1 + h_0,$$

where c is the constant 1-simplex given by restricting h to $(\Delta^1 \circ d_0) \times [0,1]$, and $h_0 := h|_{\Delta^1\times\{0\}}, h_1 := h|_{\Delta^1\times\{1\}}$ are the paths. Since a constant 1-simplex is clearly a boundary, it follows that $[h_0] = [h_1]$ in $\frac{S_1(X)}{B_1(X)}$.

Now, clearly a loop is an 1-cycle, and thus gives a homology class. Moreover, homotopic loops (with fixed basepoint) gives the same homology class. Thus, for any fixed $x_0 \in X$, we have obtained a well-defined map

$$\begin{aligned} \eta' : \pi_1(X, x_0) &\to H_1(X) \\ [\sigma] &\mapsto [\sigma], \end{aligned}$$

which is moreover a group homomorphism. Recall that in general $\pi_1(X, x_0)$ is non-Abelian, whereas $H_1(X)$ is always Abelian.

Definition 9.8: (*Abelianization*)

Given a group G, the *commutator group* $[G,G]$ is defined as the normal subgroup generated by all commutators $[x,y] = xyx^{-1}y^{-1}$ for all $x, y \in G$. The group $G^{\mathrm{ab}} := \frac{G}{[G,G]}$ is called the *Abelianization* of the group G.

Exercise 9.9: (*Universal Property of Abelianization*)

Verify that Abelianization is the left adjoint to the forgetful functor $\mathrm{Ab} \to \mathrm{Grp}$.

In particular, $\eta' : \pi_n(X, x_0) \to H_n(X)$ naturally defines a (unique) group homomorphism

$$\eta : \pi_1(X, x_0)^{\mathrm{ab}} \to H_1(X).$$

This map is called the *Hurewicz homomorphism*.

Remark 9.10:

More generally, Hurewicz homomorphism can be defined in a similar way as $\eta : \pi_n(X, x_0) \to H_n(X)$, and even as $\pi_n(X, A, a_0) \to H_n(X, A)$ for $a_0 \in A \subset X$. Hurewicz homomorphism is natural, and commutes with the suspension map.

Day 10 : 17th February, 2026

`Hurewicz isomorphism – reduced singular homology – cofibration – signular homology of cofiber – cellular decomposition`

10.1 Hurewicz Isomorphism

Let us now show that the Hurewicz map is an isomorphism for path connected spaces.

Theorem 10.1: (*Hurewicz Isomorphism*)

If X is a path connected space with a basepoint x_0, then the Hurewicz map $\eta : \pi_1(X, x_0)^{\mathrm{ab}} \to H_1(X)$ is an isomorphism.

Proof : For each $x \in X$, fix a path $u_x : x_0 \to x$, with $\overline{u}_x$ as the reversed pat $x \to x_0$. Assume u_{x_0} to be the constant path c_{x_0}. Next, for any $\sigma : \Delta^1 \to X$, denote $\sigma_0 = \sigma d_0 = \sigma(0,1)$ and $\sigma_1 = \sigma d_1 = \sigma(1,0)$. Then, $\left(u_{\sigma_1} \star \sigma\right) \star \overline{u}_{\sigma_0}$ is a loop at x_0. Let us define the linear map

$$\begin{aligned} \xi' : S_1(X) &\to \pi_1(X, x_0)^{\mathrm{ab}} \\ \sigma &\mapsto \left[\left(u_{\sigma_1} \star \sigma\right) \star \overline{u}_{\sigma_0}\right]. \end{aligned}$$

This is possible since $\pi_1(X, x_0)^{\mathrm{ab}}$ is Abelian. Now, for any 2-simplex $\omega : \Delta^2 \to X$, denote the faces $\omega_j = \omega d_j : \Delta^1 \to X$, and then observe that $\omega_2 \star \omega_0 \simeq \omega_1$. Also, it is easy to see that

$$\begin{aligned} \xi'(\omega_1) &= \left[\left(u_{\omega_{11}} \star \omega_1\right) \star \overline{u}_{\omega_{10}}\right] = \left[\left(u_{\omega_{11}} \star (\omega_2 \star \omega_0)\right) \star \overline{u}_{\omega_{10}}\right] \\ &= \left[\left(\left(u_{\omega_{21}} \star \omega_2\right) \star \overline{u}_{\omega_{20}}\right) \star \left(\left(u_{\omega_{01}} \star \omega_0\right) \star \overline{u}_{\omega_{00}}\right)\right], \text{since } \overline{u}_{\omega_{20}} \star u_{\omega_{01}} \simeq c_{x_0} \\ &= \left[\left(u_{\omega_{21}} \star \omega_2\right) \star \overline{u}_{\omega_{20}}\right] + \left[\left(u_{\omega_{01}} \star \omega_0\right) \star \overline{u}_{\omega_{00}}\right] \\ &= \xi'(\omega_2) + \xi'(\omega_0) \end{aligned}$$

Hence, $\xi'(\partial \omega) = 0$ which implies that ξ' factors as a map $\xi : \frac{S_1(X)}{B_1(X)} \to \pi_1(X)^{\mathrm{ab}}$. Clearly, $\xi \circ \eta = \mathrm{Id}$, and hence, η is injective. To show that η is surjective, let $\alpha \in H_1(X)$ be represented by some 1-cycle $a := \sum n_\sigma \sigma$ with $\partial a = 0$. By linearity, we have a map $u : S_0(X) \to S_1(X)$. Then, for any $\sigma : \Delta^1 \to X$, we have $u(\partial \sigma) = u(\sigma d_0 - \sigma d_1) = u_{\sigma_0} - u_{\sigma_1} = u_{\sigma_0} + \overline{u}_{\sigma_1}$. In particular, we have

$$\eta\left(\sum n_\sigma \xi(\sigma)\right) = \sum n_\sigma\left(\left[u_{\sigma_1}\right] + [\sigma] + \left[\overline{u}_{\sigma_1}\right]\right) = \sum n_\sigma([\sigma] + [u\partial\sigma]) = \alpha + u([\partial a]) = \alpha,$$

as $\partial a = 0$. This implies η is surjective. □

Exercise 10.2:

Compute the homology groups of $S_g := \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g}$, the torus with g-holes.

Hint : Recall, $\pi_1(S_g) = \langle a_1, b_1, \cdots, a_g, b_g \mid \prod [a_i b_i] = 1\rangle$ by Van Kampen's theorem.

Remark 10.3: (*The Hurewicz Isomorphism*)

The more general Hurewicz isomorphism states the following. Suppose X is a $(n-1)$-connected, i.e, $\pi_i(X) = 0$ for $i < n$. Then, the Hurewicz map $h : \pi_n(X) \to H_n(X)$ is an isomorphism. As the Hurewicz map commutes with the suspension map, it follows moreover that $h : \pi_{n+1}(X) \to H_{n+1}(X)$ is an epimorphism. The isomorphism lets us inductively show that

$$\pi_i(S^n) = \begin{cases} 0, & i < n \\ \mathbb{Z}, & i = n. \end{cases}$$

In general, computing the higher homotopy groups of the sphere is a nontrivial problem.

10.2 Reduced Singular Homology

Suppose X is a space, and $c : X \to \star$ is the constant map. Recall, the reduced homology is defined as the kernel $\tilde{H}_n(X) := \ker\left(H_n(X) \xrightarrow{H_n(c)} H_n(\star)\right)$. Let us interpret it in a different way for singular homology. Consider the chain map $\varepsilon_\bullet : S_\bullet(X) \to S_\bullet(\star)$ given by $\varepsilon_n = 0$ for $n \neq 0$, and

$$\varepsilon_0\left(\sum n_\sigma \sigma\right) = \left(\sum n_\sigma\right)\sigma_0,$$

where $\sigma_0 : \Delta^0 \to \star$ is the unique 0-simplex.

Exercise 10.4:

Using Proposition 6.10, verify that $H_\bullet(c) = H_\bullet(\varepsilon) : B_\bullet(X) \to H_\bullet(\star)$.

Let us now define a chain homotopy $s_n : S_n(X) \to S_{n+1}(\star)$ by setting $s_n(\sigma) = (-1)^n \sigma_{n+1}$ on generators and extending linearly. Here, $\sigma_{n+1} : \Delta^n \to \star$ is the unique $(n+1)$-simplex. Note that

$$\begin{aligned} s\partial(\sigma) + \partial s(\sigma) &= s\left(\sum_0^n (-1)^i \sigma d_i\right) + (-1)^n \partial \sigma_{n+1} \\ &= (-1)^{n-1} \sum_0^n (-1)^i \sigma_n + (-1)^n \sum_0^{n+1} (-1)^i \sigma_n \\ &= (-1)^n \left[-\sum_0^n (-1)^i \sigma_n + \sum_0^{n+1} (-1)^i \sigma_n\right] \\ &= (-1)^n (-1)^{n+1} \sigma_n = -\sigma_n = \varepsilon_n(\sigma) - S_n(c)(\sigma). \end{aligned}$$

Then, $H_\bullet(\varepsilon) = H_\bullet(c)$ follows.

Definition 10.5: (*Augmented Singular Chain Complex*)

Given a space X, the *augmented singular chain complex* is defined as the chain complex $S_2(X) \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, where $\varepsilon(\sum n_\sigma \sigma) = \sum n_\sigma$. This map ε is called the *augmentation map*.

Observe that considering $\mathbb{Z}$ as a chain complex concentrated at degree 0, we can look at the augmentation map as a chain map by setting it zero every nozero degree.

Exercise 10.6:

Verify that the reduced singular homology is the homology of the augmented singular chain complex.

10.3 Cofibration and Homology

Let us recall the notion of cofibration.

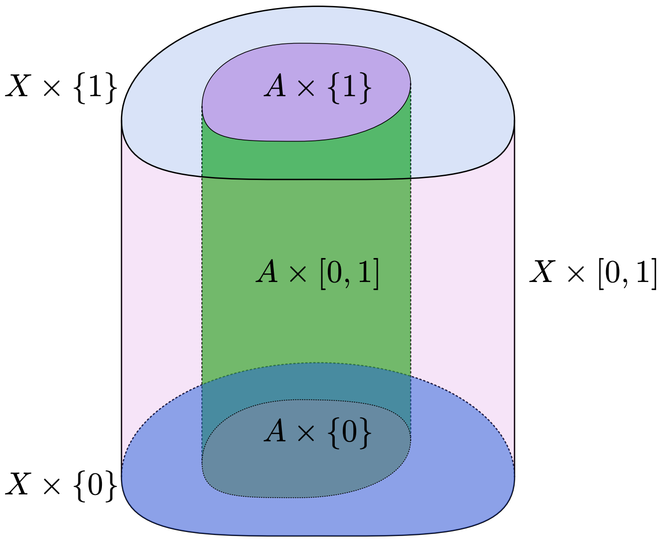
Definition 10.7: (*Cofibration*)

A map $f : A \to X$ is called a *cofibration* if it has the *homotopy extension property* against any space, i.e, given a homotopy $h : A \times [0,1] \to Y$ and a map $F : X \to Y$ with $F \circ f = h|_{A\times 0}$, there exists a homotopy $H : X \times [0,1] \to Y$ such that $H|_{X\times\{t\}} \circ f = h|_{X\times\{t\}}$. In other words, the commuting diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & Y^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "F"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

admits a solution H which commutes with other maps.

The homotopy extension is easier to visualize if we only consider an inclusion map $\iota : A \hookrightarrow X$.



Homotopy extension for $\iota : A \hookrightarrow X$

Most of the times, we will only concern ourselves about an inclusion being a cofibration. In fact, given any function $f : A \to X$, consider the *mapping space*

$$M_f := X \cup_f (A \times [0,1]),$$

where we identify $(a,0) \sim f(a)$. It is easy to see that M_f (strongly) deformation retracts onto X by collapsing $A \times [0,1]$. On the other hand, we have the inclusion $\iota : A \hookrightarrow M_f$ given by $a \mapsto (a,1)$. We then have a *homtopy* commutative diagram

$$\begin{tikzcd} & M_f \\ A \arrow[ur, hook, "\iota"] \arrow[r, "f"'] & X \arrow[u, hook, "j"', "\text{s.d.r.}"] \end{tikzcd}$$

Exercise 10.8: (*Mapping Cylinder is a Cofibration*)

Given any map $f : A \to X$, show that $\iota : A \to M_f$ is a cofibration.

Let us now observe a few easy topological consequences about cofibraions.

Lemma 10.9: (*Composition of Cofibration*)

Composition of cofibrations is again a cofibration.

__Proof__ : Suppose $f : X \to Y, g : Y \to Z$ are cofibrations. Let us consider the homotopy extension diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"] \arrow[dd, bend right, "g \circ f"'] & W^I \arrow[dd, "\mathrm{ev}_0"] \\ Y \arrow[d, "g"] & \\ Z \arrow[r, "F"] & W \end{tikzcd}$$

We first get a lift $H_1 : Y \to W^I$ such that $\mathrm{ev}_0 \circ H_1 = F \circ g$ and $H_1 \circ f = h$. Then, we get another lift $H_2 : Z \to W^I$ such that $H_2 \circ g = H_1$ and $\mathrm{ev}_0 \circ H_2 = F$. Then, we have $H_2 \circ (g \circ f) = H_1 \circ f = h$, and $\mathrm{ev}_0 \circ H_2 = F$. Thus, H_2 is the required extension, proving that $g \circ f$ is a cofibration. $\square$

Next, we consider pushout of a cofibration. Let us now recall the defintion first.

Definition 10.10: (*Pushout of Spaces*)

Given continuous maps $X \xleftarrow{f} A \xrightarrow{g} Y$, the *pushout* is defined via the following universal property.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "p"] \arrow[ddr, bend left, "\varphi"] & \\ X \arrow[r, "q"'] \arrow[drr, bend right, "\psi"'] & Z \arrow[dr, dashed, "\exists! h"] & \\ & & W \end{tikzcd}$$

Given any map $\varphi : Y \to W, \psi : X \to W$ with $\varphi \circ g = \psi \circ f$, there exists a unique map $h : Z \to W$ such that $h \circ p = \varphi, h \circ q = \psi$.

Explicitly, one can construct the pushout as the identification space $Z = (X \sqcup Y)/_{\sim}$, where $\sim$ is the smallest equivalence relation such that $f(a) \sim g(a)$ for $a \in A$.

Example 10.11:

Given a map $f : A \to X$, the mapping cone $C(f)$ is the pushout of the diagram $CA \hookleftarrow A \xrightarrow{f} X$. The pushout of the sphere as the boundary $D^n \hookleftarrow S^{n-1} \hookrightarrow D^n$ is the sphere S^n.

Lemma 10.12: (*Pushout of a Cofibration*)

Suppose $f : A \to X$ is a cofibration. Then, any pushout of f is again a cofibration.

Proof : Consider a pushout diagram and a homtopy extension problem for the middle arrow

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "f"'] & Z \arrow[r, "h"] \arrow[d, "g"] & Y^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "q"'] & W \arrow[r, "F"'] & Y \end{tikzcd}$$

As f is a cofibration, we have a lift

$$\begin{tikzcd} A \arrow[r, "h \circ p"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & Y^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "F \circ q"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

Next, we use the pushout to get a unique map

$$
\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "f"'] & Z \arrow[d, "g"] \arrow[ddr, bend left, "h"] & \\
X \arrow[r, "q"] \arrow[drr, bend right, "G"'] & W \arrow[dr, dashed, "H"] & \\
 & & Y^I
\end{tikzcd}
$$

We claim that H is the solution in

$$
\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d, "g"'] & Y^I \arrow[d, "\mathrm{ev}_0"] \\
W \arrow[r, "F"'] \arrow[ur, dashed, "H"] & Y
\end{tikzcd}
$$

We alredy have $H \circ g = h$ from the pushout diagram. Next, we again look at the pushout

$$
\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "f"'] & Z \arrow[d, "g"] \arrow[ddr, bend left, "\mathrm{ev}_0 \circ h"] & \\
X \arrow[r, "q"] \arrow[drr, bend right, "\mathrm{ev}_0 \circ G"'] & W \arrow[dr, dashed, "\mathrm{ev}_0 \circ H", "F"'] & \\
 & & Y
\end{tikzcd}
$$

It follows that both $\mathrm{ev}_0 \circ H$ and F solves the pushout diagram. Hence, by the uniqueness, we have $\mathrm{ev}_0 \circ H = F$. This proves that H is a homotopy extension, as required. Thus, the pushout $g : Z \to W$ is a cofibration. $\square$

Lemma 10.13: (*Contractible Cofibered Subspace*)

Suppose $A \subset X$ is contractible, and the inclusion $\iota : A \hookrightarrow X$ is a cofibration. Then, the quotient $q : X \to X/A$ is a homotopy equivalence.

Proof : Since A is contractible, there exists a homotopy $h : A \times [0,1] \to A$ such that $h|_{A\times\{0\}} = \mathrm{Id}_A$ and $h|_{A\times\{1\}} = \{a_0\}$ for some $a_0 \in A$. As $\iota : A \hookrightarrow X$ is a cofibration, we have a homotopy extension, $H : X \times [0,1] \to X$ with $H_0 := H|_{X\times\{0\}} = \mathrm{Id}_X$. Observe that $H_1 := H|_{X\times\{1\}} : X \to X$ maps A to a point $\{a_0\}$, and thus, we have a map $r : X/A \to X$ such that $r \circ q = H_1$. By construction, $H : r \circ q \simeq \mathrm{Id}_X$ is a homotopy. On the other hand, the homotopy $H_t : X \to X$ maps $H_t(A) \subset A$, and hence, passing to quotient we have the homotopy $\tilde{H}_t : X/A \to X/A$. Clearly $\tilde{H}_0 = \mathrm{Id}_{X/A}$. Also, $qr([x]) = qrq(x) = qH_1(x) = \tilde{H}_1 q(x) = \tilde{H}_1([x])$, i.e, $\tilde{H}_1 = q \circ r$. Thus, $\tilde{H} : q \circ r \simeq \mathrm{Id}_{X/A}$ is a homotopy. This proves that $q : X \to X/A$ is a homotopy equivalence. $\square$

Theorem 10.14: (*Homology of Quotient by a Cofibered Subspace*)

Suppose $A \subset X$ is a subspace such that the inclusion $\iota : A \hookrightarrow X$ is a cofibration. Then, the quotient map $q : X \to X/A$ induces an isomorphism $H_n(X, A) \to H_n(X/A, \star) \cong \tilde{H}_n(X/A)$.

Proof : Conider the mapping $C(\iota)$ which is given as the pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "\iota"'] & CA \arrow[d, "j"] \\ X \arrow[r] & C(\iota) \end{tikzcd}$$

As ι is a cofibration, by Lemma 10.12, we have j is a cofibration. But CA is contractible. Hence, by Lemma 10.13, we have $q : C(\iota) \to C(\iota)/CA = X/A$ is a homotopy equivalence. Now, the inclusion map $(X, A) \to (C(\iota), CA)$ induces an isomorphism in homology since we can first excise out the coning point, and then apply a deformation retract. Also, $(C(\iota), CA) \to (X/A, \star)$ induces and isomorphism as the quotient map is a homotopy equivalence. Hence, composing, it follows that the quotient $(X, A) \to (X/A, \star)$ induces an isomorphism in homology.

The isomorphism $H_n(X/A, \star) \cong \tilde{H}_n(X/A)$ follows from applying the quotient map $(Y, \emptyset) \to (Y/\emptyset, \emptyset/\emptyset) = (Y, \star)$ for any space Y. □

Example 10.15: (*CW pair*)

Let X be a CW-complex, and $A \subset X$ a subcomplex. Then the inclusion $A \hookrightarrow X$ is a cofibration. In fact, if X is a locally finite CW complex (i.e, only finitely many cells of each dimension), then we can drop the assumption that A is a sub-complex and only require that $A \subset X$ is closed and a CW complex on its own.

10.4 Cellular Decomposition

Let us recall the definition of a (relative) CW complex. Given a pair (X, A), we say X is obtained from A by *attaching an n-cell* if there is a pushout diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "\varphi"] \arrow[d, hook] & A \arrow[d, hook, "\iota"] \\ D^n \arrow[r, "\Phi"'] & X \end{tikzcd}$$

Now, the inclusion $S^{n-1} \hookrightarrow D^n$ is trivially a cofibration. Indeed, we have a retract of $D^n \times [0,1]$ to $D^n \times \{0\} \cup S^{n-1} \times [0,1]$ obtained by "pushing everything radially from the center of $D^n \times \{1\}$". Composing with this retract, we can always get the desired homotopy extension from D^n, while keeping any homotopy on S^{n-1} fixed. Then, it follows from Lemma 10.12 that $\iota : A \hookrightarrow X$ is also a cofibration. The map $\varphi : S^{n-1} \to A$ is called the *attaching map* for the n-cell, and Φ is called the *characteristic map*. Note that A is necessarily closed in X, and Φ is a homeomorphism of the interior $\mathring{D}^n$ onto $X \setminus A$. Indeed, X is homeomorphic to the mapping cone $C(\varphi) = A \cup_\varphi \underbrace{C(S^{n-1})}_{D^n}$. In other words, $\Phi : (D^n, S^{n-1}) \to (X, A)$ is a *relative homeomorphism*, i.e, the induced map

$$\tilde{\Phi} : D^n/S^{n-1} \to X/A$$

is a homeomorphism.

Example 10.16:

The n-sphere is obtained from the point $\{\star\}$ by attaching an n-cell, where the attaching map is the constant map. On the other hand, we can also realize the n-sphere from the n-disc by attaching a single n-cell, where the attaching map is the identity map on the boundary. The real projective space $\mathbb{RP}^n$ is obtained from $\mathbb{RP}^{n-1}$ by attaching an n-cell. The complex projective space $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a $2n$-cell.

Remark 10.17:

Suppose X is a T_2-space with a closed subspace $A \subset X$. Let $\Phi : D^n \to X$ be continuous map, which restricts to a homeomorphism of $\mathring{D}^n$ onto $X \setminus A$. Then, X is obtained from A by attaching an n-cell, via the attaching map $\varphi := \Phi|_{S^{n-1}=\partial D^n}$.

More generally, we can attach multiple n-cells at once. Given a pair (X, A), we say X is obtained from A by *attaching n-cells* if there exists a pushout diagram

$$\begin{tikzcd}
\bigsqcup_{\alpha\in J} S^{n-1}_\alpha \arrow[r, "\varphi"] \arrow[d, hook] & A \arrow[d, hook, "\iota"] \\
\bigsqcup_{\alpha\in J} D^n_\alpha \arrow[r, "\Phi"'] & X
\end{tikzcd}$$

Here, J is an arbitrary (possibly empty) indexing set. The restriction $\varphi_\alpha := \varphi|_{S^{n-1}_\alpha}$ is called the *attaching map* for the n-cell $E^n_\alpha := \mathring{D}^n_\alpha$. It follows that A is closed in X, and Φ induces a homeomorphism of $\sqcup E^n_\alpha$ with $X \setminus A$, in particular $X \setminus A$ is a union of components, each being an n-cell. In case $J = \emptyset$, we have $X = A$.

Exercise 10.18:

Suppose X is obtained from A by attachin n-cells, with attaching maps $\{\varphi_\alpha : S^{n-1}_\alpha \to A\}_{\alpha\in J}$. Show that a map $f : A \to Y$ extends to a map $F : X \to Y$ if and only if each $f \circ \varphi_\alpha$ is null-homotopic (whence the extension is determined by the choice of null-homotopies).

Definition 10.19: (*CW Complex*)

Let $A \subset X$ be given. A *CW decomposition* of the pair (X, A) is a sequence of subspaces

$$A = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X,$$

such that the following holds.

1. $X = \cup_{i \geq 0} X^i$, with the colimit topology (i.e, a map $f : X \to Y$ is continuous if and only if $f|_{X^n} : X^n \to X$ is continuous for all n).
2. For each $n \geq 0$, the space X^n is obtained from X^{n-1} by attaching n-cells.

The pair (X, A) equipped with a CW decomposition is called a *relative CW complex*. When $A = \emptyset$, we say X is a *CW complex*. The space X^n is called the *n-skeleton* of the CW complex. We say (X, A) is a *finite CW complex* (resp. *countable CW complex*) of there are only finitely (resp. countably) many cells attached. We say (X, A) is a *locally finite CW complex* if for each n, only finitely many n-cells are attached.

Exercise 10.20:

Suppose (X, A) is a relative CW complex. Show that the inclusion $A \hookrightarrow X$ is a cofibration. More generally, given a CW complex X, and a subcomplex $Y \subset X$ (i.e, Y is closed and consists of cells of X), the inclusion $Y \hookrightarrow X$ is a cofibration.

Hint : Use Lemma 10.12 to the cofibration $\sqcup S^{n-1} \hookrightarrow \sqcup D^n$, and observe that the composition of cofibrations is again a cofibration (Lemma 10.9). Inductively extend the homotopy $X^n \times [0, 1]$ to $X^{n+1} \times [0, 1]$. The final homotopy is continuous as the topology on X is a colimit topology.

In the above definition, there is no restriction on the topology on A. If A is a T_2-space, then X is a T_2-space, moreover the topology on X is given by the colimit topology with respect to the family consisting of A and the closure of cells. This leads to the following definition.

Definition 10.21: (*Whitehead Complex*)

A *Whitehead complex* is a space X, along with a *cell decomposition* $\{e_\lambda \mid \lambda \in \Lambda\}$, where each e_λ is homeomorphic to $\mathring{D}^n$ for some n, such that the following holds.

1. X is a T_2-space.
2. For each n-cell, there is a map $\Phi_\lambda : D^n \to X$ such that $\Phi|_{\mathring{D}^n} : \mathring{D}^n \to e_\lambda$ is a homeomorphism, and Φ maps the boundary ∂D^n into X^{n-1}, which is the union of cells of dimension $\leq n - 1$.
3. (**C**losure Finiteness) The closure $\overline{e}_\lambda$ of each cell intersects only a finitely many cells.
4. (**W**eak Topology) The topology on X is induced by the colimit topology of the family $\{\overline{e}_\lambda \mid \lambda \in \Lambda\}$, i.e, $U \subset X$ is open if and only if $U \cap \overline{e}_\lambda$ is open for all $\lambda \in \Lambda$.

It follows that any Whitehead complex carries a CW decomposition, and conversely, a CW complex is a Whitehead complex.

Remark 10.22:

Note that the phrase "X is a CW complex" technically means that "X is a space equipped with a CW decomposition, and the corresponding colimit topology". There are spaces which does not admit any CW decomposition, e.g., the Hawaiian earring. To see this, one first proves that a space admitting a CW decomposition must be locally contractible, but Hawaiian earring is not.

Finally, let us understand cell decomposition on a product of CW complexes. Suppose X, Y are given are CW complexes, with $\{e_\alpha^n, \varphi_\alpha\}$ and $\left\{f_\beta^n, \psi_b\right\}$ as the respective collection of cells and attaching maps. Then, on $X \times Y$ we have a cell decomposition given by $\left\{e_\alpha^n \times f_\beta^n, \varphi_\alpha \times \psi_b\right\}$. In particular, we have the n-skeleton $(X \times Y)_n = \cup_{i+j=n} X_i \times Y_j$. Note that the weak topology on $X \times Y$ generated by the skeletons maybe strictly finer than the product topology! The two topologies match if any of the following holds true.

- Either of X or Y is compact (or more generally, locally compact). Note, X, Y being CW complexes, is already T_2.
- If both X and Y have countably many cells, i.e, they are countable CW complexes.

In general, the weak topology on $X \times Y$ matches with the *compactly generated topology* on $X \times Y$.

Example 10.23: (*CW decomposition of $S^p \times S^q$*)

The sphere S^p can be constructed using one 0-cell e^0 and one p-cell e^p. Similarly, S^q can be constructed using one 0-cell f^0 and one q-cell f^q. Then, on $S^p \times S^q$ we have a cell decomposition:

0-Cells	**p-Cells**	**q-Cells**	**$p+q$-Cells**
$e^0 \times f^0$	$e^p \times f^0$	$e^0 \times f^q$	$e^p \times f^q$

The attaching map $\omega : S^{p+q-1} \to S^p \vee S^q$ for the top cell represents the *universeal Whitehead bracket* : given maps $[f] \in \pi_p(X), [g] \in \pi_q(X)$, this is an element $[f, g] \in \pi_{p+q-1}(X)$ represented by the composition

$$S^{p+q-1} \xrightarrow{\omega} S^p \vee S^q \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X$$

Day 11 : 20th February, 2026

`cellular maps — cellular approximation theorem — weak homotopy equivalence — Whitehead theorem — connectivity and compressibility`

11.1 Cellular Map and Cellular Approximation

Definition 11.1: (*Cellular Map*)

Let $(X, A), (Y, B)$ be two relative CW complexes. A map $f : (X, A) \to (Y, B)$ is called a *cellular map* if $f(X^n) \subset Y^n$ holds for each $n \geq -1$.

Given a CW complex X, a *subcomplex* $A \subset X$ is a subspace which consists of a sub-collection of cells of X. In particular, a subcomplex $A \subset X$ is a CW complex on its own, and the inclusion map $A \hookrightarrow X$ is cellular. We have the following important theorem, proof of which follows from inductively constructing the map cell-by-cell.

Theorem 11.2: (*Cellular Approximation Theorem*)

Let X, Y be CW complexes. Then, any map $f : X \to Y$ is homotopic to a cellular map $g : X \to Y$. Suppose $B \subset X$ is a subcomplex. If $f|_B : B \to Y$ is already cellular, then the homotopy $f \simeq g$ can be chosen to be relative to B, i.e, the homotopy stays constant on B.

Let us see an important consequence of the cellular approximation theorem.

Proposition 11.3: (*Maps $S^n \to S^{n+k}$ are Null-homotopic for $k > 0$*)

Any map $f : S^n \to S^{n+k}$ is null-homotopic provided $k > 0$. Consequently, $\pi_n(S^{n+k}) = 0$ for $k > 0$.

<u>Proof</u> : Let $f : S^n \to S^{n+k}$ be given. Fix basepoint $x_0 \in S^n$, and $y_0 = f(x_0) \in S^{n+k}$. Fix CW structure on S^k (resp. on S^{n+k}) consisting of a single 0-cell x_0 (resp. y_0), and a single n-cell (resp. $(n + k)$-cell). By Theorem 11.2, f is homotopic relative to the basepoint to a cellular map $g : S^n \to S^{n+k}$. In the CW structure considered, the n-skeleton of S^n is S^n itself, whereas that of S^{n+k} consists of the 0-cell $\{y_0\}$ since $n + k > n$. But then being cellular, we have $g(S^n) = \{y_0\}$. Hence, f is null-homtopic. Since it is a base-point preserving homotopy, it follows that $\pi_n(S^{n+k}) = 0$ for $k > 0$. □

Exercise 11.4: (*Homotopy Groups of Infinite Sphere*)

Define the infinite sphere S^∞ as the union of

$$S^0 \hookrightarrow S^1 \hookrightarrow \dots S^n \hookrightarrow S^{n+1} \hookrightarrow \dots,$$

where S^{n+1} is obtained from S^n by attaching two $(n + 1)$-cells along the equator. This makes S^∞ into a CW complex, with two n-cells in each dimension. Show that $\pi_k(S^\infty) = 0$ for all $k \geq 0$.

11.2 Weak Homotopy Equivalence

Definition 11.5: (*Weak Homotopy Equivalence*)

A map $f : X \to Y$ is called a *weak homotopy equivalence* if

- $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is group isomorphism for each $n \geq 1$ and each $x \in X$, and
- $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.

Recall that the 0^{th} homotopy "group" $\pi_0(X)$ is the set of all path components of X, in general there is no group structure on it. Moreover, $\pi_0(X)$ is independent (i.e, defined up to set bijection) of choice of basepoint. The condition $\pi_0(f)$ is a bijection essentially means that f induces a bijection between the path components. Let us now see an example of an *weak* homotopy equivalence, which cannot be a homotopy equivalence.

Example 11.6: (*Pseudocircle or Digital Circle*)

Consider the four point space $X = \{N, E, W, S\}$ with the following topology

$$\{\emptyset, \{N\}, \{S\}, \{N, S\}, \{N, E, S\}, \{N, W, S\}, X\}.$$

This space is *not* T_2 as E and W cannot be separated by open sets (in fact, it is T_0 but not T_1). Let us now consider a map $f : S^1 \to X$ which maps the open upper semicirlce (resp. lower semicircle) to N (resp. to S), and the two remaining points to E and W respectively. Clearly, f is a continuous map (Check!). In fact, X is obtained from S^1 as the identification space which maps the two open semicircles to two distinct points, and f is precisely the quotient map.

Surjectivity of f shows that X is a path connected space. Moreover, one can construct a contractible universal cover of X, and show that $\pi_1(X) = \mathbb{Z}$, $\pi_k(X) = 0$ for $k > 1$. It follows that $f : S^1 \to X$ is a weak homotopy equivalence. The space X is called the *pseudocircle*.

Note that any $g : X \to S^1$ is necessarily a constant map, as the image must be path connected and can have at most finitely many points. Thus, f (or any other map $S^1 \to X$) cannot have a homotopy inverse since a constant map $S^1 \to S^1$ is not homotopic to the identity.

Thus, we cannot in general invert (even up to homotopy) a weak homotopy equivalence. But when the space is CW, we can!

Theorem 11.7: (*Whitehead Theorem*)

Let X, Y be CW complexes. Then, any weak homotopy equivalence $f : X \to Y$ is actually a homotopy equivalence.

The next example says that having isomorphic homotopy groups is not sufficient for being homotopy equivalent (or in fact even being weak homotopy equivalent). It assumes some results that we shall see later.

Example 11.8: (*Spaces with Isomorphic Homotopy Groups*)

Consider the spaces $Y_1 = \mathbb{RP}^2 \times S^3$ and $Y_2 = S^2 \times \mathbb{RP}^3$. It is easy to see that $X = S^2 \times S^3$ is a double cover of both Y_i, and moreover, Y is an universal cover (being simply connected). Then, $\pi_1(Y_i) = \mathbb{Z}_2$. On the other hand, from the long exact sequence in homotopy groups for the fibration $\mathbb{Z}_2 \hookrightarrow X \to Y_i$ it follows that $\pi_n(X) = \pi_n(Y_i)$ for $n \geq 2$. Thus, $\pi_n(Y_1) = \pi_n(Y_2)$ for all n.

On the other hand, we can compute homology with $\mathbb{Q}$-coefficients : $H_2(Y_1; \mathbb{Q}) = 0$ and $H_2(Y_2; \mathbb{Q}) = \mathbb{Q}$. Hence, Y_1 and Y_2 are not homotopy equivalence (and moreover, not weakly homotopy equivalent).

We have the following important theorem.

Theorem 11.9: (*CW Approximation Theorem*)

Given any space Y, there exists a CW complex X and weak homotopy equivalence $\alpha : X \to Y$, which is called a *CW approximation* of Y. Moreover, if $f : Y_1 \to Y_2$ is a map, and $\alpha_i : X_i \to Y_i$ are CW approximations for $i = 1, 2$, then there exists a (cellular) map $g : X_1 \to X_2$, defined unique up to homotopy, such that $\alpha_2 \circ g \simeq f \circ \alpha_1$.

Note that CW approximation is applicable for any space, but this does not let us replace a space. However, when computing algebraic invariants like homotopy or (co)homology, it is good enough to replace any space by a CW approximation.

Remark 11.10: (*CW Type*)

A space X is said to be of *CW type* if X is homotopy equivalent to a space Y, where Y admits a CW decomposition. We have noted that the Hawaiian earring does not admit a CW decomposition, moreover it does not have CW type. Consider the *Hedgehog space*

$$X := \left\{ re^{\iota\theta} \mid 0 \leq r \leq 1, \theta \in \mathbb{Q} \right\} \subset \mathbb{C}$$

which is a dense collection of spokes. Again, X is not locally contractible at any point other than the origin, and hence, X does not admit a CW decomposition. On the other hand, X is contractible, and so, it is homotopy equivalent to a CW complex (the singleton). So, X is of CW type. It is easy to see that Whitehead theorem (Theorem 11.7) generalizes to CW tpye : a weak homotopy equivalence between spaces of CW type is a homotopy equivalence.

11.3 Connectivity and Compressibility

Let us interpret the weak homotopy equivalence in terms of connectivity.

Definition 11.11: (*Connectivity of a Map*)

A map $f : X \to Y$ is said to be *n-connected* if the following holds : for all $x \in X$, $\pi_i(f) : \pi_i(X, x) \to \pi_i(Y, f(y))$ is an isomorphism for each $i < n$ and an epimorphism for $i = n$. A pair (X, A) is called *n-connected* if the inclusion map $\iota : A \hookrightarrow X$ is n-connected.

In the above definition, it is understood that for $n = 0$, an isomorphism $\pi_0(f) : \pi_0(X, x) \to \pi_0(Y, f(x))$ is just a bijection of pointed sets. In particular, for $n > 0$, we can simply ask $\pi_0(f)$ to induce bijection on path components. Clearly, $f : X \to Y$ is a weak homotopy equivalence if and only if it is n-connected for all $n \geq 0$.

Suppose (X, A) is a pair. If there is a homotopy $\Phi_t : (D^i, S^{i-1}) \to (X, A)$ of pairs, such that Φ_1 lands in A, then we would like to have a homotopy Ψ_t of Φ which is constant relative to the boundary, and Ψ_1 lands in A. More generally, we observe the following technical lemma.

Lemma 11.12: (*Homotopy of Pair and Relative Homotopy*)

Given a map $f : X \to Y$, suppose we have a commuting diagram

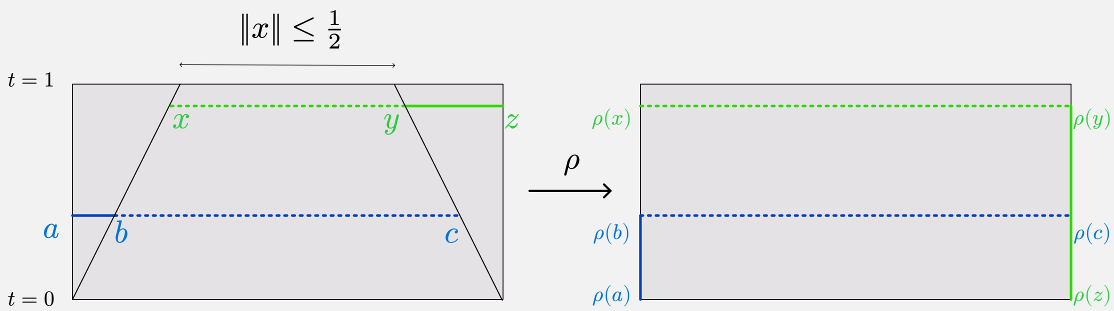
$$\begin{tikzcd} S^{i-1} \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d, "f"] \\ D^i \arrow[r, "\Phi"'] & Y \end{tikzcd}$$

Suppose we have homotopies $\Phi_t : D^i \to Y$ and $\varphi_t : S^{i-1} \to X$, such that $\Phi_0 = \Phi$, $\Phi_1 = f \circ \zeta$ for some $\zeta : D^i \to X$, and $\Phi_t|_{S^{i-1}} = f \circ \varphi_t$. Then, there exists a homotopy $\Psi_t : D^i \to Y$ and a map $\psi : D^i \to X$, such that $\Psi_0 = \Phi$, $\Psi_1 = f \circ \psi$ and $\Psi|_{S^{i-1}} = f \circ \varphi$.

Proof : The proof is immediate if one can define a conntinuous map $\rho : D^i \times I \to D^i \times I$ such that $\rho_0 = \mathrm{Id}_{D^i \times \{0\}}$ and $\rho|_{S^{i-1} \times I} = \rho_0|_{S^{i-1}}$. Here is an explicit formula :

$$\rho(x, t) := \left(\frac{2x}{\max(2\|x\|, 2 - t)}, 2 - \max(2\|x\|, 2 - t) \right),$$

where we are considering the euclidean norm. It is easy to see that $\Psi = \Phi \circ \rho$ is the desired homotopy. One can visualize the map as follows.



Visualizing deformation of $D^i \times I$ to make homotopy relative to boundary. □

Next, we compare n-connectednes of a map with a lifting problem of discs and spheres.

Lemma 11.13: (*Connectivity and Compressibility*)

A map $f : X \to Y$ is n-connected if and only if for any $i \leq n$, and given any commutative diagram

$$\begin{tikzcd} S^{i-1} \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d, "f"] \\ D^i \arrow[r, "\Phi"'] \arrow[ur, dashed, "\psi"] & Y \end{tikzcd}$$

there exists a map $\psi : D^i \to X$ such that $\psi|_{S^{i-1}} = \varphi$ and $f \circ \psi \underset{\text{rel } S^{i-1}}{\simeq} \Phi$ holds. This equivalent condition is also known as *i-compressibility*, especially when we are considering inclusion maps $\iota : A \hookrightarrow X$.

Proof : Suppose f is n-connected. Given a diagram as above, since $\Phi|_{S^{i-1}} = f \circ \varphi$, it follows that $f \circ \varphi$ is null-homotopic in Y. But $\pi_{i-1}(f) : \pi_{i-1}(X) \to \pi_{i-1}(Y)$ is injective for $i \leq n$, and hence, φ is null-homotopic in X. Thus, we have a null-homotopy $\varphi_t : S^{i-1} \to X$ with $\varphi_0 = \varphi$ and φ_1 constant. As $S^{i-1} \hookrightarrow D^i$ is a cofibration, and since $f \circ \varphi_0 = f \circ \varphi = \Phi|_{S^{i-1}}$, extending we get a homotopy $\Phi_t : D^i \to Y$ with $\Phi_0 = \Phi$ and $\Phi_t|_{S^{i-1}} = f \circ \varphi_t$. In particular, $\Phi_1 : D^i \to Y$ has $\Phi_1|_{S^{i-1}}$ constant. But then Φ_1 induces a map $\zeta : S^i \to Y$. Since ζ represents a homotopy class in $\pi_i(Y)$, by the surjectivity of $\pi_i(f)$, we have a map $\xi : S^i \to X$ with $f \circ \xi \simeq \zeta$ in Y. Let us consider this homotopy $\zeta_t : D^i \to Y$ with $\zeta_t|_{S^{i-1}}$ constant, and $\zeta_0 = \zeta, \zeta_1 = f \circ \xi$. Finally, concatenating with the homotopy Φ_t, we have a homotopy $\Psi_t : D^i \to Y$ such that $\Psi_0 = F$ and $\Psi_1 = f \circ \xi$. Moreover, $\Psi_t|_{S^{i-1}}$ always factors through f, and Ψ_1 factors through f as well. But then by Lemma 11.12 we can get a homotopy $\psi_t : D^i \to Y$ such that $\psi_0 = \Phi, \psi_t|_{S^{i-1}} = f \circ \varphi$ and ψ_1 factors through X. That is, $\psi_1 = f \circ \psi$ for some $\psi : D^i \to X$. This ψ solves desired lifting.

Conversely, suppose $f : X \to Y$ is i-compressible for $0 \leq i \leq n$. First consider the path components. Let $x_0, x_1 \in X$ be in different path components. Then, $f(x_0), f(x_1) \in Y$ are also in different path components, since otherwise by the diagram above we will have a path $D^1 = [0,1] \to X$ joining x_0 to x_1. Thus, $\pi_0(f)$ is injective on path components. Also, taking $i = 0$ (whence $D^i = D^0 = \{\star\}$ and $S^{i-1} = S^{-1} = \emptyset$), it follows that $\pi_0(f)$ is surjective on path components. Next, suppose $0 < i \leq n$. Let $\alpha : S^i \to Y$ be given. Consider the quotient $q : D^i \to D^i/S^{i-1} = S^i$. Get a point $x_0 \in X$ such that $f(x_0)$ is path connected to the point $\alpha q(S^{i-1}) \in Y$. By modifying the quotient map (think of a collar around the boundary that is collapsed to the path like a balloon with a string), we have map $\Phi : D^i \to Y$, and set $\varphi : S^{i-1} \to X$ to be the constant map at x_0. Then, get a lift $\psi : D^i \to X$ such that $f \circ \psi \simeq \Phi$. Clearly, the restriction $\psi|_{S^i}$ gives a homotopy class that is mapped to $[\alpha]$ by $\pi_i(f)$. This proves that $\pi_i(f)$ is surjective for $i \leq n$. A similar argument using shows the injectivity for $i < n$ as well. □

As a consequence, we have the following important result.

Theorem 11.14: (*Connectivity and Lift Against Relative CW Complex*)

Let (Y, B) be an n-connected pair, and (X, A) be a relative CW-complex of dimension $\leq n$. Then, any map $f : (X, A) \to (Y, B)$ is homotopic to a map into B, relative to A. Moreover, if $\dim(X, A) < n$, then the homotopy class of $X \to B$ is unique relative to A.

Proof : The proof is by induction over skeleton filtration of (X, A). Suppose first X is obtained from A by attaching q-cells, with $q \leq n$. Let $(F, f) : (X, A) \to (Y, B)$ be a map. Then, we have a diagram

$$\begin{tikzcd}
\bigsqcup S_\alpha^{q-1} \arrow[r, "\varphi"] \arrow[d, hook] & A \arrow[r, "f"] \arrow[d, hook, "\iota"'] & B \arrow[d, hook, "j"'] \\
\bigsqcup D_\alpha^q \arrow[r, "\Phi"'] & X \arrow[r, "F"'] & Y
\end{tikzcd}$$

As j is n-connected, we see by Lemma 11.13 that $F \circ \Phi$ is homotopic relative to $\sqcup S_\alpha^{q-1}$ to a map in to B. Since this homotopy is relative, by the pushout square on the left, we then have a homotopy of F into B, and they are clearly constantly equal f on A. Thus, we have homotopy of F into B relative to A.

Now, consider a relative CW complex of dimension $\leq n$. Let $f : (X, A) \to (Y, B)$ be a map. Suppose we have a homotopy of f relative to A to a map g such that $g(X^k) \subset B$. Since X^{k+1} is obtained from X^k by attaching $(k+1)$-cells, by the above argument, we have a homotopy of $g : (X^{k+1}, X^k) \to (Y, B)$ relative to X^k which sends X^{k+1} into B. Since $X^{k+1} \hookrightarrow X$ is a cofibration, we can extend this homotopy, relative to X^k, to all of X. Continuing this, we get the desired homotopy.

The above argument will work for $n < \infty$. In case of infinite dimension, at each step, we need to scale the homotopies suitably to, say, $\left[\frac{1}{n}, \frac{1}{n+1}\right]$, and then concatenate to get the desired homotopy.

Uniqueness of the homotopy class when $\dim(X, A) < n$ follows by observing that the pair $\dim(X \times I, X \times \partial I \cup A \times I) = n$, and we can apply the above argument to get a homotopy of homotopies. $\square$

As a consequence, we can now prove the Whitehead theorem for CW complexes. Let us first observe the effect of composing by an n-connected map on homotopy class.

Proposition 11.15: (*Composing n-connected Map and Bijection of Homotopy Class*)

Let $h : B \to Y$ be an n-connected map. Then, the induced map $h_\star : [X, B] \to [X, Y]$ is a bijection (resp. surjection) if X is a CW complex with $\dim X < n$ (resp. $\dim X \leq n$). If h is a map of pointed space, then the same conclusion hold for a pointed CW complex X and basepoint preserving homotopy classes.

Proof : Let us apply the *mapping cylinder construction* to replace h by an inclusion. Indeed, we have the following diagram

$$\begin{tikzcd}
& M_h \arrow[d, bend left, "\pi"] \\
B \arrow[ur, hook, "\iota"] \arrow[r, "h"'] & Y \arrow[u, hook, "j"', "\text{s.d.r}"]
\end{tikzcd}$$

Here, the mapping cylinder $M_h = Y \cup_h B \times [0, 1]$ deforms to Y via the projection $\pi : M_h \to Y$. Clearly, we have the composition

$$h_\star : [X, B] \xrightarrow{\iota_\star} [X, M_h] \xrightarrow{\pi_\star} [X, Y].$$

Since π is a deformation retract, it follows that $\pi_\star$ is a bijection. So we only need to consider $\iota_\star$. Without loss generality, let us assume that $h : B \hookrightarrow Y$ is an inclusion map.

Now, applying Theorem 11.14 for the CW-complex $(X, \emptyset)$, we immediately have the surjectivity when $\dim(X, \emptyset) = \dim X \leq n$. For injectivity, we again use Theorem 11.14 for the CW-complex $(X \times I, X \times \partial I)$. This concludes the claim. $\square$

We can now give a proof of the Whitehead theorem.

Proof of Theorem 11.7 : Suppose X, Y are CW complexes and $f : X \to Y$ is a weak homotopy equivalence. Then, f is n-connected for all n. Hence, as in Proposition 11.15, we have $f_\star : [Z, X] \to [Z, Y]$ is a bijection for any CW-complex Z. In particular, taking $Z = Y$, we have $f_\star : [Y, X] \to [Y, Y]$ is a bijection. But then there exists $g : Y \to X$ such that $f_\star([g]) = [\mathrm{Id}_Y]$, i.e, $f \circ g \simeq \mathrm{Id}_Y$. Next, consider $Z = X$ so that $f_\star : [X, X] \to [X, Y]$ is a bijection. We have

$$f_\star[g \circ f] = [f \circ (g \circ f)] = [(f \circ g) \circ f] = [\mathrm{Id}_Y \circ f] = [f] = f_\star[\mathrm{Id}_X].$$

By injectivity, we have $[g \circ f] = [\mathrm{Id}_X]$, that is, $g \circ f \simeq \mathrm{Id}_X$. Hence, $f : X \to Y$ is a homotopy equivalence, with g as a homotopy inverse. $\square$

Day 12 : 24th February, 2026

`cellular chain – cellular homology – cellular homology of ℂℙ^∞`

12.1 Cellular Chain Complex and Homology

Let $H_\star$ be an additive ordinary homology theory with $H_0(\star) = \mathbb{Z}$. As example, we can assume $H_\bullet$ to be the singular homology (with $\mathbb{Z}$-coefficient). Suppose X is obtained from $A \subset X$ by attaching n-cells via the map

$$(\Phi, \varphi) : \left(\bigsqcup_{\alpha\in\Lambda} D^n_\alpha, \bigsqcup_{\alpha\in\Lambda} S^{n-1}_\alpha \right) \to (X, A).$$

Proposition 12.1: (*Homology Isomorphism and Attaching Cells*)

The induced map

$$\sum H_\star(\Phi_\alpha) : \bigoplus_{\alpha\in\Lambda} H_\star(D^n_\alpha, S^{n-1}_\alpha) \to H_\star(X, A)$$

is an isomorphism.

Proof : Recall that Φ is a *relative homeomorphism*, i.e, a homeomorphism after taking quotient on both sides. Thus, we have the diagram

$$\begin{tikzcd}
\bigoplus H_\star(D^n_\alpha, S^{n-1}_\alpha) \arrow[r, "\cong"] & H_\star(\bigsqcup D^n_\alpha, \bigsqcup S^{n-1}_\alpha) \arrow[r, "H_\star(\Phi)"] \arrow[d, "\cong"'] & H_\star(X, A) \arrow[d, "\cong"'] \\
 & \tilde{H}_\star(\bigvee D^n_\alpha / S^{n-1}_\alpha) \arrow[r, "\cong", "\tilde{H}_\star(\tilde{\Phi})"'] & \tilde{H}_\star(X/A)
\end{tikzcd}$$

By Theorem 10.14, the vertical quotient maps induce isomorphism in homology. The commutativity shows that $H_\star(\Phi)$ is an isomorphism. The proof then follows since $H_\star$ is assumed to be an additive homology, which gives the isomorhpim on the left. □

Let us describe an inverse map as well. For $\alpha \in \Lambda$, we have an inclusion $p_\alpha : (X, A) \hookrightarrow (X, X \setminus E^n_\alpha)$, where $E^n_\alpha = \Phi\big(\mathring{D}^n_\alpha\big)$, and the relative homeomorphism $\Phi_\alpha : (D^n_\alpha, S^{n-1}_\alpha) \to (X, X \setminus E^n_\alpha)$. Thus, we have the maps

$$H_k(X, A) \xrightarrow{H_k(p_\alpha)} H_k(X, X \setminus E^n_\alpha) \xleftarrow[\cong]{H_k(\Phi_\alpha)} H_k(D^n_\alpha, S^{n-1}_\alpha).$$

The image under this map (after inverting the isomorphism), gives a map $z \mapsto (z_\alpha)_{\alpha\in\Lambda}$. It follows that this is precisely the inverse of the isomorphism in Proposition 12.1.

Now, suppose X is a CW complex. From the triple (X^{n+1}, X^n, X^{n-1}), we have the boundary map

$$\partial : H_{k+1}(X^{n+1}, X^n) \to H_k(X^n, X^{n-1}).$$

It is easy to see that $\partial^2 = 0$. Indeed, the boundary maps of the triples are obtained from the diagram

$$\begin{tikzcd}
& H_k(X^n) \arrow[dr] & & & \\
H_{k+1}(X^{n+1}, X^n) \arrow[ur] \arrow[rr, "\partial"'] & & H_k(X^n, X^{n-1}) \arrow[rr, "\partial"] \arrow[dr] & & H_{k-1}(X^{n-1}, X^{n-2}) \\
& & & H_{k-1}(X^{n-1}) \arrow[ur] &
\end{tikzcd}$$

Clearly, the consecutive red and blue maps compose to 0 being part of the long exact sequence of (X^n, X^{n-1}). Thus, $\partial^2 = 0$.

Definition 12.2: (*Cellular Chain Complex and Homology*)

Given a CW complex X, the *cellular chain complex* is defined as $C_n^{\text{cell}}(X) := H_n(X^n, X^{n-1})$, with the boundary map $\partial_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ obtained from the long exact sequence of the triple (X^{n+1}, X^n, X^{n-1}). The homology of the chain complex is called the *cellular homology* of X, and is denoted as $H_\bullet^{\text{cell}}(X)$.

Exercise 12.3: (*Cellular Chain as Free Groups on Cells*)

Verify that the the cellular chain group $C_n^{\text{cell}}(X) = H_n(X^n, X^{n-1})$ is canonically identified with the free Abelian group generated by the n-cells in X. That is, verify that $C_n^{\text{cell}}(X) = \bigoplus_{\{\alpha \in \Lambda\}} \mathbb{Z}\langle e_\alpha^n \rangle$, where $\{e_\alpha^n \mid \alpha \in \Lambda\}$ is the collection of n-cells in the CW decomposition of X.

Proposition 12.4: (*Cellular Homology is Functorial and Homotopy Invariant*)

A cellular map $f : X \to Y$ induces a chain map $C_\bullet^{\text{cell}}(X) \to C_\bullet^{\text{cell}}(Y)$. Homotopic chain maps induce a chain homotopy.

Proof : Let $f : X \to Y$ be a chain map, i.e, $f(X^n) \subset X^n$. Then, we have induced maps $H_n(f) : H_n(X^{n+1}, X^n) \to H_n(Y^{n+1}, Y^n)$ which commutes with bounary by the naturality of the long exact sequence of triple (Theorem 3.8).

Next, suppose $h : X \times I \to Y$ is a homotopy between two cellular maps $f, g : X \to Y$. Since $X \times I$ is a CW complex, by Theorem 11.2, we can assume that $\varphi : X \times I \to Y$ is a cellular map as well. Note that $(X \times I)^n = X^n \times \partial I \cup X^{n-1} \times I$. Consider the map

$$s_n : H_n(X^{n+1}, X^n) \xrightarrow{\sigma} H_{n+1}(X^{n+1} \times I, X^{n+1} \times \partial I \cup X^n \times I) \xrightarrow{H_{n+1}(\varphi)} H_{n+1}(Y^{n+1}, Y^n),$$

where the first σ is the isomorphism from Lemma 13.11 for the pair (X^{n+1}, X^n). Working through the diagrams involving the triple (X^{n+1}, X^n, X^{n-1}), one can show that $\partial s_n = H_n(g) - H_n(f) - s_{n-1}\partial$. That is, $s_\bullet$ is a chain homotopy. $\square$

Example 12.5: (*Cellular Homology of* $\mathbb{CP}^n$)

Recall the complex projective space $\mathbb{CP}^n$ is defined as the space of complex lines (i.e, 1-dimensional complex vector subspaces) of $\mathbb{C}^{n+1}$. In other words, $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus 0)/_\sim$, where

$$(z_0, \cdots, z_n) \sim (\lambda z_0, \cdots, \lambda z_n), \quad 0 \neq \lambda \in \mathbb{C}.$$

We denote the equivalence class as $[z_0 : \cdots : z_n]$, which gives rise to the *homogeneous coordinates* on $\mathbb{CP}^n$, making it into an n-dimensional complex manifold. Clearly, $\mathbb{CP}^0 = \{\star\}$ is a singleton.

Since every line is determined by the unit vector (up to sign) on the line, restricting to norm one vectors, we can equivalently get $\mathbb{CP}^n = S^{2n+1}/S^1$. Note that the unit vectors in $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is the $2n+1$ -sphere, and $S^1 = \left\{e^{\iota\theta} \mid 0 \leq \theta < 2\pi\right\}$ acts freely as an (Abelian) group. This is generalization of the antipode action which gives $\mathbb{RP}^n = S^n/\mathbb{Z}_2 = S^n/S^0$.

It is easy to see that the inclusion $\mathbb{C}^{n+1} \hookrightarrow \mathbb{C}^{n+2}$ as the hyperplane $\{z_{n+2} = 0\}$ gives rise to an inclusion

$$\begin{aligned} \mathbb{CP}^n &\hookrightarrow \mathbb{CP}^{n+1} \\ [z_0 : \cdots : z_n] &\longmapsto [z_0 : \cdots : z_n : 0]. \end{aligned}$$

Thus, we have

$$\{\star\} = \mathbb{CP}^0 \hookrightarrow \mathbb{CP}^1 \hookrightarrow \cdots \hookrightarrow \mathbb{CP}^\infty,$$

where $\mathbb{CP}^\infty$ is given the weak topology from the inclusions.

Let us now give a CW decomposition of $\mathbb{CP}^n$. We claim $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a $2n$ -cell. Note that the inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ gives the inclusion $S^{2n-1} \hookrightarrow S^{2n+1}$, which induces the same map $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$. Then, any point $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$ is *uniquely* written as $\left[z_0 : \cdots : z_{n-1} : \sqrt{1 - \|\mathbf{z}\|^2}\right]$, where $\mathbf{z} = (z_0, \ldots, z_{n-1}) \in \mathbb{C}^n$ satisfies $\|\mathbf{z}\|^2 := \sum_{j=0}^n z_j \bar{z}_j \lneq 1$. Treating $\mathbb{D}^{2n} \subset \mathbb{C}^n$ as the closed unit ball, we have a continuous map

$$\begin{aligned} \Phi : \mathbb{D}^{2n} &\longrightarrow \mathbb{CP}^n \\ \mathbf{z} &\longmapsto \left[z_0 : \cdots : z_n : \sqrt{1 - \|\mathbf{z}\|^2}\right]. \end{aligned}$$

The boundary map $\varphi := \Phi|_{S^{2n-1}}$ maps to $\mathbb{CP}^{n-1}$ since the last coordinate is 0. In the interior, Φ is a homeomorphism, since we can define the inverse map

$$\begin{aligned} \mathbb{CP}^n \setminus \mathbb{CP}^{n-1} &\longrightarrow \mathring{\mathbb{D}}^{2n} \\ [z_0 : \cdots : z_n] &\longmapsto \left(\frac{z_0}{z_n}, \ldots, \frac{z_{n-1}}{z_n}\right). \end{aligned}$$

In other words, $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a $2n$-cell via the attaching map φ. In particular, note that $\mathbb{CP}^1 = \{\star\} \cup_\varphi D^2 = S^2$ is the 2-sphere.

Now, we can immediately write the cellular chain complex $C_n = C_n^{\text{cell}}(\mathbb{CP}^n)$. It follows that

$$0 \to \underset{\mathbb{Z}}{C_{2n}} \to \underset{0}{\mathbb{C}_{2n-1}} \to \cdots \to \underset{\mathbb{Z}}{C_2} \to \underset{0}{C_1} \to \underset{\mathbb{Z}}{C_0} \to 0$$

The boundary maps are forced to be 0. Hence, it follows that

$$H_k^{\text{cell}}(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}, & 0 \leq k \leq 2n, k \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

This pattern continues for $\mathbb{CP}^\infty$ as well, and we have,

$$H_k^{\text{cell}}(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

Exercise 12.6: (*Cellular Homology of* $S^n \times S^n$)

Compute the cellular homology $X = S^n \times S^n$.

<u>**Hint**</u> **:** There is a CW decomposition of X with exactly 4 cells!

Day 13 : 27th February, 2026

`homological degree — degree of suspension — local degree — cellular boundary map — cellular and singular homology`

13.1 Homological Degree and Cellular Boundary Formula

In order to compute the cellular boundary map, we need to understand the notion of the *degree* of a self map of a sphere.

Definition 13.1: (*Degree of a Self-map of a Sphere*)

Given a map $f : S^n \to S^n$, the *homological degree* of the map $d(f)$ is defined to be the integer such that $\tilde{H}_\star(f) : \tilde{H}_\star(S^n) \to \tilde{H}_\star(S^n)$ is given as the multiplication by $d(f)$. Recall, $\tilde{H}_\star(S^n) = \begin{cases} \mathbb{Z}, & \star = n \\ 0, & \text{otherwise.} \end{cases}$

Exercise 13.2: (*Properties of Homological Degree*)

Verify the following.

1. If $f \sim g : S^n \to S^n$ are homotopic, then $d(f) = d(g)$. In paarticular, if f is null-homotopic then $d(f) = 0$.
2. Given $S^n \xrightarrow{f} S^{\xrightarrow{g}} S^n$, we have $d(g \circ f) = d(g) \cdot d(f)$.
3. If $f : S^n \to S^n$ is a homeomorphism (or even a homotopy equivalence), then $d(f) = \pm 1$.
4. What are the possible degrees of a map $S^0 \to S^0$?

Let us show some other interesting properties of the degree. Let us first re-interpret the suspension isomorphism. Given the sphere S^n, denote the two hemi-spheres by $D^n_\pm$. Then, we have two maps

$$s_+ : (D^n_-, S^{n-1}) \hookrightarrow (S^n, D^n_+), \quad s_- : (D^n_+, S^{n-1}) \hookrightarrow (S^n, D^n_-),$$

given by reflection about the equator. These give rise to the *suspension isomorphisms*

$$\sigma_\pm : \tilde{H}_{k-1}(S^{n-1}) \xleftarrow[\cong]{\partial} H_k(D^n_\mp, S^{n-1}) \xrightarrow[\cong]{H_k(s_\pm)} H_k(S^n, D^n_\pm) \xleftarrow[\cong]{j_\star} \tilde{H}_k(S^n).$$

Exercise 13.3:

Verify that $\partial, j_\star$ and $H_k(s_\pm)$ are indeed isomorphisms!

Consider the diagram

$$
\begin{tikzcd}
 & & \tilde{H}_k(S^n) \arrow[dll, "H_k(l_+)"'] \arrow[drr, "H_k(l_-)"] \arrow[d, "H_k(j_0)"] & & \\
H_k(S^n, D^n_+) & & & & H_k(S^n, D^n_-) \\
 & & H_k(S^n, S^{n-1}) \arrow[ull, "H_k(j_+)"] \arrow[urr, "H_k(j_-)"'] \arrow[dd, "\partial"] & & \\
H_k(D^n_-, S^{n-1}) \arrow[uu, "H_k(s_+)", "\cong"'] \arrow[urr, "H_k(\iota_-)"] \arrow[drr, "\partial_+"'] & & & & H_k(D^n_+, S^{n-1}) \arrow[uu, "H_k(s_-)"', "\cong"] \arrow[ull, "H_k(\iota_+)"'] \arrow[dll, "\partial_-"] \\
 & & \tilde{H}_k(S^{n-1}) & &
\end{tikzcd}
$$

Using Lemma 5.1, it follows that $\sigma_+ = -\sigma_-$.

Proposition 13.4: (*Degree of the Antipodal Map*)

Degree of any reflection map $S^n \to S^n$ about an hyperplane through origin is -1, degree of the antipodal map $\eta : S^n \to S^n$ is $(-1)^{n+1}$.

Proof : Consider the map $t : S^n \to S^n$ which reflects about S^{n-1}, i.e, $t(x_1, ..., x_n, x_{n+1}) = (x_1, ..., x_n, -x_{n+1})$. It is easy to see that $t_\star \sigma_+ = \sigma_- = -\sigma_+$, and hence, $d(t) = -1$. In particular, degree of any reflection about an hyperplane through origin is -1, since any two such reflections are homotopic. The antipodal map $\eta : S^n \to S^n$ can be written as $(n+1)$-fold composition of reflections. Hence, it follows that $d(\eta) = (-1)^{n+1}$. $\square$

Next, we compute the degree of the suspension of a map.

Proposition 13.5: (*Degree of Suspension*)

Let $f : S^n \to S^n$ be a map, and $\Sigma f : S^{n+1} = \Sigma S^n \to \Sigma S^n = S^{n+1}$ be the suspension. Then, $d(\Sigma f) = d(f)$.

Proof : Consider the pair (CS^n, S^n) and the quotient map $q : (CS^n, S^n) \to \Sigma S^n = S^{n+1}$. The induced map of (Cf, f) in the quotient is precisely the suspension Σf. By Theorem 10.14, $\tilde{H}_\star(q)$ is an isomorphism, also the boundary map for the pair is the suspension isomorphism. Naturality of the boundary map gives the commutative diagram

$$
\begin{tikzcd}
\tilde{H}_{k+1}(S^{n+1}) \arrow[d, "\tilde{H}_k(\Sigma f)"'] & H_{k+1}(CS^n, S^n) \arrow[l, "H_{k+1}(q)"'] \arrow[r, "\partial"] \arrow[d, "\tilde{H}_k(Cf, f)"'] & \tilde{H}_k(S^n) \arrow[d, "\tilde{H}_k(f)"] \\
\tilde{H}_{k+1}(S^{n+1}) & H_{k+1}(CS^n, S^n) \arrow[l, "H_{k+1}(q)"] \arrow[r, "\partial"'] & \tilde{H}_k(S^n)
\end{tikzcd}
$$

The commutativity forces that $d(f) = d(\Sigma f)$. $\square$

Exercise 13.6: (*Constructing Map of Specified Degree*)

Given any integer $k \in \mathbb{Z}$ and $n \geq 1$, construct a map $f : S^n \to S^n$ such that $d(f) = k$.

Hint : Start with the k-fold map $S^1 \to S^1$ and then suspend it $(n-1)$-times.

Next, we prove a formula to compute the degree of a map via *local degree*. Let $f : S^n \to S^n$ be a map. Suppose, there exists $y \in S^n$ such that $f^{-1}(y)$ is a finite set, say, $f^{-1}(y) = \{x_1, .., x_k\} \subset S^n$. By continuity of f, we can fix open sets $y \in\subset V \subset S^n$ and $x_i \in U_i \subset f^{-1}(V) \subset S^n$, such that $V_i \cap V_j = \emptyset$ for $i \neq j$. Now, $f(U_i \setminus x_i) \subset V \setminus i$ for $1 \leq i \leq k$. We have a commutative diagram

$$\begin{tikzcd}
& H_n(U_i, U_i \setminus x_i) \arrow[r, "f_\star"] \arrow[dl, "\cong"'] \arrow[d, "q_i"] & H_n(V, V \setminus y) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus x_i) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_\star"] & H_n(S^n, S^n \setminus y) \\
& H_n(S^n) \arrow[u, "j"] \arrow[ul, "\cong"] \arrow[r, "f_\star"'] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

The blue isomorphisms are via excision, and the red isomorphisms follow from long exact sequence as $S^n \setminus p$ is contractible for any $p \in S^n$. In particular, the two homology groups on the top row are then isomorphic to $H_n(S^n) = \mathbb{Z}$ via the isomorphisms. The other maps are induced from obvious space level maps, which gives the commutativity.

Definition 13.7: (*Local Degree*)

The local degree of $f : S^n \to S^n$ at x_i, denoted as $d(f)|_{x_i}$ is the integer that gives the map $H_n(U_i, U_i \setminus x_i) \to H_n(V, V \setminus y)$ via multiplication.

Local degree is independent of choice of U_i and V, and thus, it is indeed local. If f maps U_i homeomorphically onto V, then $d(f)|_{x_i} = \pm 1$. In practice, this situation appears many times, which helps us to compute the degree of a map using the next proposition (e.g Exercise 13.9).

Proposition 13.8: (*Degree is Sum of Local Degree*)

Let $f : S^n \to S^n$ be such that $f^{-1}(y) = \{x_1, ..., x_k\}$. Then, $d(f) = \sum d(f)|_{x_i}$.

Proof : We again consider the diagram

$$\begin{tikzcd}
 & H_n(U_i, U_i \setminus x_i) \arrow[r, "f_\star"] \arrow[dl, "\varphi_i"'] \arrow[d, "q_i"] & H_n(V, V \setminus y) \arrow[d, "\varphi"] \\
H_n(S^n, S^n \setminus x_i) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_\star"] & H_n(S^n, S^n \setminus y) \\
 & H_n(S^n) \arrow[r, "f_\star"'] \arrow[u, "j"] \arrow[ul, "\psi_i"] & H_n(S^n) \arrow[u, "\psi"]
\end{tikzcd}$$

It follows via excision that

$$H_n(S^n, S^n \setminus f^{-1}(y)) \cong H_n\left(\bigsqcup_{i=1}^k U_i, \bigsqcup_{i=1}^k (U_i \setminus x_i)\right) \cong \bigoplus_{i=1}^k H_n(U_i, U_i \setminus x_i).$$

Moreover, the inclusion of the i^{th} summand is via the map q_i, and the projection to the i^{th} summand is via the map p_i. Thus, we have the equation

$$\sum q_i \varphi_i^{-1} p_i = \text{Id}.$$

Fix a generator $z \in H_n(S^n)$. This gives generators $z_i := \varphi_i^{-1}\psi_{i(z)}$ for $H_n(U_i, U_i \setminus x_i)$, and generator $z_0 = \varphi^{-1}\psi(z)$ of $H_n(V, V \setminus y)$. By definition of (local) degree

$$f_\star(z) = d(f) \cdot z, \quad f_\star(z_i) = d(f)|_{x_i} \cdot z_0.$$

We compute,

$$\begin{aligned} d(f) \cdot \psi(z) = \psi f_\star(z) = f_\star j(z) = f_\star\Big(\sum q_i \varphi_i^{-1} p_i\Big) j(z) = \sum \varphi f_\star \varphi_i^{-1} \psi_i(z) = \varphi\Big(\sum f_\star(z_i)\Big) \\ = \varphi\Big(\sum d(f)|_{x_i} \cdot z_0\Big) = \sum d(f)|_{x_i} \cdot \varphi(z_0) = \sum d(f)|_{x_i} \, \psi(z). \end{aligned}$$

Since ψ is an isomorphism, we get the formula $d(f) = \sum d(f)|_{x_i}$. □

Exercise 13.9: (*Constructing Map of Specified Degree*)

Consider the fold map $\vee_{i=1}^k S^n \to S^n$, and the "diagonal" map $S^n \to \vee_{i=1}^k S^n$. One way to construct the diagonal map is to first fix k many distinct points and their disjoint neighborhoods, and then collapsing the complement to a point. Verify that the composition $S^n \to \vee_{i=1}^k S^n \to S^n$ has degree k. Composing with a reflection about a hyperplane gives a map with degree $-k$.

Example 13.10: (*Cell Decomposition of* $\mathbb{RP}^n$)

Let us give a cellular decomposition of $\mathbb{RP}^n$, similar to Example 12.5. Recall, $\mathbb{RP}^n$ is the collection of lines in $\mathbb{R}^{n+1}$, i.e, $\mathbb{RP}^n = \mathbb{R}^{n+1}/_{\sim}$, where $x \sim \lambda x$ for $\lambda = \pm 1$. Restricting to unit vectors, we equivalently get $\mathbb{RP}^n = S^n/_{\sim}$, where $x \sim y$ if and only if (x, y) are antipode points, i.e, $y = -x$. We can realize $\mathbb{RP}^n$ as the orbit space of the S^0 action on S^n, where the nontrvial map is the antipode map, i.e, $\mathbb{RP}^n = S^n/S^0$. Clearly $\mathbb{RP}^0 = \{\star\}$ is a singleton.

In order to get a cell decomposition, observe that there is a canonical inclusion $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$ induced by the inclusion $\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+2}$ as the hyperplane with last coordinate 0. The points in $\mathbb{RP}^{n+1} \setminus \mathbb{RP}^n$ represents the lines that are *uniquely* determined by a point in the, say, upper hemishpere $D^{n+1}_+ = \{(x_0, \ldots, x_{n+1}) \in S^{n+1} \subset \mathbb{R}^{n+2} \mid x_{n+1} > 0\}$. Thus, we see that $\mathbb{RP}^{n+1}$ is obtained from $\mathbb{RP}^n$ by attaching a single $(n+1)$-cell. Explicitly, we have the characteristic map

$$\begin{aligned} \Phi : D^{n+1} &\longrightarrow \mathbb{RP}^{n+1} \\ \mathbf{x} &\longmapsto \left[x_0 : \cdots : x_n : \sqrt{1 - \|\mathbf{x}\|^2}\right], \end{aligned}$$

where $\mathbf{x} = (x_0, \ldots, x_n)$, and $\|\mathbf{x}\|^2 = \sum x_i^2$. The attaching map $\varphi = \Phi|_{S^{n+1}}$ is in fact the universal (double) cover. Immediately, it follows that $\mathbb{RP}^1 = \mathbb{RP}^0 \cup_\varphi D^1 = S^1$, where $\varphi = \Phi|_{S^0 = \partial D^1}$ is the attaching map, which is just the constant map.

13.2 Cellular Boundary Formula

Using Proposition 12.1, for each $(n+1)$-cell E^{n+1}_α, and each n-cell E^n_β, we have components of the map ∂, denoted as

$$m(\alpha, \beta) : H_{k+1}(D^{n+1}_\alpha, S^n_\alpha) \to H_k\left(D^n_\beta, S^{n-1}_\beta\right).$$

Observe that we have the composition

$$\iota(\alpha, \beta) : S^n_\alpha \xrightarrow{\varphi_\alpha} X^n \xrightarrow{q_\beta} X^n/\left(X^n \setminus E^n_\beta\right) \xleftarrow[\cong]{\Phi_\beta} D^n_\beta/S^{n-1}_\beta.$$

Fixing a homeomorphism $\kappa_n : D^n/S^{n-1} \to S^n$, we have the composed map

$$\kappa_n \circ \iota(\alpha, \beta) : S^n_\alpha \longrightarrow S^n_\beta$$

as a self-map of spheres. We now give a formula for the cellular boundary map in terms of the *degree* of self-map of spheres. Indeed, we identify $m(\alpha, \beta)$ as the degree $d(\kappa_n \circ \iota(\alpha, \beta))$.

It is easy to see that $m(\alpha, \beta) \circ \sigma : H_n(D^n_\alpha, S^{n-1}_\alpha) \to H_{n+1}(D^{n+1}_\alpha, S^n_\alpha) \to H_n\left(D^n_\beta, S^{n-1}_\beta\right)$ is nothing but multiplication by $d(\alpha, \beta)$. Here σ is the *relative suspension isomorphism*.

Lemma 13.11: (*Relative Suspension Isomorphism*)

Given a pair (X, A), we have isomorphisms

$$H_n(X,A) \to H_n(X \times \partial I \cup A \times I, X \times \{0\} \cup A \times I) \xleftarrow{\partial} H_{n+1}(X \times I, X \times \partial I \cup A \times I),$$

where the first map is given by identifying (X, A) with $(X \times \{1\}, A \times \{1\})$, and the boundary map is from the triple $(X \times I, X \times \partial I \cup A \times I, X \times 0 \cup A \times I)$. The composition is denoted as $\sigma_{X,A}$

<u>**Proof**</u> **:** The first map is an isomorphism since we can excise out $X \times \{0\}$, and then use a strong deformation retract. The boundary map is part of the exact sequence, where the groups $H_\bullet(X \times I, X \times \{0\} \cup A \times I) = 0$ since the inclusion $X \times \{0\} \cup A \times I \hookrightarrow X \times I$ is a homotopy equivalence (both deformation retracts onto $X \times \{0\}$). But then by exactness, ∂ is an isomorphism $\square$

Let us now describe the cellular boundary formula via homological degree.

Theorem 13.12: (*Cellular Boundary Formula*)

We have a commutative diagram

$$\begin{tikzcd}
H_{k+1}(D^{n+1}_\alpha, S^n_\alpha) \arrow[r, "m(\alpha,\beta)"] \arrow[d, "\partial"'] & H_n\left(D^n_\beta, S^{n-1}_\beta\right) \arrow[d, "(q_\beta)_\star"] \\
\tilde{H}_k(S^n_\alpha) \arrow[r, "\iota(\alpha,\beta)_\star"'] & \tilde{H}_k\left(D^n_\beta / S^{n-1}_\beta\right)
\end{tikzcd}$$

Consequently, given a relative suspension isomorphism $\sigma : H_k(D^n, S^{n-1}) \to H_{k+1}(D^{n+1}, S^n)$, we have $m(\alpha, \beta) \circ \sigma$ is multiplication by the degree of the map $\kappa_n \circ \iota(\alpha, \beta)$, provided $\partial \circ \sigma = (\kappa_n)_\star \circ (q_\beta)_\star$ holds.

<u>**Proof**</u> **:** We have the following diagram, where the solid arrows are commutative

$$\begin{tikzcd}
H_{k+1}(X^{n+1}, X^n) \arrow[r, "\partial"] & \tilde{H}_k(X^n) \arrow[r, "j_\star"] & H_k(X^n, X^{n-1}) \arrow[r, "q_\star"] \arrow[d] & \tilde{H}_k(X^n / X^{n-1}) \arrow[d] \\
H_{k+1}(D^{n+1}_\alpha, S^n_\alpha) \arrow[u, "(\Phi_\alpha)_\star"', "\cong"] \arrow[r, "\partial"'] \arrow[drr, bend right, "m(\alpha,\beta)"'] & \tilde{H}_k(S^n_\alpha) \arrow[u, "(\varphi_\alpha)_\star"'] \arrow[r, "(\varphi_\alpha)_\star"'] & H_k\left(X^n, X^n \setminus e^n_\beta\right) \arrow[r, "(q_\beta)_\star"] & \tilde{H}_k\left(X^n / X^n \setminus e^n_\beta\right) \\
 & & H_k\left(D^n_\beta, S^{n-1}_\beta\right) \arrow[u, "(\Phi_\beta)_\star"', "\cong"] \arrow[r, "(q_\beta)_\star"'] \arrow[ull, dashed, bend left=60, "\sigma_{D^n, S^{n-1}}"] & \tilde{H}_k\left(D^n_\beta / S^{n-1}_\beta\right) \arrow[u, "\cong"', "(\varphi_\beta)_\star"] \arrow[d, dashed, "(\kappa_n)_\star"] \\
 & & & \tilde{H}_k\left(S^n_\beta\right)
\end{tikzcd}$$

Note that $m(\alpha, \beta)$ is given by following the blue arrows, whereas $\iota(\alpha, \beta)$ is given by following the red arrows. This gives the commutativity of the diagram in the statement.

Next, fix a relative suspension isomorphism $\sigma = \sigma_{D^n, S^{n-1}}$ and a homeomoephism $\kappa_n : D^n/S^{n-1} \to S^n$, such that $\partial \circ \sigma = (\kappa_n)_\star \circ (q_\beta)_\star$. Here, we identify $S^n_\alpha = S^n = S^n_\beta$ etc. Then, we have a diagram

$$\begin{tikzcd}
H_k(D^n, S^{n-1}) \arrow[r, "\sigma"] \arrow[d, "q_\star"'] & H_{k+1}(D^{n+1}, S^n) \arrow[r, "m(\alpha,\beta)"] \arrow[d, "\partial"] & H_k(D^n, S^{n-1}) \arrow[r, "\sigma"] \arrow[d, "q_\star"] & H_{k+1}(D^{n+1}, S^n) \arrow[d, "\partial"] \\
\tilde{H}_k(D^n/S^{n-1}) \arrow[r, "(\kappa_n)_\star"'] & \tilde{H}_k(S^n) \arrow[r, "\iota(\alpha,\beta)_\star"'] & \tilde{H}_k(D^n/S^{n-1}) \arrow[r, "(\kappa_n)_\star"'] & \tilde{H}_k(S^n)
\end{tikzcd}$$

All the blue arrows are isomorphisms. Since the left and right squares (they are identical) are commutative by assumption, it follows that $m(\alpha, \beta) \circ \sigma$ is same multiplying by the degree of the map $\kappa_n \circ \iota(\alpha, \beta)$. □

Note that the cellular boundary formula depends on the choice of the suspension isomorphism σ, as we saw earlier we can get another isomorphism which differs by a sign. On the other hand, if we fix them once and use it for all the cells, then the cellular boundary map is determined up to a sign. Indeed, we can fix $\sigma : H_0(D^0, S^{-1}) \to H_1(D^1, S^0)$, and then use its iterates. Thus, the cellular homology computation will be unaffected.

Example 13.13: (*Cellular Homology of* $\mathbb{RP}^n$)

In Example 13.10, we say a cell decomposiont $\mathbb{RP}^{n+1} = \mathbb{RP}^n \cup_\varphi e^{n+1}$, where the attaching map $\varphi : S^n \to \mathbb{RP}^n$ is given as the restriction of the map

$$\begin{aligned}
\Phi : D^{n+1} &\longrightarrow \mathbb{RP}^n \\
(x_0, \dots, x_n) &\longmapsto \left[x_0 : \cdots : x_n : \sqrt{1 - \sum x_i^2}\right].
\end{aligned}$$

We compute the cellular boundary map for this decomposition. By Theorem 13.12, we need to understand the degree of the map $\iota : S^n \xrightarrow{\varphi} \mathbb{RP}^n \to \mathbb{RP}^n/(\mathbb{RP}^n \setminus e^n) \xrightarrow{\cong} S^n$, where φ is the universal covering map. Note that $\mathbb{RP}^n/(\mathbb{RP}^n \setminus e^n) = \mathbb{RP}^n/\mathbb{RP}^{n-1}$. Fix some $y \in \mathbb{RP}^n \setminus \mathbb{RP}^{n-1}$. Then, $y = [x_0 : \dots : x_n : t]$, where $t = \sqrt{1 - \sum x_i^2} \neq 0$. Then, $\iota^{-1}(y) = \{\pm z\}$, where $z = \left(\frac{x_0}{t}, \dots, \frac{x_n}{t}\right)$. Fix the open hemi-spheres $D^n_\pm$, and without loss of generality assume that $z \in D^n_+$ (otherwise relabel z by $-z$). It follows that φ is identity on D^n_+, and we can get φ on D^n_- by by first applying the antipode map on D^n_- (which maps it to D^n_+), followed by φ. Thus, from Proposition 13.8, it follows that the degree of ι is $1 + (-1)^{n+1}$. Hence, denoting $C_n := C_n^{\text{cell}}(\mathbb{RP}^n) = H_n(\mathbb{RP}^n, \mathbb{RP}^{n-1})$, it follows that the cellular boundary map $\partial_n : C_n \to C_{n-1}$ is given by $1 + (-1)^{(n-1)+1} = 1 + (-1)^n$. In particular, we have

$$H_k(\mathbb{RP}^{2n+1}) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/2\mathbb{Z}, & 0 < k < 2n+1, k \text{ odd} \\ \mathbb{Z}, & k = 2n+1 \\ 0, & k > 2n+1. \end{cases} \qquad H_k(\mathbb{RP}^{2n}) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/2\mathbb{Z}, & 0 < k < 2n, k \text{ odd} \\ 0, & k \geq 2n. \end{cases}$$

Exercise 13.14: (*Homology of $S^p \times S^q$*)

Extend Exercise 12.6 by computing the ceullar (and hence the singular) homology of $S^p \times S^q$ for any p, q.

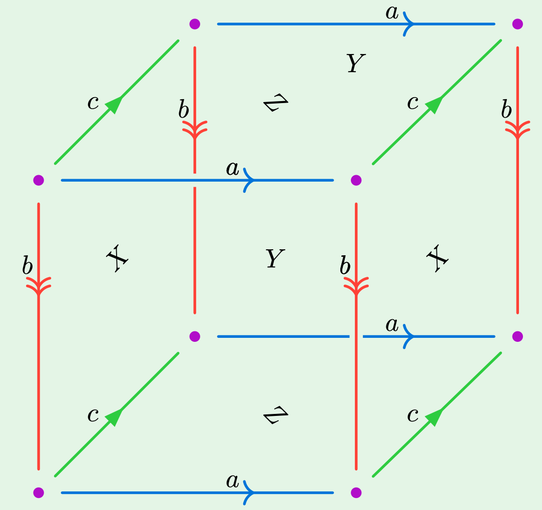
Hint : Justify $S^p \times S^q = (S^p \vee S^q) \cup_\varphi e^{p+q}$, where $\varphi : S^{p+q-1} \to S^p \vee S^q$ is the attaching map.

Exercise 13.15: (*Cellular Homology of S^n*)

Compute the celluar homology of the n-shpere with exactly two k-cells in each dimension, $0 \leq k \leq n$. Compute the cellular homology of S^∞.

Example 13.16: (*Cellular Homology of* 3*-Torus* $S^1 \times S^1 \times S^1$)

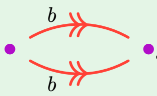
The three torus $\mathbb{T}^3 = S^1 \times S^1 \times S^1$ can be given the following CW structure.



There is **one** 0-cell, say, v, **three** 1-cells, a, b, c, **three** 2-cells, say, X, Y, Z, and **one** 3-cell, say, Δ. The two cell X denotes the left-right face, Y denotes the front-back face, and Z denotes the top-bottom face, identified accordingly. The cellular chain complex $C_n := C_n^{\text{cell}}(X)$ is given as

$$\cdots \to \underset{0}{C_4} \to \underset{\mathbb{Z}\langle \Delta \rangle}{C_3} \xrightarrow{\partial_3} \underset{\mathbb{Z}\langle X,Y,Z \rangle}{C_2} \xrightarrow{\partial_2} \underset{\mathbb{Z}\langle a,b,c \rangle}{C_1} \xrightarrow{\partial_1} \underset{\mathbb{Z}\langle v \rangle}{C_0} \to 0.$$

We compute ∂_i by Theorem 13.12.

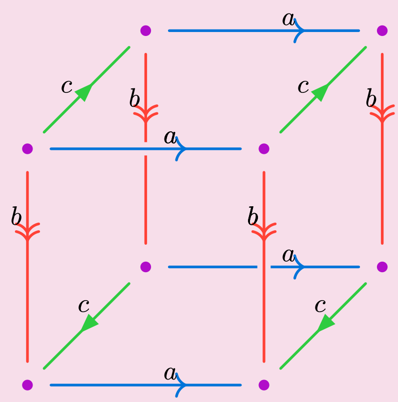
- We have $\partial_1 : H_1(X_1, X_0) \to H_0(X_0, X_{-1})$. We have X_1 is wedge of three circles and $X_{-1} = \emptyset$, and hence, $\partial_1 : H_1(X_1, \{v\}) \to H_0(\{v\}) \to H_0(\{v\})$ is simply the boundary map of the pair $(X_1, \{v\})$. As X_1 is path connected, it follows that $\partial_1 = 0$.
- To compute $\partial_2(X) = a_X a + b_X b + c_X c$, it is easy to see that $a_X = 0$. To compute b_X, we consider the map $S^1 \to S^1$ given by the attaching map . Since the two edges differ by a reflection, it follows from Proposition 13.8 and Proposition 13.4 that $b_X = 0$. Similarly, $c_X = 0$, which makes $\partial_2(X) = 0$. Repeating this argument for Y and Z, we see that $\partial_2 = 0$.
- Finally, we compute $\partial_3(\Delta)$. Say, $\partial_3(\Delta) = uX + vY + wZ$. In order to compute u, we consider the map $S^2 \to S^2$ which identfies the hemispheres via a reflection. Thus, $u = 0$, and similarly $v = w = 0$. Thus, $\partial_3 = 0$.

Since all the boundary maps are 0, we have the cellular homology groups as

$$H_k(X) = \begin{cases} \mathbb{Z}, & k = 0, 3 \\ \mathbb{Z}^3, & k = 1, 2 \\ 0, & k > 3. \end{cases}$$

Exercise 13.17: (*Cellular Homology of* $S^1 \times K$*, where* K *is the Klein's Bottle*)

Let K be the Klein's bottle, and $X = S^1 \times K$. Consider the CW structure on X given as



Compute the cellular homology of X.

13.3 Cellular Homology as Homology Theory

Let us consider $H_\bullet$ to be an ordinary homology theory, such that $H_0(\star) = \mathbb{Z}$. In the next theorem, we prove that $H_\bullet$ is naturally isomorphic to $H_\bullet^{\text{cell}}$ for CW complexes. For convenience, we are only considering singular homology theory.

Theorem 13.18: (*Cellular Homology Equals Singular Homology*)

Let X be a CW complex. Then, $H_\bullet^{\text{cell}}(X)$ is naturally isomorphic to the singular homology $H_\bullet(X)$.

Proof : Let us make some observations.

1. $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$.
2. $H_k(X^n) = 0$ for $k > n$. For $n = 0$, this is immediate by the dimension axiom. Inductively, assume it to be true for some $n \geq 0$. Say, $k > n+1$. Then, we have part of the exact sequence
$$H_k(X^n) \to H_k(X^{n+1}) \to H_k(X^n, X^{n-1}),$$
where the two endpoints are 0 : left one by induction, and the right by previous observation. So, $H_k(X^{n+1}) = 0$.
3. As $H_{n-1}(X^{n-2}) = 0$, the map $H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ is injective. Hence, the cellular n-cylces Z_n^{cell} is same as the kernel $\ker(\partial : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}))$.
4. The exact sequence $0 \to H_n(X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$ then gives an isomorphism $H_n(X^n) \cong Z_n^{\text{cell}}$.
5. $H_k(X, X^n) = 0$ for $k \leq n$. One proves by induction that $H_k(X^{n+\ell}, X^n) = 0$ for $\ell \geq 0$ and $k \leq n$. Then, passing to colimit, we get $H_k(X, X^n) = 0$. This is easy to see for singular homology, but remains true for any homology theory.
6. The map $H_n(X^{n+1}) \to H_n(X)$ is an isomorphism, which follows from the long exact sequence of the pair (X, X^n) and the fact $H_k(X, X^n) = 0$ for $k \leq n$.

Finally, we consider the following diagram.

$$
\begin{tikzcd}
H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial"] \arrow[d, dash, "="] & Z_n^{\text{cell}} \arrow[r] & H_n^{\text{cell}}(X) \arrow[r] & 0 \\
H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial"] & H_n(X^n) \arrow[r] \arrow[u, "\cong"'] & H_n(X^{n+1}) \arrow[r] \arrow[u, dashed] \arrow[d, "\cong"'] & 0 \\
& & H_n(X) &
\end{tikzcd}
$$

The commuttivity of the left square frollows from the definition of the cellular boundary map. Then, we have an induced map $H_n(X^{n+1}) \to H_n^{\text{cell}}(X)$, which is an isomorphism by 5-lemma. But then it follows that $H_n(X) \cong H_n^{\text{cell}}(X)$. Clearly, the isomorphism is natural since all the maps and diagrams involved are natural. $\square$

Thus, cellular homology is a powerful tool to compute many homology groups.

Day 14 : 24th March, 2026

`rings — modules — tensor product — free module — projective module — injective module — flat module`

14.1 Rings and Modules

Let us recall some basic definition from (commutative) algebra.

Definition 14.1: (*Ring*)

A *ring* R is a set R equipped with two binary operations, usually denoted as *addition* and *multiplication*, such that $(R, +)$ is an *Abelian group,* $(R, \cdot)$ is a *monoid,* and the multiplication distributes over the addition, i.e, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in R$.

The above definition assumes that R has a multiplicative identity, usually denoted 1 (or 1_R), since a *monoid* is just a set M equpped with an associative binary operation and a both-sided identity for the same operation. In the lieterature, a ring without a multiplicative identity is sometimes called a *rng*.

Remark 14.2: (*Rings as Monoid Objects*)

In category theoretic language, one can say that a ring is just a monoid object in the monoidal category of Abelian groups! This abstract generalization leads to the notion of *ring spectra,* which are monoid object in the category of *spectra.*

Example 14.3: (*Endomorphism Ring*)

Given an Abelian group G, consider $E = \mathrm{End}(G, G)$ to be the collection of endomorphisms $G \to G$. Then, E is clearly a ring, where the multiplication is given by composition and the unit is the idenity map $1_G : G \to G$.

We have a natural notion of *ring homomorphism* $f : R \to S$ which preseves both the Abelian group structure, and the monoid structure. Explicitly, we have $f(r_1 - r_2) = f(r_1) - f(r_2)$, $f(r_1 r_2) = f(r_1)f(r_2)$ and $f(1_R) = 1_S$.

Definition 14.4: (*Left Module over a Ring*)

Given a ring R, a *left module* over R is an Abelian group M, equipped with a ring homomorphism $\varphi : R \to \mathrm{End}(M, M)$.

Unpacking the defintion, we are given an action map $\bullet : R \times M \to M$ given by $r \bullet m := \varphi(r)(m)$, such that the following holds.

1. $r \bullet \_$ is a group homomorphism, i.e, $r \bullet (m_1 + m_2) = r \bullet m_1 + r \bullet m_2$.
2. $r \mapsto r \bullet \_$ is a ring map, i.e, $(r_1 \cdot r_2) \bullet m = r_1 \bullet (r_2 \bullet m)$, and $1_R \bullet m = m$.

Definition 14.5: (*Right Module over a Ring*)

A *right module* over R is an Abelian group M equipped with a ring *anti*-homomorphism $R \to \mathrm{End}(M, M)$, i.e, there is a right action $M \times R \to M$ satisfying $m \bullet (r_1 \cdot r_2) = (m \bullet r_1) \bullet r_2$. The category of left (resp. right) R-modules is denoted as $R\text{-}\mathrm{Mod}$ (resp. $\mathrm{Mod}\text{-}R$).

Given a ring R, we have the notion of the *opposite ring* R^{op}, where the multiplication is flipped, i.e, $r \mathbin{\tilde{\cdot}} s := s \cdot r$. Now, given a left R-module M, we can define a natural right R^{op}-module structure on M given by $m \mathbin{\tilde{\bullet}} r := r \bullet m$. If R is a commutative ring, then R^{op} is naturally isomorphic to R, and hence, every left R-module is naturally a right R-module and vice versa. In this case, we only consider modules without specifying left or right, and denote the category of R-modules as $R\text{-}\mathrm{Mod}$.

Example 14.6: (*Modules over Rings*)

Here are some example of modules over some known rings.

1. Every ring R is a module over itself. A submodule of R is precisely an *ideal* of R.
2. A module over $\mathbb{Z}$ is precisely an Abelian group.
3. A module over a field k is precisely a vector space over k.
4. A module over the polynomial ring $k[X]$ is a vector space over k, equipped with a linear endomorphism.
5. (Restriction of scalars) If $R \to S$ is a ring map, then every S-module M is naturally an R-module by considering the composition $R \to S \to \mathrm{End}(M, M)$.

The category $R\text{-}\mathrm{Mod}$ is additive (Definition 6.19). It has arbitrary *coproduct*: explicitly, given a collection $\{M_\alpha\}_{\alpha \in \Lambda}$ of R-modules, the direct sum $\bigoplus_{\alpha \in \Lambda} M_\alpha$ is defined as the collection of formal sums $\sum_{\alpha \in \Lambda} r_\alpha m_\alpha$, where $r_\alpha \in R, m_\alpha \in M_\alpha$, and all but finitely many $r_\alpha = 0$. We also have arbitrary *product,* namely the direct product $\prod_{\alpha \in \Lambda} M_\alpha$, where the action is component-wise. In fact, $R\text{-}\mathrm{Mod}$ has all limits and colimits, and moreover, it is Abelian (Definition 6.27).

14.2 Free Module

We have the *forgetful functor* $U : R\text{-}\mathrm{Mod} \to \mathrm{Sets}$, which admits a right adjoint $F : \mathrm{Sets} \to R\text{-}\mathrm{Mod}$, called the *free R-module* functor. Explicitly, given a set S, the free R-module $F(S)$ can be constructed as the direct sum $\bigoplus_S R$. We can canonically identify $S \hookrightarrow F(S)$ by considering each s as the formal sum $1_R \cdot s \in \bigoplus_S R$. Recall that the free module has the following universal property: given any R-module M and a set map $\varphi : S \to M$, there exists a unique R-module map $\Phi : F(S) \to M$ such that the following commutes

$$\begin{tikzcd}
F(S) \arrow[dr, dashed, "\exists!\Phi"] & \\
S \arrow[u, hook] \arrow[r, "\varphi"'] & M
\end{tikzcd}$$

One can think of the free module admitting a basis. In particular, as a consequence of the axiom of choice, it follows that for a field k every k-module is a free module (i.e, every k-vector space admits a basis). Recall that a ring is called a *PID* (i.e, a *principal ideal domain*) if it is an integral domain such that every ideal is generated by a single element.

Proposition 14.7: (*Submodule of Free Module over a PID*)

Let R be a PID and F be a free R-module. If $M \subset F$ is a submodule then M is again free.

<u>**Proof**</u> **:** Let $\{e_i \mid i \in I\}$ be a basis of F over R. By the well-ordering principal, get an well-order on the index set I. Let $F_i \subset F$ be the (free) submodule generated by all $\{e_j \mid j \in I, j \leq i\}$, and set $U_i := U \cap F_i$. Denote by $\pi_i : F \to R$ the projection along the basis element e_i. Then, $\pi_i(U_i)$ is a submodule of R, i.e, an ideal of R. Since R is a PID, we have $\pi_i(U_i) = Ra_i$ for some $a_i \in R$. Choose some $u_i \in U_i$ such that $\pi_i(u_i) = a_i$; if $a_i = 0$ choose $u_i = 0$.

By transfinite induction, one can show that $\{u_i \mid u_i \neq 0, i \in I\}$ is a basis of U. Hence, U is a free R-module $\square$

14.3 Tensor Product

Given two R-modules M, N, the hom set $\hom_R(M, N)$ is naturally an R-module. In particular, we have a functor

$$\hom_R : R\text{-Mod} \times R\text{-Mod} \to R\text{-Mod},$$

which is contravariant on the first position, and covariant on the second position. Given $N \in R\text{-Mod}$, the *tensor product* functor $\_ \otimes_R N$ is defined as the left adjoint to the covariant $\hom$ functor $\hom_R(N, \_)$. In particular, we have a natural R-module isomorphism

$$\hom_R(M \otimes_R N, P) \cong \hom_R(M, \hom_R(N, P))$$

for any $M, N, P \in R\text{-Mod}$. In particular, every bilinear map $M \times N \to P$ admits a unique lift to a linear map $M \otimes_R N \to P$. Explicitly, one constructs $M \otimes_R N$ as the quotient of the free module generated by formal symbols $m \otimes n$ via the bilinearity relations. The tensor product is naturally associative, and moreover tensoring by R (on both sides) is naturally isomorphic to the identity functor. If R is commutative, then $M \otimes_R N$ is naturally isomorphic to $N \otimes_R M$. In particular, $(R\text{-Mod}, \otimes_R, R)$ is a monoidal category, which is a *symmetric* monoidal category if R is commutative.

Exercise 14.8: (*Tensor Product as Cokernel*)

Let $f : R \to S$ be a ring map, and $M, N \in S\text{-Mod}$. Justify that we have an exact sequence

$$M \otimes_R S \otimes_R N \xrightarrow{\varphi} M \otimes_R N \to M \otimes_S N \to 0,$$

where $\varphi(m \otimes s \otimes n) := ms \otimes n - m \otimes sn$. Here we are identifying M, N, S as R-modules by restriction of scalars via the map $f : R \to S$.

When the underline ring is understood, we shall simply denote the tensor product as $\otimes$.

Exercise 14.9: (*Tensor Product of Cyclic Groups*)

Compute the tensor product $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ for integers $m, n \geq 1$. Given the $\mathbb{Z}/4\mathbb{Z}$-module structure on $\mathbb{Z}/2\mathbb{Z}$ via restriction of scalar by the canonical ring map $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, compute $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}/4\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.

In general, if $I \subset R$ is an ideal such that $MI = 0 = IN$, it follows that $M \otimes_R N \cong M \otimes_{R/I} N$. Tensor pruduct is additive, even for arbitrary sum. In fact, $\_ \otimes N$ being a left adjoint functor, preserves all colimits.

Proposition 14.10: (*Tensor Product and Arbitrary Coproduct*)

Given an arbitrary direct sum $\bigoplus_{i\in I} M_i$ of R-modules, we have $\left(\bigoplus_{i\in I} M_i\right) \otimes N = \bigoplus_{i\in I}(M_i \otimes N)$ and $N \otimes \left(\bigoplus_{i\in I} M_i\right) = \bigoplus_{i\in I}(N \otimes M_i)$

Proof : We proof one of them via the universal property. Let $\varphi : \left(\bigoplus_{i\in I} M_i\right) \times N \to \bigoplus_{i\in I}(M_i \otimes N)$ be given as $(\sum m_i, n) \mapsto \sum m_i \otimes n$, which is easily seen to be an R-bilinear map. Hence, there is a unique lift $\Phi : \left(\bigoplus_{i\in I} M_i\right) \otimes N \to \bigoplus_{i\in I}(M_i \otimes N)$. On the other hand, for each $j \in I$, we have a bilinear map $\psi_j : M_j \times N \to \left(\bigoplus_{i\in I}\right) \times N$ given by $(m_j, n) \mapsto m_j \otimes n$, which lifts to a map $\Psi_j : M_j \otimes N \to \left(\bigoplus_{i\in I} M_i\right) \otimes N$. Taking direct sum, we have $\Psi : \bigoplus_{i\in I}(M_i \otimes N) \to \left(\bigoplus_{i\in I} M_i\right) \otimes N$. It is easy to see that Φ and Ψ are inverses of each other, proving the claim. $\square$

14.4 Projective, Injective and Flat Modules

Definition 14.11: (*Exact Functor*)

An additve functor $F : R\text{-Mod} \to R\text{-Mod}$ is called *left exact* if given any short exact sequence $0 \to M \to N \to P \to 0$ we have $0 \to F(M) \to F(N) \to F(P)$ is again exact, and is called *right exact* if $F(M) \to F(N) \to F(P) \to 0$ is exact.

In the definition of the exactness, we do not need to start with a *short* exact sequence. Indeed, F is left exact if and only for any exact sequence $0 \to M \to N \to P$, we have $0 \to F(M) \to F(N) \to F(P)$ is exact, i.e, if F preserves kernel. Similarly, righ exactness can be characterized as F is right exact if and only if it preserves cokernel.

Exercise 14.12: (*Exactness and Adjoint*)

Verify that given an adjoint pair of additive functors $R\text{-Mod} \underset{\mathcal{R}}{\overset{\mathcal{L}}{\rightleftarrows}} S\text{-Mod}$, we have $\mathcal{L}$ is *right* exact, and $\mathcal{R}$ is *left* exact. In particular, the covariant $\hom$ functor $\hom(N, \_)$ is left exact, and the tensor product functor $\_ \otimes N$ is right exact.

Exercise 14.13: (*Exactness of Hom*)

Verify that both the covariant and contravariant $\hom$ functors are left exact.

Exercise 14.14: (*Exactness of Tensor*)

Verify that both $M \otimes \_$ and $\_ \otimes M$ are right exact functors.

Observe that if F is a free R-module, then $\hom(F, \_)$ is right exact as well. In general, $\hom$ is *not* right exact. This leads to the following important notion.

Definition 14.15: (*Projective Module*)

A module $P \in R\text{-Mod}$ is called *projective* if the $\hom$ functor $\hom(P, \_)$ is right exact. In other words, given any epimorphism $f : A \to B$ and any map $\varphi : P \to B$, there is a lift $\Phi : P \to A$ such that the diagram commutes

$$\begin{tikzcd} & P \arrow[dl, dashed, "\Phi"'] \arrow[d, "\varphi"] & \\ A \arrow[r, two heads, "f"'] & B \arrow[r] & 0 \end{tikzcd}$$

As noted earlier, every free module is projective. If R is a PID, then every projective module is also free.

Proposition 14.16: (*Characterization of Projective Modules*)

Given an R-module P, the following are equivalent.

1. P is projective.
2. Any short exact sequence $0 \to M \to N \to P \to 0$ splits.
3. P is a direct summand of a free R-module, i.e, there exists an R-module Q such that $P \oplus Q$ is free.

Proof : Suppose P is projective. Let $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ be a short exact sequence. Then, we have a map $s : P \to N$ in the commutative diagram

$$\begin{tikzcd} & & & P \arrow[dl, dashed, "s"'] \arrow[d, "\mathrm{Id}"] & \\ 0 \arrow[r] & M \arrow[r, "f"'] & N \arrow[r, "g"'] & P \arrow[r] & 0 \end{tikzcd}$$

Clearly, s is then a splitting.

Next, assume that any short exact sequence $0 \to M \to N \to P \to 0$ splits. Let $S \subset P$ be a set of generators $F = F(S)$ be the free R-module. Then, we have a epimorphism $p : F \twoheadrightarrow P$. Let $Q = \ker(p)$, so that we have a short exact sequence $0 \to Q \to F \to P \to 0$. But then this splits, and we have $F = P \oplus Q$.

Finally, suppose there is a module Q such that $F = P \oplus Q$. We have the isomorphism $\hom(F, C) = \hom(P \oplus Q, C) = \hom(P, C) \oplus \hom(Q, C)$ for any $C \in R\text{-Mod}$. Now, given an epimorphism $B \to C$, we have the commutative diagram

$$\begin{tikzcd} \hom(F, B) \arrow[r] \arrow[d, "\cong"'] & \hom(F, C) \arrow[d, "\cong"] \\ \hom(P, B) \oplus \hom(Q, B) \arrow[r] & \hom(P, C) \oplus \hom(Q, C) \end{tikzcd}$$

Since F is free, the blue arrow is surjective, and hence, the green arrow is also an epimorphism. But the green arrow is surjective precisely when each component arrow is surjective. In particular, it follows that P (and Q as well) are projective. $\square$

Exercise 14.17: (*Direct Summand of Projective*)

Let $P = \bigoplus_{i \in I} P_i$ be an arbitrary direct sum of R-modules. Show that P is projective if and only if each P_i is projective.

Remark 14.18: (*Eilenberg-Mazur Swindle*)

Given a projective R-module P, one can get *free* module A such that $P \oplus A$ if free. Indeed, by Proposition 14.16, we have some module Q such that $F := P \oplus Q$ is free. Then, consider A to be infinite sum

$$A := (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots = F \oplus F \oplus \cdots,$$

which is clearly a free module. Now,

$$P \oplus A = P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots = (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots = F \oplus F \oplus \cdots,$$

which is again free. This *rearranging* is counterintuitive but completely valid, and is called Mazur's swindle! The same does not work in real analysis with infinite series sum, as evident by Riemann rearrangement theorem. Indeed, it leads to the *proof* that $1 = 1 + (-1+1) + (-1+1) + \cdots = (1-1) + (1-1) + \cdots = 0$.

As an immediate corollary to the above characterization, we have the following example.of a projective module which is *not* free.

Example 14.19: (*Projective $\not\Rightarrow$ Free*)

Consider $R = \mathbb{Z}/6\mathbb{Z}$. Then, R itself is trivially free over R. Now, $R = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is a direct sum decomposition. Hence, both $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are projective R-modules. But neither can be free, since any (nonzero) free R-module must have at least 6 elements.

A consequence of Proposition 14.7 and Proposition 14.16 is the following.

Corollary 14.20: (*Projective over PID*)

Let R be a PID. Then any projective R-module is free.

Similarly, the failure of left exactness of the tensor functor leads to the following definition.

Definition 14.21: (*Flat Module*)

An R-module M is called *flat* if the functor $\_ \otimes M$ is left exact. In other words, M is flat if given any monomorphism $f : A \hookrightarrow B$, the induced map $f \otimes \mathrm{Id}_M : A \otimes M \to B \otimes M$ is injective.

It follows that any projective module is a flat module.

Proposition 14.22: (*Projective $\Rightarrow$ Flat*)

A projective module is flat.

<u>Proof</u> : Let $f : A \to B$ be a monomorphism. Let us first consider a free module $F = F(S)$, where S is the generating set. Consider the map $\Phi := f \otimes \mathrm{Id}_F : A \otimes F \to B \otimes F$. Now, $F = \bigoplus_{s\in S} R$, so that $A \otimes F = A \otimes \left(\bigoplus_{s\in S} R\right) = \bigoplus_{s\in S}(A \otimes R) = \bigoplus_{s\in S} A$, and similarly, $B \otimes F = \bigoplus_{s\in S} B$. The map Φ is seen to be identified with the map $\bigoplus_{s\in S} f$. Since f is monic, it follows that Φ is monic, proving that F is flat. Next, let P be a projective R-module. Then, there exists a module Q such that $F = P \oplus Q$ is free. Now, we have the commutative diagram

$$\begin{tikzcd}
A \otimes F \arrow[r, "f \otimes \mathrm{Id}_F"] \arrow[d, "\cong"'] & B \otimes F \arrow[d, "\cong"] \\
(A \otimes P) \oplus (A \otimes Q) \arrow[r, "f \otimes \mathrm{Id}_P + f \otimes \mathrm{Id}_Q"'] & (B \otimes Q) \oplus (B \otimes Q)
\end{tikzcd}$$

The vertical isomorphisms follow from Proposition 14.10. Now, F being a free module is a flat module and hence the top row is a monomorphism. But then the bottom row is a monomorphism as well. This is possible if and only if each component is a monomorphism. In particular, P is a flat module. $\square$

Just as Exercise 14.17, we have the following fact: given a direct summand $M = \oplus_{i\in I} M_i$, the module M is flat if and only if M_i is flat.

Example 14.23: (*Flat $\not\Rightarrow$ Projective*)

Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module (i.e, an Abelian group). It follows that $\mathbb{Q}$ is flat, but not a projective module.

Remark 14.24:

We have the following (strict) implications, where the reverse inclusion holds when the underlying ring has certain extra properties.

$$\begin{tikzcd}
\text{Free} \arrow[r, Rightarrow] & \text{Projective} \arrow[r, Rightarrow] \arrow[l, Rightarrow, dashed, bend left, "\text{PID}" near start, "\text{or Local ring}"'] & \text{Flat} \arrow[r, Rightarrow] \arrow[l, Rightarrow, dashed, bend left, "\text{Perfect ring}"'] & \text{Torsion-free} \arrow[l, Rightarrow, dashed, bend left, "\text{Dedekind ring}"']
\end{tikzcd}$$

Finally, the failure of the right exactness of the contravariant $\hom$ functor leads to the following definition.

Definition 14.25: (*Injective Module*)

An R-module I is called *injective* if $\hom(\_, I)$ is right exact. In other words, given any monomorphism $f : A \to B$, and any map $\varphi : A \to I$, there is a map $\Phi : B \to I$ such that the diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\varphi"'] & B \arrow[dl, dashed, "\Phi"] \\ & I & \end{tikzcd}$$

Just as in Proposition 14.16, it follows that I is an injective R-module if and only if any short exact sequence $0 \to I \to M \to N \to 0$ splits. As an example, $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. Unlike the projective modules, injective modules are harder to construct.

Day 15 : 27th March, 2026

`resolution – horseshoe lemma – left derived functors – Tor functors – long exact sequence of left derived functors`

15.1 Resolutions

The definition of projective and injective objects make sense in any category via the diagrams. More interestingly, they make sense in any Abelian categories via the exactness of the $\hom$ functors. This leads to the following definition.

Definition 15.1: (*Enough Projective and Enough Injective*)

An Abelian categroy $\mathcal{A}$ is said to have

- *enough projectives* if given any $A \in \mathcal{A}$ there is an epimorphism $\pi : P \twoheadrightarrow A$, where $P \in \mathcal{A}$ is a projective object, and
- *enough injective* if given any $A \in \mathcal{A}$ there is a monomorphism $\iota : A \hookrightarrow I$, where $I \in \mathcal{A}$ is an injective object.

The category $R\text{-Mod}$ has enough projectives, since we can easily construct epimorphism from free (and hence, projective) modules. $R\text{-Mod}$ has enough injectives as well, although it is considerably harder to prove and involves the axiom of choice.

The point of *resolving an object* (or a map) in a category is to replace it by something *equivalent* which behaves well with computation. As a concrete example, recall that in Theorem 11.9 we say that given any topology space Y, there is a CW complex X and a map $f : X \to Y$ such that f is a weak homotopy equivalence. CW complexes are well-suited to compute singular homology (Theorem 13.18), and homology remains invariant under weak homotopy equivalence. Thus, the CW approximation theorem gives a *resolution* in the category of topological spaces. To make this precise, one needs the notion of *model category,* where one can perform (co)fibrant replacement of maps and objects. We shall focus on the category $\mathrm{Ch} = \mathrm{Ch}(R)$ of chain complexes of R-modules.

Definition 15.2: (*Left and Right Resolution*)

Given an R-module M, a *left resolution* is an exact sequence

$$\cdots \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} M \to 0,$$

which we express as $C_\bullet \xrightarrow{\varepsilon} M \to 0$. Similarly, a *right resolution* is an exact sequence

$$0 \to M \xrightarrow{\varepsilon} D_0 \xrightarrow{d_0} D_1 \to \cdots \to D_n \xrightarrow{d_n} D_{n+1} \to \cdots,$$

which we express as $0 \to M \xrightarrow{\varepsilon} D_\bullet$. The map ε is called the *augmentation map.*

The R-module M can be realized as a chain complex concentrated at degree 0. Thus, a left resolution $C_\bullet \to M \to 0$ can be understood as the chain map

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r, "d_2"] \arrow[d] & C_1 \arrow[r, "d_1"] \arrow[d] & C_0 \arrow[r, "0"] \arrow[d, "\varepsilon"'] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

The fact that $C_\bullet \xrightarrow{\varepsilon} M \to 0$ is an exact sequence is equivalent to the fact that the chain map $\varepsilon_\bullet : C_\bullet \to M$ is a chain equivalence. Thus, getting a resolution is to replace the object M by a chain equivalent complex, which is exact everywhere except at degree 0. Note that if $C_\bullet \xrightarrow{\varepsilon} M \to 0$ and $D_\bullet \xrightarrow{\eta} N \to 0$ are two resolutions, then $C_\bullet \oplus D_\bullet \to (\varepsilon + \eta) \to M \oplus N \to 0$ is again a resolution.

Definition 15.3: (*Free/Projective/Flat/Injective Resolution*)

Given an R-module, a left resolution $C_\bullet \xrightarrow{\varepsilon} M \to 0$ is called a *free resolution* (resp. *projective resolution*, *flat resolution*, *injective resolution*) if each C_n is a free (resp. projective, flat, injective) module. Similarly, we define free/projective/flat/injective right resolution.

In practice, we are interested in free/projective/flat left resolutions, and injective right resolutions.

Proposition 15.4: (*Existence of Projective and Injective Resolutions*)

Given any R-module M, there exists a free (and hence projective) left resolution and injective right resolution. If R is a a PID, then there is a free left resolution $0 \to F_1 \to F_0 \to M \to 0$.

Proof: There is an epimorphism $\varepsilon : F_0 \to M$, where F_0 is a free module (e.g, take $F_0 = R[M]$). Let $K_0 = \ker(\varepsilon)$ and get an epimorphism $d_1 : F_1 \to K_0$. Then, we have an exact sequence $F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \to 0$. Inductively continuing we have a free resolution. If R is a PID, then K_0 is free being a submodule of a free module (Proposition 14.7), and hence we can simply set $F_1 = K_0$.

Since $R\text{-Mod}$ has enough injectives, we have an injection $0 \to M \xrightarrow{\varepsilon} I_0$, where I_0 is an injective module. Set $L_0 = \operatorname{coker}(M \to I_0)$ and get monomorphism $\delta_0 : L_0 \to I_1$, where I_1 is injective. Composing, we have $d_0 : I_0 \to L_0 \to I_1$. Then, $0 \to M \xrightarrow{\varepsilon} I_0 \xrightarrow{d_0} I_1$ is exact. Continuing inductively, we have an injective resolution. □

In general, in any Abelian category with enough projectives, we have projective left resolutions. Similarly, if the category has enough injectives, we have injective right resolutions. As an example, the category Sh of Ab-valued sheaves over a topological space have enough injectives, which leads to the *Godement resolution*, but in general Sh does not have enough projectives.

In any case, the existence of a resolution is not unique as we made choices at each step. Thus, we need to be able to compare different resolutions.

Theorem 15.5: (*Comparison of Resolutions*)

Let $P_\bullet \xrightarrow{\varepsilon} M$ be a projective resolution of M, and $Q_\bullet \xrightarrow{\eta} N$ be any resolution of N. Say, $f' : M \to N$ is a given map. Then, there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ such that the diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f'"'] & 0 \\
\cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"'] & N \arrow[r] & 0
\end{tikzcd}$$

Moreover, any such chain map is unique up to chain homotopy. In fact the statement remains true under the assumption that $P_\bullet$ is just a chain complex of projective modules.

__Proof__ **:** The proof is via induction. Denote $P_{-1} = M, Q_{-1} = N$ and $f_{-1} = f'$ for notational convenience. Assume that f_n is constructed so that the diagram commutes

$$\begin{tikzcd}
P_{n+1} \arrow[r] \arrow[d, dashed, "f_{n+1}"'] & P_n \arrow[r] \arrow[d, "f_n"'] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f'"] & 0 \\
Q_{n+1} \arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_0 \arrow[r, "\eta"'] & N \arrow[r] & 0
\end{tikzcd}$$

Denoting the cycles $Z_n = \ker(P_n \to P_{n-1})$ and $W_n = \ker(Q_n \to Q_{n-1})$ it follows that we have an induced map

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "f_n'"] & P_n \arrow[r] \arrow[d, "f_n"'] & P_{n-1} \arrow[d, "f_{n-1}"'] \\
0 \arrow[r] & W_n \arrow[r] & Q_n \arrow[r] & Q_{n-1}
\end{tikzcd}$$

Sicne $Q_\bullet$ is a resolution, we have an exact sequence $Q_{n+1} \to W_n \to 0$. Since P_{n+1} is projective, we have a lift

$$\begin{tikzcd}
P_{n+1} \arrow[dr, bend left, "d"] \arrow[dd, dashed, "f_{n+1}"'] & & \\
& Z_n \arrow[d, "f_n'"] & \\
Q_{n+1} \arrow[r] & W_n \arrow[r] & 0
\end{tikzcd}$$

Clearly, f_{n+1} satisfies the chain map condition. Thus, inductively we can get a lift. Note: we only used the fact that the differential $d : P_{n+1} \to P_n$ maps in to Z_n, i.e, $P_\bullet$ need only be a chain complex.

Next, suppose $g_\bullet : P_\bullet \to Q_\bullet$ be another lift. We inductively construct the chain homotopy $s_n : P_n \to Q_{n+1}$. For $n < 0$, we can set $s_n = 0$. Indeed, for $s_{-1} : P_{-1} = M \to Q_0$, we check $\eta 0 = 0 = f' - f'$. Consider the diagram

$$\begin{tikzcd}
& P_0 \arrow[r, "\varepsilon"] \arrow[dl, dashed, "s_0"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & M \arrow[r] \arrow[d, "f'"] \arrow[dl, dashed, "0"'] & 0 \\
Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"'] & N \arrow[r] & 0
\end{tikzcd}$$

Note that $\eta(f_0 - g_0) = (f' - f')\varepsilon = 0$, and thus, $f_0 - g_0 : P_0 \to Z_0 = Z_0 := \ker(\eta)$. Since P_0 is projecive, and since $Q_1 \to Z_0 \to 0$ is exact, we have a lift $s_0 : P_0 \to Q_1$ such that $ds_0 = f_0 - g_0$. Inductively assume that $s_k : P_k \to Q_{k+1}$ has been constructed for $k \leq n$. In particular, we have

$$f_n - g_n = ds_n + s_{n-1}d \Rightarrow ds_n = f_n - g_n - s_{n-1}d.$$

Let us compute

$$d(f_{n+1} - g_{n+1} - s_n d) = (f_n - g_n)d - ((f_n - g_n) - s_{n-1}d)d = 0.$$

Thus, $f - g - sd$ lands in $W_n = \ker(d : Q_{n+1} \to Q_n)$. But since P_{n+1} is projective and $Q_\bullet$ is a resolution, we have a lift $s_{n+1} : P_{n+1} \to Q_{n+2}$. Clearly this fits together as a chain homotopy. This completes the proof. □

As a corollary, we have the following.

Corollary 15.6: (*Resolutions are Unique Upto Chain Homotopy Equivalence*)

Given two projective resolutions $P_\bullet \xrightarrow{\varepsilon} M$ and $Q_\bullet \xrightarrow{\varepsilon} M$, there is a chain homotopy equivalence $f_\bullet : P_\bullet \to Q_\bullet$.

Proof : Using Theorem 15.5, we can lift Id_M to get two chain maps $f : P_\bullet \to Q_\bullet$ and $g : Q_\bullet \to P_\bullet$. Then, $g \circ f : P_\bullet \to P_\bullet$ lifts Id_M as well. But $\mathrm{Id}_{P_\bullet} : P_\bullet \to P_\bullet$ is trivially a lift. Hence, again by Theorem 15.5, we have $g \circ f$ is chain homotopic to $\mathrm{Id}_{P_\bullet}$. By the same argument, we have $f \circ g$ is chain homotopic to $\mathrm{Id}_{Q_\bullet}$. Consequently, f is a chain homotopy equivalence, with inverse g. □

The next lemma gets its name from the shape of the diagram!

Lemma 15.7: (*Horseshoe Lemma*)

Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence of R-modules. Suppose $P'_\bullet \xrightarrow{\varepsilon'} A$ and $P''_\bullet \xrightarrow{\varepsilon''} A''$ are two projective resolutions of A' and A'' resppectively. Then, setting $P_n := P'_n \oplus P''_n$ yields a projective resolution $P_\bullet \xrightarrow{\varepsilon} A$. Moreover, we get the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P'_2 \arrow[r] \arrow[d, "\iota"] & P'_1 \arrow[r] \arrow[d, "\iota"] & P'_0 \arrow[r, "\varepsilon'"] \arrow[d, "\iota"] & A' \arrow[r] \arrow[d, "\iota"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\pi"] & P_1 \arrow[r] \arrow[d, "\pi"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi"] & 0 \\
\cdots \arrow[r] & P''_2 \arrow[r] \arrow[d] & P''_1 \arrow[r] \arrow[d] & P''_0 \arrow[r, "\varepsilon''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

where each column is a short exact sequence, and each row is exact.

__Proof__ **:** Since P''_0 is projective, we can lift ε'' to a map $P''_0 \to A$. As P_0 is the direct sum $P'_0 \oplus P''_0$, we can take the sum of the map $P''_0 \to A$ with the map $\iota\varepsilon'$ to get the map $\varepsilon : P_0 \to A$. Then, we have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(\varepsilon') \arrow[r] \arrow[d, dashed] & P'_0 \arrow[r, "\varepsilon'"] \arrow[d, "\iota"] & A' \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & \ker(\varepsilon) \arrow[r] \arrow[d, dashed] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & \ker(\varepsilon'') \arrow[r] \arrow[d] & P''_0 \arrow[r, "\varepsilon''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The left column is induced naturally and it is a short exact sequence by the snake lemma (Lemma 6.29). Since $P'_\bullet$ and $P''_\bullet$ are exact everywhere, we have the diagram, where the column is exact.

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
\cdots \arrow[r] & P'_1 \arrow[r] & \ker(\varepsilon') \arrow[r] \arrow[d, dashed] & 0 \\
 & & \ker(\varepsilon) \arrow[d, dashed] & \\
\cdots \arrow[r] & P''_1 \arrow[r] & \ker(\varepsilon'') \arrow[r] \arrow[d] & 0 \\
 & & 0 &
\end{tikzcd}$$

We can now continue inductively. This completes the proof. $\square$

Remark 15.8: (*Explicit Formula for $P_\bullet \to A$*)

Let us justify an explicit formula for the differential appearing in the resolution $P_\bullet$. Let us write in the matrix form

$$d_n = \begin{pmatrix} f_n & \lambda_n \\ \mu_n & g_n \end{pmatrix} : \begin{matrix} P'_n \\ \oplus \\ P''_n \end{matrix} \longrightarrow \begin{matrix} P'_{n-1} \\ \oplus \\ P''_{n-1} \end{matrix}.$$

Following the construction in Lemma 15.7, we see that $f_n = d'_n, \mu_n = 0$ and $g_n = d''_n$. In other words, $d = \left(\begin{smallmatrix} d' & \lambda \\ 0 & d'' \end{smallmatrix}\right)$ for some $\lambda_n : P''_n \to P'_{n-1}$.

Exercise 15.9: (*Comparison and Horseshoe Lemma of Injective Resolution*)

State and proof the comparison theorem and the horseshoe lemma for injective right resolutions.

15.2 Left Derived Functors and Tor

Let us fix two Abelian categories $\mathcal{A}, \mathcal{B}$, so that $\mathcal{A}$ has enough projectives. In particular, we can consider $\mathcal{A} = R\text{-Mod}$ and $\mathcal{B} = S\text{-Mod}$. Let $F : \mathcal{A} \to \mathcal{B}$ be a *right* exact additive functor. Given $A \in \mathcal{A}$, fix some projective resolution $P_\bullet \xrightarrow{\varepsilon} A$. Then, the i^{th} *left derived functor* of F is defined as

$$\mathfrak{L}_i F(A) := H_i(F(P_\bullet)) = H_i(\cdots \to F(P_2) \to F(P_1) \to F(P_0) \to 0).$$

In other words, if we consider a projective resolution to be a chain complex $\cdots \to P_1 \to P_0 \to 0$ with a weak equivalence $\varepsilon : P_\bullet \to M$, then the left derived functor is precisely the homology of $P_\bullet$. Note that a priori, the definition depends on the choice of the projective resolution.

Proposition 15.10: (*$\mathfrak{L}_i F$ is a Well-defined Functor*)

$\mathfrak{L}_i F : \mathcal{A} \to \mathcal{B}$ is a well-defined functor (up to isomorphism).

Proof : Suppose $Q_\bullet \xrightarrow{\eta} A$ is another projective resolution. Then, by Corollary 15.6, we can lift $\mathrm{Id}_A : A \to A$ to a chain map $f : P_\bullet \to Q_\bullet$, which is unique up to chain homotopy (hence gives *same map* at homology), and moreover, which is a chain homotopy equivalence (hence gives homology isomorphism). In particular, $H_\bullet(f)$ is a uniquely defined isomorphism, which justifies that $\mathfrak{L}_i F(A)$ is well-defined.

Given any $\varphi : A \to B$, we can again use Theorem 15.5 to get a chain map $\Phi : P_\bullet \to Q_\bullet$, which induces a map $H_i(\Phi) : \mathfrak{L}_i F(A) \to \mathfrak{L}_i F(B)$. As Φ is unique up to chain homotopy, it follows that $H_i(\Phi)$ is uniquely defined. Thus, we have $\mathfrak{L}_i F(\varphi) = H_i(\Phi) : \mathfrak{L}_i F(f) \to \mathfrak{L}_i F(B)$. Let us check $\mathfrak{L}_i F$ is actually a functor.

- Since Id_A lifts to identity chain map, it follows that $\mathfrak{L}_i F(\mathrm{Id}_A) = \mathrm{Id}_{\mathfrak{L}_i F(A)}$.
- Say, $A \xrightarrow{f} B \xrightarrow{g} C$ is given. Get reolutions $P_\bullet \to A, Q_\bullet \to B, R_\bullet \to C$, and lifts $F : P_\bullet \to Q_\bullet, G : Q_\bullet \to R_\bullet$ and $H : P_\bullet \to R_\bullet$ of f, g, and $g \circ f$ respectively. Now, $G \circ F$ is also a lift of $g \circ f$, and hence, $G \circ F$ is chain homotopic to H. Thus, $\mathfrak{L}_i F(g \circ f) = \mathfrak{L}_i F(g) \circ \mathfrak{L}_i F(f)$.

Thus, we see that $\mathfrak{L}_i F(A)$ is defined up to a canonical isomorphism. Moreover, the composition also holds true, compatible with these isomorphisms. □

Remark 15.11: (*Derived Functor and Choice*)

As noted, $\mathfrak{L}_i F$ is only defined up to isomorphism. To make it a true functor, we need to *fix* a projective resolution for each object, which is moreover functorial. This is possible in most model categories. As an example, when working with $R\text{-Mod}$, given any R-module M, we have a *canonical* free resolution $0 \to F_1 = \ker(\varepsilon) \to F_0 \xrightarrow{\varepsilon} M \to 0$, where $F_0 = F(M)$ is the free module generated by M (as a set). It is easy to see that any $f : M \to N$ induces a natural map

$$\begin{tikzcd} 0 \arrow[r] & F_1(M) \arrow[r] \arrow[d, "f_1"'] & F_0(M) \arrow[r] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\ 0 \arrow[r] & F_1(N) \arrow[r] & F_0(N) \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

Thus, $\mathfrak{L}_i F$ is a uniquely defined functor here.

Since F is right exact, given any resolution $P_\bullet \xrightarrow{\varepsilon} M$, we have $F(P_1) \to F(P_0) \to F(A) \to 0$ is exact. Hence,

$$\begin{aligned} \mathfrak{L}_0 F(A) = H_0(\cdots \to F(P_1) \to F(P_0) \to 0) &= \frac{\ker(F(P_0) \to 0)}{\operatorname{im}(F(P_1) \to F(P_0))} \\ &= \frac{F(P_0)}{\ker\left(F(P_0) \xrightarrow{F(\varepsilon)} F(A)\right)} = F(A). \end{aligned}$$

That is, $\mathfrak{L}_0 F = F$ naturally. The goal of derived functor is to measure the failure of exactness of the functor F.

Theorem 15.12: (*Long Exact Sequence of Left Derived Functors*)

Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact additive functor between Abelian categories, where $\mathcal{A}$ has enough projectives. Then, given any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, there exists a natural long exact sequence

$$\cdots \to \mathfrak{L}_i F(A) \to \mathfrak{L}_i F(B) \to \mathfrak{L}_i F(C) \xrightarrow{\delta} \mathfrak{L}_{i-1} F(A) \cdots \to \mathfrak{L}_1 F(C) \xrightarrow{\delta} \underbrace{\mathfrak{L}_0 F(A)}_{F(A)} \to \underbrace{\mathfrak{L}_0 F(B)}_{F(B)} \to \underbrace{\mathfrak{L}_0 F(C)}_{F(C)} \to 0$$

<u>**Proof**</u>**:** Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence. Get projective resolutions $P'_\bullet \xrightarrow{\varepsilon'} A'$ and $P''_\bullet \xrightarrow{\varepsilon''} A''$. Using Lemma 15.7, we get the projective resolution $P_\bullet \xrightarrow{\varepsilon} A$, so that $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ fits together as a short exact sequence. Now, $0 \to P'_n \to P_n \to P''_n \to 0$ splits by Proposition 14.16. Hence, $0 \to F(P'_n) \to F(P_n) \to F(P''_n) \to 0$ is a (split) exact sequence. Thus, we have a short exact sequence $0 \to F(P'_\bullet) \to F(P_\bullet) \to F(P''_\bullet) \to 0$ of chain complexes. By Lemma 6.29, we get the long exact sequence of homology

$$\cdots \to \mathfrak{L}_i F(A) \to \mathfrak{L}_i F(B) \to \mathfrak{L}_i F(C) \xrightarrow{\delta} \mathfrak{L}_{i-1} F(A) \to \cdots$$

We need to justify he naturaility of the boundary map δ. Consider the commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

Get projective resolutions $P'_\bullet \xrightarrow{\varepsilon'} A'$, $P''_\bullet \xrightarrow{\varepsilon''} A''$, $Q'_\bullet \xrightarrow{\eta'} B'$, $Q''_\bullet \xrightarrow{\eta''} B''$ by Proposition 15.4. Next, using Theorem 15.5, lift f' and f'' to chain maps $F'_\bullet : P'_\bullet \to Q'_\bullet$ and $F''_\bullet : P''_\bullet \to Q''_\bullet$. Using Lemma 15.7, get the projective resolutions $P_\bullet \xrightarrow{\varepsilon} A$ and $Q_\bullet \xrightarrow{\eta} B$, so that we have the short exact sequences $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$, and $0 \to Q'_\bullet \to Q_\bullet \to Q''_\bullet \to 0$. We need to fill in $F_\bullet : P_\bullet \to Q_\bullet$ lifting $f : A \to B$ so that the following diagram commutes.

$$
\begin{tikzcd}
0 \arrow[rr] & & P'_\bullet \arrow[rr] \arrow[dr, "\varepsilon'"] \arrow[dd, "F'_\bullet"'] & & P_\bullet \arrow[rr] \arrow[dr, "\varepsilon"] \arrow[dd, dashed, "F_\bullet"'] & & P''_\bullet \arrow[rr] \arrow[dr, "\varepsilon''"] \arrow[dd, "F''_\bullet"'] & & 0 & \\
& 0 \arrow[rr] & & A' \arrow[rr, "\iota_A"] \arrow[dd, "f'"] & & A \arrow[rr, "\pi_A"] \arrow[dd, "f"] & & A'' \arrow[rr] \arrow[dd, "f''"] & & 0 \\
0 \arrow[rr] & & Q'_\bullet \arrow[rr] \arrow[dr, "\eta'"] & & Q_\bullet \arrow[rr] \arrow[dr, "\eta"] & & Q''_\bullet \arrow[rr] \arrow[dr, "\eta''"] & & 0 & \\
& 0 \arrow[rr] & & B' \arrow[rr, "\iota_B"] & & B \arrow[rr, "\pi_B"] & & B'' \arrow[rr] & & 0
\end{tikzcd}
$$

Note that every square in the above diagram commutes, provided $F_\bullet$ have been constructed. We construct a map $\gamma_n : P''_n \to Q'_n$ so that written in a matrix form, we have

$$
F_n = \begin{pmatrix} F'_n & \gamma_n \\ 0 & F''_n \end{pmatrix} : \begin{matrix} P'_n \\ \oplus \\ P''_n \end{matrix} \longrightarrow \begin{matrix} Q'_n \\ \oplus \\ Q''_n \end{matrix}
$$

Recall that $P_n = P'_n \oplus P''_n$ and $Q_n = Q'_n \oplus Q''_n$. Now F needs to be a lift of f. So, $\eta \circ F = f \circ \varepsilon$ must hold. Restricting to the summands, we get the equations

$$
f\varepsilon|_{P'_0} = \iota_B \eta' F'_0, \qquad\qquad f\varepsilon|_{P''_0} = \iota_B \eta' \gamma_0 + \eta|_{Q''_0} F''_0.
$$

The first relation is already satisfied as F'_0 is a lift of f', and hence,

$$
f\varepsilon|_{P'_0} = f\iota_A \varepsilon' = \iota_B f' \varepsilon' = \iota_B \eta' F'_0.
$$

For the second relation, we need to find $\gamma_0 : P''_0 \to Q'_0$ such that

$$
\iota_B \eta' \gamma_0 = f\varepsilon|_{P''_0} - \eta|_{Q''_0} F''_0.
$$

Note that

$$
\pi_B\left(f\varepsilon|_{P''_0} - \eta|_{Q''_0} F''_0\right) = f''\pi_A \varepsilon|_{P''_0} - \pi_B \eta|_{Q''_0} F''_0 = f''\varepsilon'' - \eta'' F''_0 = 0,
$$

and hence, we have a well-defined map

$$
\beta := \iota_B^{-1}\left(f\varepsilon|_{P''_0} - \eta|_{Q''_0} F''_0\right) : P''_0 \to B'.
$$

As P''_0 is projective, we can now get a lift γ_0 in the diagram

$$
\begin{tikzcd}
& P''_0 \arrow[dl, dashed, "\gamma_0"'] \arrow[d, "\beta"] & \\
Q'_0 \arrow[r, "\eta'"'] & B' \arrow[r] & 0
\end{tikzcd}
$$

This immediately gives $\iota_B\eta'\gamma_0 = \iota_B\beta = f\varepsilon|_{P_0''} - \eta|_{Q_0''} F_0''$, as required. Next, for F to be lifted to a chain map, we require $dF = Fd$. Using Remark 15.8, we get

$$\begin{aligned}0 = dF - Fd &= \begin{pmatrix} d' & \lambda \\ 0 & d'' \end{pmatrix}\begin{pmatrix} F' & \gamma \\ 0 & F'' \end{pmatrix} - \begin{pmatrix} F' & \gamma \\ 0 & F'' \end{pmatrix}\begin{pmatrix} d' & \lambda \\ 0 & d'' \end{pmatrix} \\ &= \begin{pmatrix} d'F' & d'\gamma + \lambda F'' \\ 0 & d''F'' \end{pmatrix} - \begin{pmatrix} F'd' & F'\lambda + \gamma d'' \\ 0 & F''d'' \end{pmatrix} \\ &= \begin{pmatrix} d'F' - F'd' & d'\gamma - \gamma d'' + \lambda F'' - F'\lambda \\ 0 & d''F'' - F''d'' \end{pmatrix}\end{aligned}$$

Thus, inductively, we need to solve for $\gamma_n : P_n'' \to Q_n'$ such that

$$d'\gamma_n = \gamma_{n-1}d'' + \lambda_n F_n'' - F_{n-1}'\lambda_n.$$

We compute,

$$\begin{aligned}d'(\gamma_{n-1}d'' + \lambda_n F_n'' - F_{n-1}'\lambda_n) &= (\gamma_{n-2}d'' + \lambda_{n-1}F_{n-1}'' - F_{n-2}'\lambda_{n-1})d'' \\ &\quad + d'\lambda_n F_n'' - F_{n-2}'d'\lambda_n \\ &= \lambda_{n-1}d''F_n'' - F_{n-2}'\lambda_{n-1}d'' + d'\lambda_n F_n'' - F_{n-2}'d'\lambda_n \\ &= (\lambda_{n-1}d'' + d'\lambda_n)F_n'' - F_{n-2}'(\lambda_{n-1}d'' + d'\lambda_n) \\ &= (d'\lambda_n + \lambda_{n-1}d'')(F_n'' - F_{n-2}') = 0,\end{aligned}$$

where the last equality follows from $0 = d^2 = \left(\begin{smallmatrix} d' & \lambda_n \\ 0 & d'' \end{smallmatrix}\right)\left(\begin{smallmatrix} d' & \lambda_{n-1} \\ 0 & d'' \end{smallmatrix}\right) \Rightarrow d'\lambda_{n-1} + \lambda_n d''$. Thus, we have the map

$$\beta_n := \gamma_{n-1}d'' + \lambda_n F_n'' - F_{n-1}'\lambda_n : P_n'' \to \ker(d') \subset Q_{n-1}'.$$

Now, there is a lift $\gamma_n : P_n'' \to Q_n'$ such that the diagram commutes

$$\begin{tikzcd} & P_n'' \arrow[dl, dashed, "\gamma_n"'] \arrow[d, "\beta_n"] & \\ Q_n' \arrow[r, "d'"'] & \operatorname{im}(d') = \ker(d') \arrow[r] & 0. \end{tikzcd}$$

Clearly, this lets us define $F_\bullet : P_\bullet \to Q_\bullet$ so that we have a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & P_\bullet' \arrow[r] \arrow[d, "F_\bullet'"] & P_\bullet \arrow[r] \arrow[d, "F_\bullet"] & P_\bullet'' \arrow[r] \arrow[d, "F_\bullet''"] & 0 \\ 0 \arrow[r] & Q_\bullet' \arrow[r] & Q_\bullet \arrow[r] & Q_\bullet'' \arrow[r] & 0 \end{tikzcd}$$

But then the naturality of the boundary map in Lemma 6.29 shows that the boundary map in the long exact sequence of derived functors is also natural. □

One of the most useful right exact functor that appears in the category of R-modules are the tensor functors.

Definition 15.13: (*Tor Functor*)

The left derived functors of the (right exact) tensor functor $\_ \otimes_R N : R\text{-Mod} \to R\text{-Mod}$ is called the *Tor functors*, and is denoted as $\operatorname{Tor}_n^R(\_, N) : R\text{-Mod} \to R\text{-Mod}$.

Example 15.14: (*Group Homology*)

Let G be a group, and R be a ring. One defines the *group ring* $R[G]$ as the free R-module generated by G (as a set), along with the multiplication $\left(\sum_{g\in G} r_g g\right) \cdot \left(\sum_{h\in G} s_h h\right) = \sum_{g\in G} \sum_{g_1 g_2 = g} \left(r_{g_1} s_{g_1}\right) g$. This makes $R[G]$ in to an algebra over R. An $\mathbb{Z}[G]$-module is known as a *G-module*. Now, given an $R[G]$ -module M, define the *coinvariants* as

$$M_G := M/\{m \in M \mid g \cdot m = m, \forall g \in G\}.$$

This defines a functor $(.)_G : R[G]\text{-Mod} \to R[G]\text{-Mod}$, which can be verified to be *right* exact. The left derived functors of the coinvariants is denoted as $H_i(G, M)$, the *group homology* of G with coefficients in the $R[G]$-module M. One can naturally identify $M_G = R \otimes_{R[G]} M$, where R is given the trivial $R[G]$-module structure: $\left(\sum r_g g\right) \cdot r = \sum r_g r$. In other words, taking coinvariants is same as taking the tensor product $R \otimes_{R[G]} \_$. Thus, group homology is essentially a Tor functor, i.e, $H_i(G, M) = \operatorname{Tor}_i^{R[G]}(R, M)$.

Exercise 15.15: (*Tor over PID*)

Let R be a PID. Given R-modules M, N, show that $\operatorname{Tor}_n^R(M, N) = 0$ for $n \geq 2$. In particular, over a PID R we can simply denote $\operatorname{Tor}^R(\_, \_) := \operatorname{Tor}_1^R(\_, \_)$. Moreover, when $R = \mathbb{Z}$, we can simply right $\operatorname{Tor}(\_, \_) := \operatorname{Tor}_1^{\mathbb{Z}}(\_, \_)$.

Since $M \otimes \_ : R\text{-Mod} \to R\text{-Mod}$ is also right exact (by the symmetry of the tensor when R is commutative), we can left derive $M \otimes \_$ as well and get another definition of the Tor functor. It turns out the two definitions match, and is known as *balancing the Tor*, i.e, $\mathfrak{L}_i(\_ \otimes_R N)(M) \cong \operatorname{Tor}_i^R(M, N) \cong \mathfrak{L}_i(M \otimes_R \_)(N)$. Commutativity of R also leads to the isomorphism $\operatorname{Tor}_n^r(M, N) \cong \operatorname{Tor}_n^R(M, N)$ for all $n \geq 0$.

Exercise 15.16: (*Computation of Tor*)

Compute the following.

1. $\operatorname{Tor}_n^{\mathbb{Z}}(\mathbb{Z}, G)$ for any Abelian group G.
2. $\operatorname{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, G)$ for any Abelian group G and $a > 1$.

> **<u>Hint</u>** : Given an Abelian group G and an integer a, we have two subgroups:
>
> - $aG = \{g \in G \mid g = ah \text{ for some } h \in G\}$, i.e, the subgroup of elements divisible by a.
> - ${}_aG = \{g \in G \mid ag = 0\}$, i.e, the subgroup of elements of order dividing a.
>
> Check that $\mathbb{Z}/a\mathbb{Z} \otimes G = aG$ and $\operatorname{Tor}(\mathbb{Z}/a\mathbb{Z}, G) = {}_aG$.

3. $\operatorname{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z})$ for $a, b > 1$.
4. $\operatorname{Tor}_n^{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ where $\mathbb{Z}/2\mathbb{Z}$ is given a $\mathbb{Z}/4\mathbb{Z}$-module structure in the obvious way.

5. $\mathrm{Tor}_n^{\mathbb{Z}/r\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z})$, where $a, b, r > 1$ are integers, and $\mathrm{lcm}(a, b) \mid r$.
6. $\mathrm{Tor}_n^R(M, N)$ for a flat R-module M, and arbitrary R-module N.
7. $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Q}, G)$ for any Abelian group G.
8. $\mathrm{Tor}_n^k(V, W)$ for a field k, and k-modules V, W.

Remark 15.17: (*Tor over Noncommutative Ring*)

Let R, S, T be arbitrary rings (possibly noncommutative). We can treat the tensor as a bifunctor

$$\_ \otimes_S \_ : R\text{-}S\text{-}\mathrm{BiMod} \times S\text{-}T\text{-}\mathrm{BiMod} \to R\text{-}T\text{-}\mathrm{BiMod}.$$

It follows that $\_ \otimes_S \_$ is right exact in both places, and one can define the (left) derived Tor functor as

$$\mathrm{Tor}_n^S : R\text{-}S\text{-}\mathrm{BiMod} \times S\text{-}T\text{-}\mathrm{BiMod} \to R\text{-}T\text{-}\mathrm{BiMod},$$

which is also *balanced*.

Corollary 15.18: (*Long Exact Sequence of Tor*)

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules, and N be a fixed R-module. Then, there exists a natural long exact sequence

$$\cdots \to \mathrm{Tor}_1^R(A, N) \to \mathrm{Tor}_1^R(B, N) \to \mathrm{Tor}_1^R(C, N) \to A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0.$$

Moreover, if R is a PID, then we have the 6-term long exact sequence

$$0 \to \mathrm{Tor}_1^R(A, N) \to \mathrm{Tor}_1^R(B, N) \to \mathrm{Tor}_1^R(C, N) \to A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0.$$

Proof : The existence of the long exact sequence is immediate from Theorem 15.12. When R is a PID, given any R-module M, we have a long exact sequence $0 \to 0 \to P_1 \to P_0 \xrightarrow{\varepsilon} M$, which shows that $\mathrm{Tor}_n^R(M, N) = 0$ for $n \geq 2$. This justifies the 6-term exact sequence. □

Day 16 : 31st March, 2026

`Tor via flat resolution – right derived functors – Ext functors – injective Abelian groups – injective` R`-modules – universal coefficient theorems`

16.1 A Digression : Left Derived Functor by Flat Resolution

Let us show that Tor can also be computed using flat resolution. We need an algebraic lemma first.

Lemma 16.1: (*Snake Lemma à la Cartier-Weil*)

Let $f : A \to B$ and $g : B \to C$ be maps in an Abelian category $\mathcal{A}$. Then, there exists an exact sequence

$$0 \to \ker(f) \to \ker(g \circ f) \to \ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(g \circ f) \to \operatorname{coker}(g) \to 0.$$

Visualized another way, we have

$$\begin{tikzcd}
& & \ker(g) \arrow[rr, bend left] \arrow[dr, hook] & & \operatorname{coker}(f) \arrow[ddrr] & & \\
& & & B \arrow[ur, two heads] \arrow[dr, "g"] & & & \\
\ker(g \circ f) \arrow[uurr] \arrow[rr, hook] & & A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & C \arrow[rr, two heads] \arrow[dr, two heads] & & \operatorname{coker}(g \circ f) \arrow[dl, bend left] \\
& \ker(f) \arrow[ul, bend left] \arrow[ur, hook] & & 0 \arrow[ll] & & \operatorname{coker}(g) \arrow[ll] &
\end{tikzcd}$$

<u>Proof</u> : Consider the diagram of exact sequences

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "g \circ f"] & B \arrow[r] \arrow[d, "g"] & \operatorname{coker}(f) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r, equal] & C \arrow[r] & 0 &
\end{tikzcd}$$

Applying Lemma 6.29, we get the exact sequence

$$\begin{tikzcd}
& \ker(g \circ f) \arrow[r] \arrow[d] & \ker(g) \arrow[r] \arrow[d] & \operatorname{coker}(f) \arrow[d] \arrow[dddll, rounded corners] & \\
\ker(f) \arrow[ur, bend left] \arrow[r] & A \arrow[r, "f"] \arrow[d, "g \circ f"] & B \arrow[r] \arrow[d, "g"] & \operatorname{coker}(f) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r, equal] \arrow[d] & C \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
& \operatorname{coker}(g \circ f) \arrow[r] & \operatorname{coker}(g) \arrow[r] & 0 \arrow[ur, bend right] &
\end{tikzcd}$$

Similarly, consider the diagram of exact sequence,

$$
\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r] \arrow[d, "g \circ f"] & 0 \\
0 \arrow[r] & \ker(g) \arrow[r] & B \arrow[r, "g"'] & C &
\end{tikzcd}
$$

Again, applying Lemma 6.29, we get the exact sequence

$$
\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \ker(f) \arrow[r] \arrow[d] & \ker(g \circ f) \arrow[d] \arrow[dddll] & \\
0 \arrow[r] \arrow[ur, bend left] & 0 \arrow[r] \arrow[d] & A \arrow[r, equal] \arrow[d, "f"] & A \arrow[r] \arrow[d, "g \circ f"] & 0 \\
0 \arrow[r] & \ker(g) \arrow[r] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & \operatorname{coker(g)} \\
 & \ker(g) \arrow[r] & \operatorname{coker}(f) \arrow[r] & \operatorname{coker}(g \circ f) \arrow[ur, bend right] &
\end{tikzcd}
$$

Splicing the two exact sequences, we get

$$
0 \to \ker(f) \to \ker(g \circ f) \to \ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(g \circ f) \to \operatorname{coker}(g) \to 0,
$$

which proves the claim. $\square$

We now have the following useful fact.

Proposition 16.2: (*Tor via Flat Resolution*)

Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact additive functor between two Abelian categories, and suppose $\mathcal{A}$ has enough projectives. Then the Tor functors can be computed by flat resolutions, i.e, given $M, N \in \mathcal{A}$ and a *flat* resolution $F_\bullet \to M$, we have $\operatorname{Tor}_n^R(M, N) \cong H_n(F_\bullet \otimes N)$, and similarly in the second variable.

Proof : Let $M, N \in \mathcal{A}$ be fixed. Suppose $F_\bullet \xrightarrow{\varepsilon} M$ is a flat resolution. Since tensor is right exact, we have an exact sequence

$$
F_1 \otimes N \xrightarrow{d_1 \otimes \mathrm{Id}_N} F_0 \otimes N \xrightarrow{\varepsilon \otimes \mathrm{Id}_N} M \otimes N \to 0.
$$

It follows that

$$
H_0(F_\bullet \otimes N) = \frac{F_0 \otimes N}{\operatorname{im}(d_1 \otimes \mathrm{Id}_N)} \cong M \otimes N \cong \operatorname{Tor}_0^R(M, N).
$$

For $n = 1$, we have the diagram

$$
\begin{tikzcd}
F_2 \arrow[r, "d_2"] & F_1 \arrow[rr, "d_1"] \arrow[dr, "d_1'"'] & & F_0 \arrow[r, "\varepsilon"] & M \\
 & & K_0 = \ker \varepsilon \arrow[ur, hook, "\iota_0"'] & &
\end{tikzcd}
$$

Tensoring by N, we have the commutative diagram

$$\begin{tikzcd}
F_1 \otimes N \arrow[rr, "d_1 \otimes \mathrm{Id}_N"] \arrow[dr, "d_1' \otimes \mathrm{Id}_N"'] & & F_0 \otimes N \\
& K_0 \otimes N \arrow[ur, "\iota_0 \otimes \mathrm{Id}_N"'] &
\end{tikzcd}$$

The right exactness of tensor shows that $d_1' \otimes \mathrm{Id}_N$ is an epimorphism, and hence, $\mathrm{im}(d_1 \otimes \mathrm{Id}_N) = \mathrm{im}(\iota_0 \otimes \mathrm{Id}_N)$ follows. Let

$$f := \widetilde{d_1' \times \mathrm{Id}_N} : \frac{F_1 \otimes N}{\mathrm{im}(d_2 \otimes \mathrm{Id}_N)} \to K_0 \otimes N, \qquad g = \iota_0 \otimes \mathrm{Id}_N : K_0 \otimes N \to F_0 \otimes N.$$

The exact sequence $F_2 \to F_1 \to K_0 \to 0$ gives the exact sequence

$$F_2 \otimes N \xrightarrow{d_2 \otimes \mathrm{Id}_N} F_1 \otimes N \xrightarrow{d_1' \otimes \mathrm{Id}_N} K_0 \otimes N \to 0,$$

which implies

$$\ker(f) = \frac{\ker(d_1' \otimes \mathrm{Id}_N)}{\mathrm{im}(d_2 \otimes \mathrm{Id}_N)} = 0.$$

Also, f is an epimorphism, as it is induced by the epimorphism $d_1' \times \mathrm{Id}_N$. Now, by Lemma 16.1, we have the exact sequence

$$\underbrace{\ker(f)}_{0} \to \ker(g \circ f) \to \ker(g) \to \underbrace{\mathrm{coker}(f)}_{0}.$$

Hence, we have an isomorphism $\ker(g \circ f) \cong \ker(g)$. But $\ker(g \circ f) = \frac{\ker(d_1 \otimes \mathrm{Id}_N)}{\mathrm{im}(d_2 \times \mathrm{Id}_N)} = H_1(F_\bullet \otimes N)$. Hence, $H_1(F_\bullet \otimes N) \cong \ker(g) = \ker(\iota_0 \times \mathrm{Id}_N)$. Now, applying Theorem 15.12 to the exact sequence $0 \to K_0 \to F_0 \to M \to 0$ we have (part of) the exact sequence for Tor

$$\mathrm{Tor}_1^R(F_0, N) \to \mathrm{Tor}_1^R(M, N) \to K_0 \otimes N \overset{\iota_0 \otimes \mathrm{Id}_N}{\longrightarrow} F_0 \otimes N.$$

Since F_0 is flat, we have $\mathrm{Tor}_1^R(F_0, N) = 0$, which implies

$$\mathrm{Tor}_1^R(M, N) = \ker(\iota_0 \otimes \mathrm{Id}_N) \cong H_1(F_\bullet \otimes N).$$

Let us now inductively assume that $\mathrm{Tor}_n^R(M, N) \cong H_n(F_\bullet \otimes N)$ for any flat resolution of any object in $\mathcal{A}$. We have part of the exact sequence

$$\mathrm{Tor}_{n+1}^R(F_0, N) \to \mathrm{Tor}_{n+1}^R(M, N) \to \mathrm{Tor}_n^R(K_0, N) \to \mathrm{Tor}_n(F_0, N).$$

As F_0 is flat, we have $\mathrm{Tor}_{n+1}^R(M, N) \cong \mathrm{Tor}_n(K_0, N)$. Now, $\cdots \to F_2 \to F_1 \to K_0 \to 0$ is a flat resolution of K_0, say, $F_\bullet' \to K_0$. Hence, by induction

$$\mathrm{Tor}_{n+1}^R(M, N) \cong \mathrm{Tor}_n^R(K_0) \cong H_n(F_\bullet' \otimes N) = H_{n+1}(F_\bullet \otimes N).$$

This proves the claim. $\square$

16.2 Right Derived Functors and Ext

Simlar to the left derived functors of a right exact functor, we can compute the right derived functors of a left exact functor. In particular, suppose $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Suppose $\mathcal{A}$ has enough injectives. Given $A \in \mathcal{A}$, fix some injective resolution $A \overset{\varepsilon}{\to} I^\bullet$. Then, the i^{th} *right derived functor* of F is defined as

$$\mathfrak{R}^i F(A) := H^i(F(I_\bullet)) = H^i(0 \to F(I^0) \to F(I^1) \to \cdots).$$

Just like the left derived functors, it follows that $\mathfrak{R}^i F : \mathcal{A} \to \mathcal{B}$ is a well-defined functor (up to isomorphism), and $\mathfrak{R}^0 F = F$ naturally. Moreover, we have the following.

Theorem 16.3: (*Long Exact Sequence of Right Derived Functors*)

Left $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor between Abelian categories, where $\mathcal{A}$ has enough injectives. Then, given a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, there exists a natural long exact sequence

$$0 \to \underbrace{\mathfrak{R}^0 F(A)}_{F(A)} \to \underbrace{\mathfrak{R}^0 F(B)}_{F(B)} \to \underbrace{\mathfrak{R}^0 F(C)}_{F(C)} \to \mathfrak{R}^1 F(A) \to \mathfrak{R}^1 F(B) \to \mathfrak{R}^1 F(C) \to \cdots.$$

The (internal) $\hom$ functor in the category of modules, both the covariant and contravariant ones, are well-known left exact functors.

Definition 16.4: (*Ext Functor*)

For a fixed R-module, the right derived functors of the (covariant) $\hom$-functor $\hom_R(M, \_) : R\text{-Mod} \to R\text{-Mod}$ are called the *Ext functors*, and is denoted as $\mathrm{Ext}^n_R(M, \_) : R\text{-Mod} \to R\text{-Mod}$.

Similarly, for fixed R-module N, the right derived functors of the *contravariant* $\hom$-functor $\hom_R(\_, N) : R\text{-Mod} \to R\text{-Mod}$ are also called Ext functors, denoted as $\mathrm{Ext}^n_R(\_, N) : R\text{-Mod} \to R\text{-Mod}$. We can do this using a left projective resolution as well. Indeed, given a projective resolution $P_\bullet \to M$, it follows that $\mathrm{Ext}^i_R(M, N) = H_i(\hom(P_\bullet, N))$. As a consequence, if R is a PID, it follows that $\mathrm{Ext}^R_i(M, N) = 0$ for $i \geq 2$, and for $R = \mathbb{Z}$ we simply denote

$$\mathrm{Ext}(M, N) := \mathrm{Ext}^1_{\mathbb{Z}}(M, N), \qquad M, N \in \mathbb{Z}\text{-Mod} = \mathrm{Ab}.$$

In general, we have a bi-functor

$$\mathrm{Ext}^n_R(\_, \_) : R\text{-Mod} \to R\text{-Mod},$$

which is contravariant on the first vairable, and covariant in the second. Of couse, this requires *balancing of Ext*, i.e, we need to show that $\mathfrak{R}^i(\hom_R(M, \_))(N) \cong \mathrm{Ext}^i_R(M, N) \cong \mathfrak{R}^i(\hom_R(\_, N))(M)$.

16.3 Ext over a PID

Over a PID, injective modules are characterized as *divisible* ones.

Definition 16.5: (*Divisible Module*)

An R-module M is called *divisible* if given any regular element $0 \neq r \in R$ (i.e, $r \in R$ is a non-zero-divisor), and any $m \in M$, there is some $x \in M$ such that $rx = m$ holds. In other words, $rM = M$ for any regular element $r \in R$.

In general, when considering divisible modules, we assume that the ring is an *integral domain* (i.e, whenever $r_1 r_2 = 0$ for some $r_1, r_2 \in R$, we have $r_1 = 0$ or $r_2 = 0$). In this case, every nonzero element is a non-zero-divisor. Note that a PID is by definition an integral domain.

Theorem 16.6: (*Injective Modules over PID are Divisible*)

If R is an integral domain, then an injective R-module I is divisible. Moreover, if R is a PID, a divisible R-module D is injective.

Proof : Suppose R is an integral domain, and I is an injective R-module. Let $0 \neq r \in R$ and $m \in I$ be fixed. Consider the map $f : R \to R$ given by (left) multiplication by. Since $r \neq 0$ is not a zero-divisor, it follows that f is injective. Also consider the map $\varphi : R \to M$ given by $\varphi(1) = x$ (and extending linearly). As I is injective (Definition 14.25), we have an R-linear map $\Phi : R \to I$ solving the diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "f"] \arrow[d, "\varphi"'] & R \arrow[dl, dashed, "\Phi"] \\ & I & \end{tikzcd}$$

Consider $x = \Phi(1)$. Then,

$$rx = r\Phi(1) = \Phi(r.1) = \Phi \circ f(1) = \varphi(1) = m.$$

As $0 \neq r \in R$ and $m \in I$ are arbitrary, we have I is divisible.

Now, suppose R is a PID and D is a divisible R-module. Let us consider a monomorphism $f : A \to B$ and a map $\varphi : A \to D$. To show the existence of $\Phi : B \to D$ such that $\Phi \circ f = \varphi$, we need to apply Zorn's lemma. As f is monic, without loss of generality, by identifying A with $f(A)$, we can assume that A is a submodule of B. Consider the family

$$\mathscr{F} = \{(C, \psi) \mid A \subset C \subset B \text{ as submodules}, \psi : C \to D, \psi|_A = \varphi\}.$$

The family is nonempty since $(A, \varphi) \in \mathscr{F}$. We have a partial order $\preceq$ on $\mathscr{F}$: for $(C_1, \psi_1), (C_2, \psi_2) \in \mathscr{F}$, we have $(C_1, \psi_1) \leq (C_2, \psi_2)$ if and only if $C_1 \subset C_2$ and $\psi_2|_{C_1} = \psi_1$. Suppose $\mathscr{J} = \{(C_i, \psi_i)\}$ is a chain in $\mathscr{F}$ with respect to $\preceq$. Let $C = \bigcup C_i$, and define $\psi : C \to D$ by $\psi|_{C_i} = \psi_i$. It is easy to see that C is a submodule of B with $A \subset C \subset B$, and ψ is a well-defined R-module map, such that $\psi|_A = \varphi$. Thus $(C, \psi) \in \mathscr{F}$ is an upper bound of the chain $\mathscr{J}$. Then by Zorn's lemma, there exists a maximal element $(C_0, \psi_0) \in \mathscr{F}$. We claim that $C_0 = B$. If not, then there is an element $x \in B \setminus C_0$. Let $C \subset B$ be the submodule generated by $C_0 \cup \{x\}$. Clearly, $A \subset C_0 \subsetneq C \subset B$. Let $I = \{r \in R \mid rx \in C_0\}$. Clearly, I is an ideal in R, and hence, $I = \langle a \rangle$ for some $a \in R$ (as R is a PID).

- If $a \neq 0$, divisibility of D implies that there is some $y \in D$ such that $ay = \psi_0(ax)$.
- If $a = 0$, choose some arbitrary $y \in D$.

In any case, we can now extend ψ_0 to $\psi : C \to D$ by setting $\psi(x) = y$. That is, define $\psi(c_0 + rx) = \psi_0(c_0) + ry$ for $c_0 + rx \in C$. Let us verify that this is well-defined. Indeed, if $rx \in C_0$ for some $r \in R$, we have $r \in I = \langle a \rangle \Rightarrow r = sa$ for some $s \in R$. Then, $\psi(rx) = ry = say = s\psi_0(ax) = \psi_0(sax) = \psi_0(rx)$. Thus, $\psi : C \to D$ is well-defined. This contradicts the maximality of (C_0, ψ_0). Hence, $C_0 = B$, and clearly $\Phi = \psi_0$ is an extension of φ as required. This proves that D is an injective module. $\square$

Example 16.7: (*Injective Abelian Groups*)

It is easy to verify that $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{R}$ are divisible Abelian groups. Since $\mathbb{Z}$ is a PID, injective $\mathbb{Z}$-modules are precisely the divisible groups (Theorem 16.6). Hence, we have $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{R}$ as examples of injective $\mathbb{Z}$-modules.

Recall the fact that every R-module embeds in an injective module, i.e, $R\text{-Mod}$ has enough injectives (Definition 15.1). Let us give a proof of this for $\mathbb{Z}$. The general statement is proved in Corollary 16.17.

Proposition 16.8: (*Enough Injective Abelian Groups*)

Every Abelian group can be embedded in a direct product of $\mathbb{Q}/\mathbb{Z}$, which is seen to be a divisible (and hence an injective) Abelian group.

Proof : Let A be an Abelian group. Fix $0 \neq A$, and consider the subgroup $\langle a \rangle \hookrightarrow A$.

- If a has infinite order in A, choose arbitrary $0 \neq x_a \in \mathbb{Q}/\mathbb{Z}$.
- If a has finite order, say, $n \geq 2$ in A, choose $0 \neq x_a \in \mathbb{Q}/\mathbb{Z}$ such that $\operatorname{ord}(x_a) \mid n$.

In any case, we have a map $\varphi_a : \langle a \rangle \to \mathbb{Q}/\mathbb{Z}$ defined by setting $\varphi(a) = x_a$. Since $\mathbb{Q}/\mathbb{Z}$ is divisible and hence injective, extend the map φ_a to a map $\Phi_a : A \to \mathbb{Q}/\mathbb{Z}$. Then, using the universal property of product, define a map $\Phi : A \to \prod_{0 \neq a \in A} \mathbb{Q}/\mathbb{Z}$ where each component is Φ_a. Clealry direct product of divisible groups is again divisible, and hence $\prod_{0 \neq a \in A} \mathbb{Q}/\mathbb{Z}$ is injective. Moreover, Φ is clearly injective. Indeed, if for some $0 \neq a \in A$ we have $\Phi(a) = 0$, then in particular, $0 = \varphi_a(a) = x_a$, a contradiction. Thus, we can embed A in to an injective Abelian group. $\square$

We now note the following interesting result for a PID.

Proposition 16.9: (*Quotient of Injective*)

Let R be a PID. Then, any quotient of an injective R-module is again injective.

Proof : Since injective modules are precisely the divisible ones for a PID, let us consider D to be a divisible R-module, and let $\varphi : D \to E$ be an epimorphism. Let us show that E is divisible. Let $0 \neq r \in R$ and $e \in E$ be given. Choose some $d \in D$ such that $\varphi(d) = e$. Since D is divisible, there is some $y \in D$ such that $ry = d$. Let $x = \varphi(x) \in E$. Clearly, $rx = r\varphi(y) = \varphi(ry) = \varphi(d) = e$. Thus, E is divisible, and hence, an injective R-module. $\square$

As a corollary, we have the following.

Corollary 16.10: ($\operatorname{Ext}_R^n = 0$ *for* $n \geq 2$ *over a PID* R)

Let R be a PID. Given any R-module N, there is an injective resolution $0 \to N \to I^0 \to I^1 \to 0$, and consequently, $\operatorname{Ext}_R^n(M, N) = 0$ for any $M \in R\text{-Mod}$ and $n \geq 2$.

Proof : Let N be an arbitrary R-module. We embed $0 \to N \overset{\eta}{\hookrightarrow} I^0$, where I^0 is injective. Let $I^1 = \operatorname{coker}(\eta)$, which is then again injective. Clearly, we have an exact sequence $0 \to N \to I^0 \to I^1 \to 0$, which is then an injective resolution.

As we have an injective resolution of length 2, it follows from the construction that $\mathrm{Ext}^n_R(M, N) = 0$ for any R-module M, and $n \geq 2$. $\square$

Exercise 16.11: (*Computation of Ext*)

Compute the following.

1. $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, G)$ for any Abelian group G and $a > 1$.
2. $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z})$ for $a, b > 1$.
3. $\mathrm{Ext}^n_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$ for any Abelian group G.
4. $\mathrm{Ext}^n_k(V, W)$ for a field k and k-modules V, W.

Example 16.12: (*Group Cohomology*)

Let us give the dual example to Example 15.14. Let G be a group, R be a ring, and M be an $R[G]$-module. Define the *invariants* as

$$M^G := \{m \in M \mid g \cdot m = m, \forall g \in G\}.$$

This defines a functor $(\cdot)^G : R[G]\text{-Mod} \to R[G]\text{-Mod}$, which can be identified with $\hom_{R[G]}(R, \_)$. Thus, invariants being a hom functor is left exact. The right derived functors, denoted as $H^i(G, M)$ is called the *group cohomology* of G with coefficients in M. Clearly, $H^i(G, M) = \mathrm{Ext}^i_{R[G]}(R, M)$.

Remark 16.13: (*Hochschild (co)Homology*)

Another interesting class of derived functors appear in Algebra known as Hochschild (co)homology. In particular, given a possibly noncommutative algebra A over a ring R, and an A-bimodule, one defines the *Hochschild cohomology* as $HH^i(A, M) := \mathrm{Ext}^i_{A^{\mathrm{op}} \otimes_k A}(A, M)$, and the *Hochschild homology* as $HH_i(A, M) := \mathrm{Tor}_i^{A^{\mathrm{op}} \otimes_k A}(A, M)$. These are important tools in deformation theory of algebras.

Exercise 16.14: (*Ext and Projective/Injective*)

Let A be an R-module. Show that

1. A is projective if and only if $\mathrm{Ext}^1_R(M, A) = 0$ for all R-module M, and
2. A is injective if and only if $\mathrm{Ext}^1_R(A, N) = 0$ for all R-moduls N.

Remark 16.15: (*Interpreting* Ext^n_R *As Extensions*)

Given R-modules A, C, we call $B \in R\text{-Mod}$ to be an *extension* of C by A if we have a short exact sequence $0 \to A \to B \to C \to 0$. Call two extensions B_1, B_2 *equivalent* if there is an isomorphism $\varphi : B_1 \to B_2$ such that the following diagram commutes

$$\begin{tikzcd} & & B_1 \arrow[dr] \arrow[dd, "\varphi"'] & & \\ 0 \arrow[r] & A \arrow[ur] \arrow[dr] & & C \arrow[r] & 0 \\ & & B_2 \arrow[ur] & & \end{tikzcd}$$

One can verify that the equivalence classes of extensions is in bijection with the set $\mathrm{Ext}^1_R(C, A)$, where 0 corresponds to split extensions. In particular, if $\mathrm{Ext}^1_R(C, A) = 0$, then every extension of C by A splits.

Higher Ext modules are related to n-extensions, namely, an exact sequence $0 \to A \to B_1 \to \cdots \to B_n \to C$. Consider the equivalence relation on all n-extensions *generated* by the relation that there is a chain map

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & B_n \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & B_1 \arrow[r] & \cdots \arrow[r] & B_n \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Then, $\mathrm{Ext}^n_R(C, A)$ is in bijection with the equivalence classes of all n-extensions of C by A.

16.4 A Digression : Enough Injective R-Modules

Let us give a category theoretic proof of the fact that for any ring R, the category of R-modules has enough injectives! First, we have the following useful result.

Proposition 16.16: (*Adjoint and Injectives*)

Suppose we have an adjunction $\mathfrak{L} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathfrak{R}$ of additive functors $\mathfrak{L}, \mathfrak{R}$ between two Abelian categories $\mathcal{A}, \mathcal{B}$. Assume that

- $\mathfrak{L}$ is *exact*, i.e, takes short exact sequence in $\mathcal{A}$ to short exact sequence in $\mathcal{B}$ (in particular, assume $\mathfrak{L}$ is left exact as any left adjoint is alsways right exact), and
- $\mathfrak{L}$ is *faithful*, i.e, given any $A_1, A_2 \in \mathcal{A}$ the induced map $\hom_{\mathcal{A}}(A_1, A_2) \to \hom_{\mathcal{B}}(\mathfrak{L}(A_1), \mathfrak{L}(A_2))$ is injective.

Then, $\mathcal{A}$ has enough injectives, provided $\mathcal{B}$ has enough injectives.

Proof : Suppose $\mathcal{B}$ has enough injectives. Let $A \in \mathcal{A}$. Then, there exists an injective object $I \in \mathcal{B}$ and a monomorphism $\iota : \mathfrak{L}(A) \hookrightarrow I$. By the adjuction $\hom_{\mathcal{B}}(\mathfrak{L}(A), I) \cong \hom_{\mathcal{A}}(A, \mathfrak{R}(I))$, we have a morphism $\tilde{\iota} : A \to \mathfrak{R}(I)$. We claim the following.

- **$\mathfrak{R}(I)$ is injective in $\mathcal{A}$:** We show that the right adjoint of an exact functor preserves injectives. Indeed, consider a monomorphism $f : A_1 \hookrightarrow A_2$ and a map $\varphi : A_1 \to \mathfrak{R}(I)$ in $\mathcal{A}$. Applying the adjunction, and the fact $\mathfrak{L}$ is exact (and in particular, preseves monomorphisms), we have the diagram

$$\begin{tikzcd}
\mathfrak{L}(A_1) \arrow[r, hook, "\mathfrak{L}(f)"] \arrow[d, "\tilde{\varphi}"'] & \mathfrak{L}(A_2) \arrow[dl, dashed, "\Psi"] \\
I &
\end{tikzcd}$$

The extension Ψ exists since I is injective. But then again by the adjunction, we have a map $\Phi = \tilde{\Psi} : A_2 \to I$ solving

$$\begin{tikzcd}
A_1 \arrow[r, hook, "f"] \arrow[d, "\varphi"'] & A_2 \arrow[dl, dashed, "\Phi"] \\
\mathfrak{R}(I) &
\end{tikzcd}$$

- **$\tilde{\iota}$ is a monomorphism:** We have an exact sequence $0 \to \ker(\tilde{\iota}) \to A \xrightarrow{\tilde{\iota}} \mathfrak{R}(I)$. Since $\mathfrak{L}$ is exact, we have an exact sequence

$$0 \to \mathfrak{L}(\ker(\tilde{\iota})) \to \mathfrak{L}(A) \xrightarrow{\mathfrak{L}(\tilde{\iota})} \mathfrak{L}(\mathfrak{R}(I)).$$

Now, using the unit $\varepsilon : \mathfrak{L} \circ \mathfrak{R} \Rightarrow \mathrm{Id}_{\mathcal{B}}$ of the adjunction, it folows that the composition is

$$\begin{tikzcd}
\mathfrak{L}(A) \arrow[r, "\mathfrak{L}(\tilde{\iota})"] \arrow[rr, bend right, "\iota"'] & \mathfrak{L}(\mathfrak{R}(I)) \arrow[r, "\varepsilon(I)"] & I
\end{tikzcd}$$

Since the composition is monic, it follows that $\mathfrak{L}(\tilde{\iota})$ is also monic. But then $\mathfrak{L}(\ker(\tilde{\iota})) \to \mathfrak{L}(A)$ is the 0 map. Since $\mathfrak{L}$ is assumed to be faithful, it follows that $\ker(\tilde{\iota}) \to A$ is also the 0 map. In other words, ι is monic.

Thus, we have an embedding $\iota : A \hookrightarrow \mathfrak{R}(I)$, where $\mathfrak{R}(I)$ is injective in $\mathcal{A}$. Since $A \in \mathcal{A}$ was arbitrary, it follows that $\mathcal{A}$ has enough injectives. □

Then, as corollary we have that any module category has enough injectives.

Corollary 16.17: ($R\text{-Mod}$ *has Enough Injectives*)

Given any ring R, the category $R\text{-Mod}$ has enough injectives.

Proof : Consider the *forgetful functor* $U : R\text{-Mod} \to \mathbb{Z}\text{-Mod} = \mathrm{Ab}$.

- U is evidently additive.
- U has a right adjoint $\mathfrak{R} : \mathrm{Ab} \to R\text{-Mod}$ given by *coextension of scalars*. Explicitly, we have $\mathfrak{R}(G) = R \otimes_{\mathbb{Z}} G$, where we treat the ring R as a $\mathbb{Z}$-module; then $\mathfrak{R}(G)$ is an R-module via the multiplication in R. Clearly $\mathfrak{R}$ is additvie as well.
- U being a left adjoint, is right exact. Moreover, any monomorphism of R-module is clearly an injective Abelian group map. Hence, U is exact.
- U is clearly faithful since any R-module map being 0 as a group map implies the map itself is 0.

Now, by Proposition 16.8, we have Ab has enough injectives. But then by Proposition 16.16, we have $R\text{-}\mathrm{Mod}$ has enough injectives. □

Unlike Proposition 16.9, over a general ring R it does not follow that a quotient of an injective module is again injective. Thus, for an aribitrary ring R, higher Ext modules may not vanish.

16.5 Universal Coefficient Theorems

The goal of this section is to compute the (co)homology of a a free chain complex after tensoring (or taking $\hom$) by an R-module. This leads to the definition of (co)homology with coefficients. We shall always assume that R is a PID; there are analogous theorems for a general ring R, which are usually stated via spectral sequences. If there is no confusion, we shall simply denote $\mathrm{Tor} = \mathrm{Tor}_1^R$ and $\mathrm{Ext} = \mathrm{Ext}_R^1$.

Theorem 16.18: (*UCT for Homology with Coefficients*)

Let R be a PID, $C_\bullet$ be a chain complex of free R-modules, and M be an arbitrary R-module. Then, there exists a short exact sequence

$$0 \to H_n(C_\bullet) \otimes M \to H_n(C_\bullet \otimes M) \to \mathrm{Tor}(H_{n-1}(C_\bullet), M) \to 0,$$

which is natural in both $C_\bullet$ and the coefficient module M. Moreover, the short exact sequence splits, the splitting is natural in the coefficient module, but may not be natural with respect to the chain complex.

Proof : Let us denote the module of cycles, boundaries, and the homology as

$$Z_n := \ker\left(C_n \xrightarrow{\partial} C_{n-1}\right), \qquad B_n := \mathrm{im}\left(C_{n+1} \xrightarrow{\partial} C_n\right), \qquad H_n := H_n(C_\bullet) = B_n/Z_n.$$

Since R is a PID, both Z_n and B_n are free, being submodules of the free R-module C_n (Proposition 14.7). In particular, we have a short exact sequence

$$0 \to Z_n \to C_n \to B_{n-1} \to 0,$$

which is split as B_{n-1} is a free (and hence projective) R-module. Hence, tensoring by M, we again get a split exact sequence

$$0 \to Z_n \otimes M \to C_n \otimes M \to B_{n-1} \otimes M \to 0.$$

Treating $Z_\bullet \otimes M$ and $B_\bullet \otimes M$ as a chain complex with 0-differentials, and $C_\bullet \otimes M$ with the differential $\partial \otimes \mathrm{Id}_M$, we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes M \to C_\bullet \otimes M \to B_\bullet \otimes M[-1] \to 0,$$

where $B_\bullet \otimes M$ is shifted by degree -1. Applying Theorem 7.1, we then have a long exact seuqence in homology

$$\cdots \to B_n \otimes M \xrightarrow{\delta} Z_n \otimes M \to H_n(C_\bullet \otimes M) \to B_{n-1} \otimes M \xrightarrow{\delta} Z_{n-1} \otimes M \to \cdots$$

Here δ is the boundary map of the long exact sequence. Recall from Remark 7.2, we have the diagram

$$
\begin{tikzcd}
 & C_n\otimes M \arrow[rr, two heads, "\partial\otimes\mathrm{Id}_M"] \arrow[dd, "\partial\otimes\mathrm{Id}_M"'] & & B_{n-1}\otimes M \\
 & & y \arrow[r, mapsto] \arrow[dd, mapsto] & x \arrow[ddlll, bend left, "\delta"] \\
Z_{n-1}\otimes M \arrow[r, hook] & C_{n-1}\otimes M & & \\
x \arrow[rr, mapsto] & & x &
\end{tikzcd}
$$

Hence, δ is nothing but the map $\iota\otimes\mathrm{Id}_M$, where $\iota : B_n \hookrightarrow Z_n$ is the inclusion.

We also have a short exact sequence

$$0 \to B_n \to Z_n \to H_n \to 0.$$

Since B_n, Z_n are free, we can treat this as a free (hence projective) resolution of H_n. In particular, it follows that

$$\mathrm{Tor}(H_n, M) \cong \ker(\iota\otimes\mathrm{Id}_M : B_n\otimes M \to Z_n\otimes M) = \ker(\delta).$$

Also, exactness of $B_n\otimes M \to Z_n\otimes M \to H_n\otimes M \to 0$ implies that

$$\mathrm{coker}(\delta) = \mathrm{coker}(\iota\otimes\mathrm{Id}_M) \cong H_n\otimes M.$$

Then, splicing the homology long exact sequence we get

$$0 \to H_n\otimes M \to H_n(C_\bullet\otimes M) \to \mathrm{Tor}\ (H_n, M) \to 0,$$

which is the required short exact sequence.

The naturality of the short exact sequence follows from the functoriality of Tor and the above construction. As for the splitting, *choose* a splitting $s_n : B_{n-1} \to C_n$ of the exact sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$, which gives a splitting $\sigma_n := s_n\otimes\mathrm{Id}_M$ of the exact sequence after tensoring. In particular, $\sigma_\bullet$ is a splitting for the short exact sequence of the chain complexes, which induces a splitting in the homology exact sequence. Since s_n was a choice independent of the module M, it is natural with respect to the coefficients, but may not be natural with respect to the chain complex. $\square$

Let us explain the meaning of the splitting being *not* natural.

Example 16.19: (*Unnaturality of the Splitting in Homology UCT*)

Let us consider the map of chain complexes

$$\begin{tikzcd}
 & & {\scriptstyle (4)} & {\scriptstyle (3)} & {\scriptstyle (2)} & {\scriptstyle (1)} & {\scriptstyle (0)} & \\
C_\bullet \arrow[d, "f_\bullet"'] & : & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "0"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
D_\bullet & : & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"'] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Note that the homology is given as

$$H_n(C_\bullet) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}/2\mathbb{Z}, & n = 1 \\ 0, & \text{otherwise.} \end{cases} \qquad H_n(D_\bullet) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}/2\mathbb{Z}, & n = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the map $f_\bullet$ induces 0 everywhere except Id on $H_0(C_\bullet) \to H_0(D_\bullet)$. Next, tensoring by $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, we have the diagram

$$\begin{tikzcd}
 & {\scriptstyle (4)} & {\scriptstyle (3)} & {\scriptstyle (2)} & {\scriptstyle (1)} & {\scriptstyle (0)} & \\
C_\bullet \otimes \mathbb{Z}_2 : \arrow[d, "f_\bullet \times \mathrm{Id}_{\mathbb{Z}_2}"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z}_2 \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z}_2 \arrow[r, "0"] \arrow[d] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
D_\bullet \otimes \mathbb{Z}_2 : & 0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "0"'] & \mathbb{Z}_2 \arrow[r] & 0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

We have the homology with coefficients

$$H_n(C_\bullet \otimes \mathbb{Z}_2) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & n = 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases} \qquad H_n(D_\bullet \otimes \mathbb{Z}_2) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & n = 0, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

Passing to homology, we now have an extra identity map $\mathrm{Id} : H_2(C_\bullet \otimes \mathbb{Z}_2) \to H_2(D_\bullet \otimes \mathbb{Z}_2)$ induced by $f_\bullet \otimes \mathrm{Id}_{\mathbb{Z}_2}$.

Looking at the short exact sequence from Theorem 16.18, we have the diagram

$$\begin{tikzcd}
 & {\scriptstyle 0} & {\scriptstyle \mathbb{Z}_2} & {\scriptstyle \mathbb{Z}_2} & \\
0 \arrow[r] & H_2(C_\bullet) \otimes \mathbb{Z}_2 \arrow[r] \arrow[d] & H_2(C_\bullet \otimes \mathbb{Z}_2) \arrow[r] \arrow[d] & \mathrm{Tor}(H_1(C_\bullet), \mathbb{Z}_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_2(D_\bullet) \otimes \mathbb{Z}_2 \arrow[r] & H_2(D_\bullet \otimes \mathbb{Z}_2) \arrow[r] & \mathrm{Tor}(H_1(D_\bullet), \mathbb{Z}_2) \arrow[r] & 0 \\
 & {\scriptstyle \mathbb{Z}_2} & {\scriptstyle \mathbb{Z}_2} & {\scriptstyle 0} &
\end{tikzcd}$$

If the splitting were natural, then we must have a commutative diagram

$$\begin{tikzcd}
H_2(C_\bullet \otimes \mathbb{Z}_2) \arrow[r, "\cong"] \arrow[d, "\mathrm{Id}"'] & H_2(C_\bullet) \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}\,(H_1(C_\bullet), \mathbb{Z}_2) \arrow[d, "0 \oplus 0"] \\
H_2(C_\bullet \otimes \mathbb{Z}_2) \arrow[r, "\cong"'] & H_2(C_\bullet) \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}\,(H_1(C_\bullet), \mathbb{Z}_2)
\end{tikzcd}$$

where the isomorphisms are induced by the splitting. But clearly the diagram does not commute, which shows that the splitting in UCT is not natural in the chain complexes.

Day 17 : 31st March, 2026

`universal coefficient theorem for cohomology – singular (co)homology with coefficients – universal coefficient theorem for singular (co)homology – graded modules – tensor of chain complexes – Koszul sign rule`

17.1 Universal Coefficient Theorems (Cont.)

Similar to Theorem 16.18, we now prove a universal coefficient theorem for cohomology, which lets us compute the cohomology with coefficients from the homology.

Theorem 17.1: (*UCT for Cohomology*)

Let R be a PID, $C_\bullet$ be a chain complex of free R-modules, and M be an arbitray R-module. Then, there exists a short exact sequence

$$0 \to \operatorname{Ext}\,(H_{n-1}(C_\bullet), M) \to H^n(\hom_R(C_\bullet, M)) \to \hom_R(H_n(C_\bullet), M) \to 0,$$

which is natural in both $C_\bullet$ and the coefficient module M. Moreover, the short exact sequence splits, the splitting is natural in the coefficient module, but may not be natural with respect to the chain complex.

Proof : Let us again denote the modules

$$Z_n := \ker\left(C_n \xrightarrow{\partial} C_{n-1}\right), \qquad B_n := \operatorname{im}\left(C_{n+1} \xrightarrow{\partial} C_n\right), \qquad H_n := H_n(C_\bullet) = B_n / Z_n.$$

We have the short exact sequence

$$0 \to Z_n \to C_n \to B_{n-1} \to 0,$$

which is split exact as R is a PID. Applying $\hom_R(\_, M)$, we have a split exact sequence

$$0 \to \hom_R(B_{n-1}, M) \to \hom_R(C_n, M) \to \hom_R(Z_n, M) \to 0.$$

Considering $\hom_R(Z_\bullet, M)$ and $\hom_R(B_\bullet, M)$ as cochain complexes with 0-codifferentials, we have a short exact sequence of cochain complexes

$$0 \to \hom_R(B_\bullet, M)[-1] \to \hom_R(C_\bullet, M) \to \hom_R(Z_\bullet, M) \to 0.$$

Applying Theorem 7.1, we have a long exact sequence in cohomology

$$\cdots \to \hom_R(B_{n-1}, M) \to H^n(\hom_R(C_\bullet, M)) \to \hom_R(Z_n, M) \to \hom_R(B_n, M) \to H^{n+1}(\hom_R(C_\bullet, M)) \to \cdots.$$

The boundary map can be easily identified with $\iota^\star : \hom_R(Z_n, M) \to \hom_R(B_n, M)$, where $\iota : Z_n \hookrightarrow B_n$ is the inclusion map. This leads to the short exact sequences

$$0 \to \operatorname{coker}(i^\star) \to H^n(\hom_R(C_\bullet, M)) \to \ker(\iota^\star) \to 0.$$

Now, we have another short exact sequence

$$0 \to B_n \xrightarrow{\iota} Z_n \to H_n \to 0,$$

which can be treates a free resolution of H_n. This gives,

$$\operatorname{Ext}(H_n, M) \cong \operatorname{coker}(\iota^\star), \qquad \ker(\iota^\star) = \hom_R(H_n, M).$$

Hence, we have the short exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}, M) \to H^n(\hom_R(C_\bullet, M)) \to \hom_R(H_n, M) \to 0,$$

which is the required exact sequence. The sequence is natural as Ext is natural. A splitting can be induced from a choice of a splitting for $0 \to Z_n \to C_n \to B_{n-1} \to 0$; the splitting is natural with respect to the coefficient module, but may not be natural with respect to the chain complex. $\square$

Exercise 17.2: (*UCT for Projectives*)

State and prove the UCT for (co)homology of a chain complex of *projective* R-modules, where R is a PID.

Exercise 17.3: (*Unnaturality of the Splitting in Cohomology UCT*)

Give an example to demonstrate that the splitting in Theorem 17.1 may not be natural with respect to the chain complex.

Remark 17.4: (*The Integral Morphism*)

The natural map $H^n(\hom_R(C_\bullet, M)) \to \hom_R(H_n(C_\bullet), M)$ in Theorem 17.1 is induced by

$$\int : \hom_R(C_n, M) \to \hom_R(H_n(C_\bullet), M)$$
$$\varphi \mapsto \left([\sigma] \mapsto \int_\sigma \varphi = \varphi(\sigma)\right).$$

The reason to denote this map by the *integral* symbol becomes apparent when the cohomology theory is the deRham cohomology and the homology is the singular homology with compact support.

Let us state the universal coefficient theorem for cochain complexes.

Theorem 17.5: (*UCT for Cochain Complex*)

Let R be a PID, $C^\bullet$ be a cochain complex of free R-modules, and M be an arbitrary R-module. Then, there exists a short exact sequence

$$0 \to H^n(C^\bullet) \otimes M \to H^n(C^\bullet \otimes M) \to \mathrm{Tor}(H^{n+1}(C^\bullet), M) \to 0,$$

which is natural in both $C^\bullet$ and the coefficient module M. Moreover, the short exact sequence splits, the splitting is natural in the coefficient module, but may not be natural with respect to the chain complex.

Proof : We can identify the cochain complex (C^n, δ^n) as the chain complex (C_n, ∂_n), where $C_n = C^{-n}$ and $\partial_n : C_n \to C_{n-1}$ as $\delta^{-n} : C^{-n} \to C^{-n+1}$. The proof is then immediate from Theorem 16.18. $\square$

In order to compute the cohomology with coefficients in a module from cohomology without coefficients, we need some extra hypothesis.

Definition 17.6: (*Finite Type Chain Complex*)

A chain complex $C_\bullet$ of R-module is called *finite type* if C_n is a finitely generated R-module for each n.

Exercise 17.7: ($\hom$ *Commutes with Tensor if Finitely Generated*)

Let R be a commutative ring and $C, M \in R\text{-Mod}$. Shoh that there exists a canonical isomorphism

$$\hom_R(C, M) \cong \hom_R(C, R) \otimes M,$$

provided either C or M is finitely generated R-module.

We have the following *rectification* result.

Proposition 17.8: (*Finite Type Homology*)

Let R be a PID, and $C_\bullet$ be a chain complex of free R-modules such that $H_n(C_\bullet)$ is finitely generated for each n. Then, there exists a chain complex $C'_\bullet$ of finite type, such that $C_\bullet$ is chain homotopy equivalent to $C'_\bullet$.

Theorem 17.9: (*UCT for Cohomology with Coefficients*)

Let R be a PID, $C_\bullet$ be a chain complex of free R-modules, and M be an R-module. Asuume that either $H_n(C_\bullet)$ is finitely generated for each n, or that M is finitely generated. Then, there exists a short exact sequence

$$0 \to H^n(\hom_R(C_\bullet, R)) \otimes M \to H^n(\hom_R(C_\bullet, M)) \to \mathrm{Tor}(H^{n+1}(\hom_R(C_\bullet, R)), M) \to 0,$$

which is natural in both $C_\bullet$ and the coefficient module M (whenever the additional hypothesis holds). Moreover, the sequence splits, the splitting is natural in the coefficient module.

Proof : If M is finitely generated, then we have a natural isomorphism

$$\hom_R(C_\bullet, R) \otimes M \cong \hom_R(C_\bullet, M).$$

We can then use Theorem 17.5. If $H_n(C_\bullet)$ is finitely generated for each n, then using Proposition 17.8, we can get a free chain complex $C'_\bullet$ of finite type such that $C'_\bullet$ is chain homotopy equivalent to $C_\bullet$. Again, we have the isomorphism $\hom_R(C'_\bullet, R) \otimes M \cong \hom_R(C'_\bullet, M)$, and using Theorem 17.5 we get the short exact sequence for $C'_\bullet$. Since all the functors involved is invariant for chain homotopy equivalence, we have the required short exact sequence for $C_\bullet$. $\square$

17.2 Singular (Co)Homology With Coefficients

Let R be a PID. Then, given a space X, one can define the singular chain complex

$$S_n(X; R) = \text{Free } R\text{-module generated by singular } n\text{-simplices.}$$

This can also be understood as

$$S_n(X; R) = S_n(X) \otimes_{\mathbb{Z}} R.$$

Next, for an R-module M, we have the singular chain complex with coefficients in M as

$$S_n(X; M) := S_n(X; R) \otimes M.$$

The homology groups of $S_n(X; M)$ are called the *singular homology with coefficients in M*, and is denoted as $H_n(X; M)$. Similarly, we have singular cochain complex with coefficients in M defined as

$$S^n(X; M) := \hom_R(S_n(X; R), M).$$

The cohomology groups are called *singular cohomology with coefficients in M*, and is denoted as $H^n(X; M)$. Using the universal coefficient theorems, we can now derive the following theorems.

Theorem 17.10: (*UCT for Singular Homology with Coefficients*)

Let R be a PID, and M be an R-module. Then, for a space X, there exists a short exact sequence

$$0 \to H_n(X; R) \otimes M \to H_n(X; M) \to \mathrm{Tor}(H_{n-1}(X; R), M) \to 0,$$

which is natural in both X and in M. Moreover, the sequence splits, the splitting being natural in M but possibly not in X.

Proof : Proof is immediate from Theorem 16.18. □

Theorem 17.11: (*UCT for Singular Cohomology*)

Let R be a PID, and M be an R-module. Then, for a space X, there exists a short exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(X; R), M) \to H^n(X; M) \to \hom_R(H_n(X; R), M) \to 0,$$

which is natural in both X and in M. Moreover, the sequence splits, the splitting being natural in M but possibly not in X.

Proof : Proof is immediate from Theorem 17.1. □

Theorem 17.12: (*UCT for Singular Cohomology with Coefficients*)

Let R be a PID, M be an R-module, and X be a space. Suppose that either M is finitely genrated, or that each $H_n(X; R)$ is finitely generated. Then, for a space X, there exists a short exact sequence

$$0 \to H^n(X; R) \otimes M \to H^n(X; M) \to \mathrm{Tor}(H^{n+1}(X; R), M) \to 0,$$

which is natural in both X and in M. Moreover, the sequence splits, the splitting being natural in M but possibly not in X.

Proof : Proof is immediate from Theorem 17.9. □

17.3 Graded Modules and Tensor Product

Let us recall the notions of graded ring and graded modules over it.

Definition 17.13: (*Graded Ring*)

A *$\mathbb{Z}$-graded ring* R is a ring equipped with a direct sum decomposition $R = \bigoplus_{i\in\mathbb{Z}} R_i$, such that each R_i is an ideal, and $R_i \cdot R_j \subset R_{i+j}$.

Example 17.14: (*Example of Graded Rings*)

Here a few examples of graded ring that appears naturally.

- A prototypical example of a graded ring is the ring of polynomials $R[T]$, which is graded by degree of the polynomial (with 0 in the negative degrees); i.e, $R[T] = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} R\langle T^i\rangle$.
- The *Laurent polynomial ring* $R[T, T^{-1}]$ (obtained by localizing $R[T]$ at T) is also a graded ring: $R[T, T^{-1}] = \bigoplus_{i\in\mathbb{Z}} R\langle T^i\rangle$.
- Any ring R can be understood as a graded ring with $R_0 = R$ and $R_i = 0$ for all $i \neq 0$.

Definition 17.15: (*Graded Module*)

Given a graded ring R, a *graded module* over R is an R-module M equipped with a direct sum decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$, such that $R_i \cdot M_j \subset M_{i+j}$ holds. An R-module map $f : M \to N$ of graded modules is given as the sum $f = \sum_{i\in\mathbb{Z}} f_i$, where $f_i : M_i \to N_i$. More generally, a *degree-k graded map* is an R-module map $f : M \to N$, where $f = \sum f_i$ for $f_i : M_i \to N_{i+k}$.

Note that a graded module can be thought of as a chain complex with 0-differential, and then a degree k map is precisely a degree k chain map. The homological algebra we developed so far carries over to the category of graded modules with graded map (of degree 0).

Exercise 17.16: (*Projective and Injective Graded Modules*)

Let R be a ring. Justify the following.

1. A graded R-module $P = \bigoplus_{i\in\mathbb{Z}} P_i$ is a projective R-module if and only if each P_i is a projective R -module
2. A graded R-module $I = \bigoplus_{i\in\mathbb{Z}} I_i$ is a projective R-module if and only if each I_i is an injective R-module

Definition 17.17: (*Tensor Product of Graded Modules*)

Given graded modules M, N over a graded ring R, their tensor product is the R-module $M \otimes_R N$ (computed by forgetting the grading) with the grading given by

$$(M \otimes N)_k = \frac{\bigoplus_{p+q=k} M_p \otimes_{\mathbb{Z}} N_q}{\langle mr \otimes n - m \otimes rn \mid m \in M_a, r \in R_b, n \in N_c, \quad a+b+c=n\rangle}.$$

For a usual ring R (with grading concentrated at 0), this simplifies to

$$(M \otimes N)_k = \bigoplus_{p+q=k} M_p \otimes_R N_q.$$

Now, given two chain complexes, we can take tensor product of the underlying graded modules. We need to define a differential on it; of course we can 0 as the differential, but that does not give it the desired properties!

Definition 17.18: (*Tensor Product of Chain Complexes*)

Let $(C_\bullet, \partial_\bullet^C), (D_\bullet, \partial_n^D)$ be two chain complexes of R-modules (or more generaly, of objects in any monoidal Abelian category). Then, the *tensor product* is defined as the chain complex $((C \otimes D)_\bullet, \partial_\bullet^{C\otimes D})$, where

$$(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j,$$

and the differential is defined on *homogeneous* degree by the formula

$$\partial_n^{C\otimes D}(x \otimes y) = \partial_i^C(x) \otimes y + (-1)^i x \otimes \partial_{j(y)}^D, \qquad x \otimes y \in C_i \otimes D_j, \quad i + j = n.$$

Exercise 17.19: (*Boundary of Tensor Product*)

Verify that the boundary map $\partial_\bullet^{C\otimes D}$ squares to zero, so that $((C \otimes D)_\bullet, \partial_\bullet^{C\otimes D})$ is indeed a chain complex.

Remark 17.20: (*Tensor*-$\hom$ *Adjunction*)

Let R be a module, and $(D_\bullet, \partial_\bullet^D)$ be a chain complex of R-modules. Then the functor $\_ \bigotimes D_\bullet : \mathrm{Ch} \to \mathrm{Ch}$ is left adjoint to the internal $\hom$-functor $[D_\bullet, \_] : \mathrm{Ch} \to \mathrm{Ch}$. That is, there is a natural isomorphism

$$\hom_{\mathrm{Ch}}(C \otimes D, E) \cong \hom_{\mathrm{Ch}}(C, [D, E]), \quad \forall \quad C, E \in \mathrm{Ch}.$$

Thus, the tensor product is actually defined via an universal property.

Let us finally mention a rule that governs the signs appearing everywhere!

Remark 17.21: (*Koszul Sign Rule*)

The boundary map in the tensor product can be defined symbolically as

$$\partial_n^{C\otimes D} = \sum_{i+j=n} \partial_i^C \otimes \mathrm{Id}_{D_j} + \mathrm{Id}_{C_i} \otimes \partial_j^D.$$

The appearance of the sign can be justified by the *Koszul sign rule*. Suppose $A = \bigoplus_{n\in\mathbb{Z}} A_n, B = \bigoplus_{n\in\mathbb{Z}} B_n$ are graded objects, and $f, g : A \to B$ are graded maps. We define $f \otimes g : A \otimes A \to B \otimes B$ as the sum $\sum_{i+j=n} f_i \otimes g_j$. The sign appears whenever we *evaluate* a tensor of maps on an element :

$$(f \otimes g)(x \otimes y) := (-1)^{|x||g|} f(x) \otimes g(y).$$

Here, $|x|$ denotes the homogeneous degree of x, and $|g|$ denotes the degree of the map.

Heurestically, whenever g jumps over an element x, it collects the sign $(-1)^{|x||g|}$. It generalizes to multiple tensors as well. The sign rule is designed in such way that boundary map squares to zero in any graded context.

Example 17.22: (*The Tensor Product Boundary Formula*)

As mentioned in Remark 17.21, the boundary formula can be written as

$$\partial_n^{C\otimes D} = \sum_{i+j=n} \partial_i^C \otimes \mathrm{Id}_{D_j} + \mathrm{Id}_{C_i} \otimes \partial_j^D.$$

Let us justify that it squares to 0 using the Koszul sign rule. We have

$$\begin{aligned}
\partial_n^{C\otimes D} \circ \partial_{n+1}^{C\otimes D} &= \left(\sum_{i+j=n} \partial_i^C \otimes \mathrm{Id}_{D_j} + \mathrm{Id}_{C_i} \otimes \partial_j^D\right) \circ \left(\sum_{i+j=n+1} \partial_i^C \otimes \mathrm{Id}_{D_j} + \mathrm{Id}_{C_i} \otimes \partial_j^D\right) \\
&= \sum_{i+j=n} \left[\left(\partial_i^C \otimes \mathrm{Id}_{D_j}\right) \circ \left(\mathrm{Id}_{C_i} \otimes \partial_{j+1}^D\right) + \left(\mathrm{Id}_{C_i} \otimes \partial_j^D\right) \circ \left(\partial_{i+1}^C \otimes \mathrm{Id}_{D_j}\right)\right] \\
&= \sum_{i+j=n} \left[(-1)^{\overbrace{0}^{\left|\mathrm{Id}_{D_j}\right|} \cdot \overbrace{0}^{\left|\mathrm{Id}_{C_i}\right|}} \partial_i^C \otimes \partial_{j+1}^D + (-1)^{\overbrace{-1}^{\left|\partial_j^D\right|} \cdot \overbrace{-1}^{\left|\partial_{i+1}^C\right|}} \partial_{i+1}^C \otimes \partial_j^D\right] \\
&= \sum_{i+j=n} \partial_i^C \otimes \partial_{j+1}^D - \sum_{i+j=n} \partial_{i+1}^C \otimes \partial_j^D \\
&= \sum_{i+j=n+1} \partial_i^C \otimes \partial_j^D - \sum_{i+j=n+1} \partial_i^C \otimes \partial_j^D \\
&= 0.
\end{aligned}$$

Note that any other composition is automatically 0, and we can ignore the signs.

As another example, recall the internal $\hom$ given as $[C,D]_n := \prod_{k\in\mathbb{Z}} \hom(X_k, Y_{k+n})$, which consists of degree n maps between the underlying graded modules $C_\bullet, D_\bullet$. The boundary map $d_n : [C,D]_n \to [C,D]_{n-1}$ is given via the formula

$$d_n\left((h_k)_{k\in\mathbb{Z}}\right) = \left(\partial_{k+n}^D \circ h_k - (-1)^n h_{k-1} \circ \partial_k^C\right)_{k\in\mathbb{Z}}.$$

The sign is justified since we are swapping the degree n map (h_k) and the degree -1 boundary map.

Day 18 : 7th April, 2026

`Künneth theorem for chain complex — Eilenberg-Zilber theorem — Künneth theorem for spaces — homological cross product`

18.1 Künneth Theorems in Homological Algebra

We now prove the Künneth theorems, which helps us compute the (co)homology of tensor product of chain complexes.

Theorem 18.1: (*Künneth Theorem for Homology*)

Let R be a PID, and $C_\bullet$ be a chain complex of free R-modules. Then, for any chain complex $D_\bullet$ of R-modules, we have a natural short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(C_\bullet) \otimes H_j(D_\bullet) \to H_n((C \otimes D)_\bullet) \to \bigoplus_{i+j=n-1} \operatorname{Tor}\big(H_i(C_\bullet), H_j(D_\bullet)\big) \to 0.$$

Moreover, if $D_\bullet$ consists of free R-modules, then the short exact sequence splits (not necessarily naturally).

Proof : Let us consider the graded modules $Z(C_\bullet), B(C_\bullet)$ and $H(C_\bullet)$ of respectively cycles, boundaries and homology of $C_\bullet$ as chain complexes with *trivial* boundary maps. Since R is a PID, and $C_\bullet$ is a complex of *free* R-modules, we have $Z(C_\bullet)$ and $B(C_\bullet)$ are also complexes of free R-modules. Now, we have the chain complex $\big((Z(C) \otimes D)_\bullet, \partial_\bullet^{Z(C)\otimes D}\big)$, where the boundary is given as

$$\partial_n^{Z(C)\otimes D} = \sum_{i+j=n} \mathrm{Id}_{Z_i(C_\bullet)} \otimes \partial_j^D.$$

Since $Z_i(C_\bullet)$ is free, we have $\operatorname{Tor}(Z_i(C_\bullet), \_) = 0$, and hence it follows that

$$\ker\big(\mathrm{Id} \otimes \partial_j^D : Z_i(C_\bullet) \otimes D_j \to Z_i(C_\bullet) \otimes D_{j-1}\big) = Z_i(C_\bullet) \otimes Z_j(D_\bullet),$$

and

$$\operatorname{im}\big(\mathrm{Id} \otimes \partial_{j+1}^D : Z_i(C_\bullet) \otimes D_{j+1} \to Z_i(C_\bullet) \otimes D_j\big) = Z_i(C_\bullet) \otimes B_j(D_\bullet).$$

In particular, since (co)kernels for graded modules are computed degreewise, we have the equalities

$$(Z(C_\bullet) \otimes Z(D_\bullet))_n = \ker\Big(\mathrm{Id} \otimes \partial^D : (Z(C_\bullet) \otimes D_\bullet)_n \to (Z(C_\bullet) \otimes D_\bullet)_{n-1}\Big),$$

and

$$(Z(C_\bullet) \otimes B(D_\bullet))_n = \operatorname{im}\Big(\mathrm{Id} \otimes \partial^D : (Z(C_\bullet) \otimes D_\bullet)_{n+1} \to (Z(C_\bullet) \otimes D_\bullet)_n\Big).$$

Moreover, tensoring the short exact sequence $0 \to B_j(D_\bullet) \to Z_j(D_\bullet) \to H_j(D_\bullet) \to 0$ by $Z_i(C_\bullet)$ gives a short exact sequence, and consequently, by the first isomorphism theorem,

$$\frac{Z_i(C_\bullet) \otimes Z_j(D_\bullet)}{Z_i(C_\bullet) \otimes B_j(D_\bullet)} \cong Z_i(C_\bullet) \otimes H_j(D_\bullet).$$

This implies that

$$H(Z(C_\bullet) \otimes D_\bullet) = Z(C_\bullet) \otimes H(D_\bullet),$$

as graded modules. By a similar argument, since $B(C_\bullet)$ is a free (graded) module, we have

$$H(B(C_\bullet) \otimes D_\bullet) = B(C_\bullet) \otimes H(D_\bullet)$$

as graded modules. We now have a *degreewise free* resolution

$$0 \to B(C_\bullet) \to Z(C_\bullet) \to H(C_\bullet) \to 0.$$

Tensoring by $H(D_\bullet)$, and taking homology, we have the exact sequence

$$0 \to \mathrm{Tor}(H(C_\bullet), H(D_\bullet)) \to B(C_\bullet) \otimes H(D_\bullet) \to Z(C_\bullet) \otimes H(D_\bullet) \to H(C_\bullet) \otimes H(D_\bullet) \to 0.$$

On the other hand, we have the *split* short exact sequence of chain complex of free R-modules

$$0 \to Z(C_\bullet) \to C_\bullet \to B(C_\bullet)[-1] \to 0.$$

Tensoring by $D_\bullet$ gives the (split) short exact sequence

$$0 \to Z(C_\bullet) \otimes D_\bullet \to C_\bullet \otimes D_\bullet \to (B(C_\bullet) \otimes D_\bullet)[-1] \to 0.$$

Passing to the long exact sequence in homology, and identifying the (co)kernel of the maps $\iota \otimes \mathrm{Id} : B(C_\bullet) \otimes D_\bullet \to Z(C_\bullet) \otimes D_\bullet$, we get the short exact sequence

$$0 \to H(C_\bullet) \otimes H(D_\bullet) \to H(C_\bullet \otimes D_\bullet) \to \mathrm{Tor}(H(C_\bullet), H(D_\bullet))[-1] \to 0.$$

Since Tor is additive, we get the required short exact sequence.

Finally, assume $D_\bullet$ is a chain complex of free R-modules. Then, we can *choose* retractions $r_\bullet : C_\bullet \to Z_n(C_\bullet)$ and $s_\bullet : D_\bullet \to Z_n(D_\bullet)$ in the respective split exact sequences. This induces a map $(C_\bullet \otimes D_\bullet)_n \to H(C_\bullet) \otimes H(D_\bullet)$ given by $c \otimes d \mapsto [r(c)] \otimes [s(d)]$. One can verify that this map sends boundaries in $C_\bullet \times .D_\bullet$ to 0, and hence, induces a map $\rho : H_n(C_\bullet \otimes D_\bullet) \to H(C_\bullet) \otimes H(D_\bullet)$. Clearly, ρ is a retraction in the short exact sequence, which then splits. As $r_\bullet, s_\bullet$ are choices, it follows that the splitting may not be natural. □

In the above theorem we have freely used homological algebraic machinary developed for R-modules to graded R-modules by degreewise consideration. This naive approach does not work when we concisder chain complexes with nontrivial differentials!

Remark 18.2: (*Künneth Theorems for Spaces*)

Can we directly use Theorem 18.1 for spaces to compute homology of product?! Suppose X, Y are two spaces, and denote $C_\bullet = S_\bullet(X), D_\bullet = S_\bullet(Y)$ to be the singular chain complexes (or the cellular ones, if we are working with CW complexes). Then, Künneth theorem is applicable for $C_\bullet, D_\bullet$, and we are able to compute $H(C_\bullet \otimes D_\bullet) = H(S_\bullet(X) \otimes S_\bullet(Y))$. On the other hand, the homology of the product $X \times Y$ is given as $H(S_\bullet(X \times Y))$. The challenge is then to relate $S_\bullet(X) \otimes S_\bullet(Y)$ with $S_\bullet(X \times Y)$! We shall see later (Theorem 18.6) that these two chain complexes are in fact quasi-isomorphic.

Exercise 18.3: (*Algebraic Cross Product*)

Verify that the first map in Theorem 18.1 is given component wise as

$$\times_{\text{alg}} : H_p(C_\bullet) \otimes H_q(D_\bullet) \to H_{p+q}(C_\bullet \otimes D_\bullet)$$
$$[\varphi] \otimes [\psi] \mapsto [\varphi \otimes \psi].$$

This is usually known as the *algebraic cross product* in homology.

Similar to Theorem 17.9, we have the following.

Theorem 18.4: (*Künneth Theorem for Cohomology*)

Let R be a PID, and $C_\bullet, D_\bullet$ be cochain complexes of free R-modules. Assume that either $H(C_\bullet)$ or $H(D_\bullet)$ is of finite type. Then, for the dual complex $C^\bullet = \hom(C_\bullet, R), D^\bullet = \hom(D_\bullet, R)$ there exists a natural short exact sequence

$$0 \to \bigoplus_{i+j=n} H^i(C^\bullet) \otimes H^j(D^\bullet) \to H^n(C^\bullet \otimes D^\bullet) \to \bigoplus_{i+j=n+1} \operatorname{Tor}(H^i(C^\bullet), H^j(D^\bullet)) \to 0,$$

which split.

18.2 Künneth Theorem for Spaces

Fix a PID R, and denote by $\mathrm{Ch}_{\geq 0}$ the category of chain complexes of R-modules, with 0 in the strictly negative degrees. We then have two functors

$$\mathscr{F} : \mathrm{Top} \times \mathrm{Top} \to \mathrm{Ch}_{\geq 0} \qquad \mathscr{G} : \mathrm{Top} \times \mathrm{Top} \to \mathrm{Ch}_{\geq 0}$$
$$(X, Y) \to S_\bullet(X) \otimes S_\bullet(Y), \qquad (X, Y) \to S_\bullet(X \times Y).$$

Here, $S_\bullet(\_)$ is the singular chain complex functor with R-coefficients. It is easy to see that

$$S_0(X) \otimes S_0(Y) = R\langle X \rangle \otimes R\langle X \rangle \cong R\langle X \times Y \rangle = S_0(X \times Y).$$

Indeed, 0-simplicies are nothing but points, and hence, sending the 0-simplices $\sigma_x \in S_0(X), \sigma_y \in S_0(Y)$ to the simplex $\sigma_{x,y} \in S_0(X \times Y)$ defines a natural isomorphism, with inverse given by $\sigma_{x,y} \mapsto \sigma_x \otimes \sigma_y$. But this strategy does not generalize immediately.

Definition 18.5: (*Eilenberg-Zilber Maps*)

A pair of natural transformations

$$\mathfrak{P} : S_\bullet(-) \otimes S_\bullet(-) \Rightarrow S_\bullet(- \times -) \qquad \text{and} \qquad \mathfrak{Q} : S_\bullet(- \times -) \Rightarrow S_\bullet(-) \otimes S_\bullet(-)$$

are called *Eilenberg-Zilber maps* (or *EZ-maps*) if they extend the canonical isomorphism at the 0^{th} level.

By an application of the *acyclic model theorem,* one gets the following.

Theorem 18.6: (*Eilenberg-Zilber Theorem*)

EZ-maps exist. Moreover, the following holds.

- For any pair $(\mathfrak{P}, \mathfrak{Q})$ of EZ-maps, we have $\mathfrak{P} \circ \mathfrak{Q}$ and $\mathfrak{Q} \circ \mathfrak{P}$ are naturally chain homotopic to identity. In particular, $\mathfrak{P}_{X,Y}$ and $\mathfrak{Q}_{X,Y}$ are natural chain equivalences.
- Given two pairs $(\mathfrak{P}, \mathfrak{Q})$ and $(\mathfrak{P}', \mathfrak{Q}')$ of EZ-maps, we have $\mathfrak{P}, \mathfrak{P}'$ (and similarly, $\mathfrak{Q}, \mathfrak{Q}'$) are naturally chain homotopic.
- An EZ map $\mathfrak{P}$ is
  - *associative* up to natural homotopy, i.e, the following diagram commutes up to natural chain homotopy:

$$\begin{tikzcd}
S_\bullet(X) \otimes S_\bullet(Y) \otimes S_\bullet(Z) \arrow[r, "\mathfrak{P}_{X,Y} \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes \mathfrak{P}_{Y,Z}"'] & S_\bullet(X \times Y) \otimes S_\bullet(Z) \arrow[d, "\mathfrak{P}_{X\times Y,Z}"] \\
S_\bullet(X) \otimes S_\bullet(Y \times Z) \arrow[r, "\mathfrak{P}_{X,Y\times Z}"'] & S_\bullet(X \times Y \times Z)
\end{tikzcd}$$

  - *commutative* up to natural homotopy, i.e, the following diagram commutes up to natural chain homotopy:

$$\begin{tikzcd}
S_\bullet(X) \otimes S_\bullet(Y) \arrow[r, "\tau_{X,Y}"] \arrow[d, "\mathfrak{P}_{X,Y}"'] & S_\bullet(Y) \otimes S_\bullet(X) \arrow[d, "\mathfrak{P}_{Y,X}"] \\
S_\bullet(X \times Y) \arrow[r, "S_\bullet(t_{X,Y})"'] & S_\bullet(Y \times X)
\end{tikzcd}$$

- An EZ map $\mathfrak{Q}$ is
  - *coassociative* up to natural homotopy, i.e, the following diagram commutes up to natural chain homotopy:

$$\begin{tikzcd}
S_\bullet(X \times Y \times Z) \arrow[r, "\mathfrak{Q}_{X\times Y,Z} \otimes \mathrm{Id}"] \arrow[d, "\mathfrak{Q}_{X,Y\times Z}"'] & S_\bullet(X \times Y) \otimes S_\bullet(Z) \arrow[d, "\mathfrak{Q}_{X,Y} \otimes \mathrm{Id}"] \\
S_\bullet(X) \otimes S_\bullet(Y \times Z) \arrow[r, "\mathrm{Id} \otimes \mathfrak{Q}_{Y,Z}"'] & S_\bullet(X) \otimes S_\bullet(Y) \otimes S_\bullet(Z)
\end{tikzcd}$$

  - *cocommutative* up to natural homotopy, i.e, the following diagram commutes up to natural chain homotopy:

$$\begin{tikzcd}
S_\bullet(X \times Y) \arrow[r, "S_\bullet(t_{X,Y})"] \arrow[d, "\mathfrak{Q}_{X,Y}"'] & S_\bullet(Y \times X) \arrow[d, "\mathfrak{Q}_{Y,X}"] \\
S_\bullet(X) \otimes S_\bullet(Y) \arrow[r, "\tau_{X,Y}"'] & S_\bullet(Y) \otimes S_\bullet(X)
\end{tikzcd}$$

Here, $t_{X,Y} : X \times Y \to Y \times X$ is natural flip map, and $\tau_{X,Y} : S_\bullet(X) \otimes S_\bullet(Y) \to S_\bullet(Y) \otimes S_\bullet(X)$ is given by $\tau_{X,Y}(\alpha \otimes \beta) = (-1)^{|\alpha||\beta|}\beta \otimes \alpha$.

Although Theorem 18.6 is existential in nature, there are explicit constructions of EZ-maps, which were found out later by Eilenberg and MacLane.

Definition 18.7: (*Alexander-Whitney Map*)

Given spaces X, Y, the *Alexander-Whitney maps*

$$\mathrm{AW}_{X,Y} : S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$$

are defined on an n-simplex $\sigma : \Delta^n \to X \times Y$ by the formula

$$\mathrm{AW}(\sigma) = \sum_{p+q=n} \pi_X \circ \sigma \circ \lambda_p^n \otimes \pi_Y \circ \sigma \circ \rho_q^n,$$

where $\lambda_p^n : \Delta^p \to \Delta^n$ and $\rho_q^n : \Delta^q \to \Delta^n$ are defined linearly as $\lambda_p^n(e_i) = e_i$ for $0 \le i \le p$, and $\rho_q^n(e_j) = e_{n-q+j}$ for $0 \le j \le q$. Intuitively, λ_p^n embeds Δ^p as the *front* p-face of Δ^n, and ρ_q^n embeds Δ^q as the *back* q-face of Δ^n.

One can verify that AW is a natural chain map, extending the natural isomorphism at the 0^{th} level, i.e, AW is an EZ map. Moreover, AW is *strictly* coassociative.

Definition 18.8: (*Eilenberg-MacLane Map*)

Given spaces X, Y the (explicit) *Eilenberg-Zilber maps* (or *Eilenberg-MacLane formula*)

$$\mathrm{EM}_{X,Y} : S_\bullet(X) \otimes S_\bullet(Y) \to S_\bullet(X \times Y)$$

are defined via *shuffle products*. Explicitly, treat $\Delta^n \subset \mathbb{R}^n$ as the subset $\Delta^n := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid 0 \le x_1 \le \cdots \le x_n \le 1\}$. Recall, a (p, q)-shuffle σ is a bijection $\sigma : \{1, ..., p + q\} \to \{1, ..., p + q\}$ such that $\sigma(1) \le \cdots \le \sigma(p)$ and $\sigma(p + 1) \le \cdots \le \sigma(p + q)$; thus $\mathrm{Sh}(p, q) \subset S_{p+q}$ is a subset of permutaions and in particular the signum function $\mathrm{sgn}(\sigma)$ makes sense. Given a (p, q)-shuffle σ, consider the linear embedding $\ell_\sigma : \Delta^{p+q} \to \Delta^p \times \Delta^q$ given by

$$\ell_\sigma(x_1, ..., x_p, x_{p+1}, ..., x_{p+q}) = \Big(\big(x_{\sigma(1)}, ..., x_{\sigma(p)}\big), \big(x_{\sigma(p+1)}, ..., x_{\sigma(p+q)}\big)\Big).$$

Then, for $\alpha : \Delta^p \to X, \beta : \Delta^q \to Y$ one can define

$$\mathrm{EM}(\alpha \otimes \beta) = \sum_{\sigma \in \mathrm{Sh}(p,q)} \mathrm{sgn}(\sigma)(\alpha \times \beta) \circ \ell_\sigma.$$

Again, it can be verified that EM is a natural chain map, extending the natural isomorphism at the 0^{th} level, making it an EZ-map! Moreover, EM is strictly coassociative. Note that we chaged the definition of simplicies, but they are naturally homeomorphic to Definition 6.1, which lets us translate the definition of EM back to the usual singular chain complexes where the formula becomes even more complicated. There are other combinatorial ways to define such maps, e.g, by identifying a simplex with paths in an integer grids.

Example 18.9: (*EZ-map from* AW)

The composition

$$\begin{tikzcd}
S_\bullet(X \times Y) \arrow[rr, dashed, "\mathrm{EZ}_{X,Y}"] \arrow[d, "S_\bullet(\Delta_{X\times Y})"'] & & S_\bullet(X) \otimes S_\bullet(Y) \\
S_\bullet(X \times Y \times X \times Y) \arrow[rr, "\mathrm{AW}_{X\times Y, X\times Y}"'] & & S_\bullet(X \times Y) \otimes S_\bullet(X \times Y) \arrow[u, "S_\bullet(\pi_X) \otimes S_\bullet(\pi_Y)"']
\end{tikzcd}$$

defines a strictly associative EZ-map as well. In fact, for any EZ-map $S_\bullet(- \times -) \Rightarrow S_\bullet(-) \otimes S_\bullet(-)$ the above composition gives a EZ-map, which may not be strictly coassociative.

As a consequence of Theorem 18.6, we can now get the Künneth theorem for spaces.

Theorem 18.10: (*Künneth Theorem for Singular Homology*)

Let R be a PID, and X, Y be spaces. Then, there exists a natural short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(X;R) \otimes H_j(Y;R) \to H_n(X \times Y; R) \to \bigoplus_{i+j=n-1} \mathrm{Tor}\big(H_i(X;R), H_j(Y;R)\big) \to 0,$$

which is split, but not naturally.

Proof : Proof is immediate from the Künneth theorem in homological algebra (Theorem 18.1) and the Eilenberg-Zilber theorem (Theorem 18.6). Indeed, naturality of the short exact sequence lets us replace $S_\bullet(X) \otimes S_\bullet(Y)$ by the naturally chain equivalent complex $S_\bullet(X \times Y)$. $\square$

The first map in the Künneth theorem gives a product stucture on singular homology.

Definition 18.11: (*Homology Cross Product*)

Given two spaces X, Y, the *homology cross product* for homology with coefficients in some ring is defined as the composition

$$\times : H_p(X) \otimes H_q(Y) \to H_{p+q}(S_\bullet(X) \otimes S_\bullet(Y)) \xrightarrow{\cong} H_{p+q}(X \times Y),$$

where the natural isomorphism is via an EZ-map.

Explicitly, given $\sigma : \Delta^p \to X, \tau : \Delta^q \to Y$, and a choice of an EZ-map $\mathfrak{Q}$, we have the element $\mathfrak{Q}_{X,Y}(\sigma \otimes \tau) \in S_{p+q}(X \times Y)$, which induces the cross product in homology.

In order to get the Künneth theorem in singular cohomology, we need to take some extra assumptions.

Theorem 18.12: (*Künneth Theorem for Singular Cohomology*)

Let R be a PID, and X, Y be spaces. Suppose that either $H_\bullet(X;R)$ or $H_\bullet(Y;R)$ are of finite type. Then, there exists a natural short exact sequence

$$0 \to \bigoplus_{i+j=n} H^i(X;R) \otimes H^j(Y;R) \to H^n(X \times Y;R) \to \bigoplus_{i+j=n+1} \operatorname{Tor}(H^i(X;R), H^j(Y;R)) \to 0,$$

which splits, but not naturally.

Proof : The proof follows from the algebraic Künneth theorem for cohomology (Theorem 18.4), and the Eilenber-Zilber theorem (Theorem 18.6). □

Remark 18.13: (*Relative Künneth Theorems*)

There are appropriate versions of Künneth theorems for *relative* singular (co)homology as well.

Day 19 : 10th April, 2026

`cohomological cross product — cup product — cochain cup product`

19.1 The Cup Product

Let us revisit the Künneth theorem for cohomology. Suppose $C_\bullet, D_\bullet$ are some arbitrary chain complex of R-modules, and denote the dual cochain complexes $C^\bullet := \hom_R(C_\bullet, R), D^\bullet := \hom_R(D_\bullet, R)$. We have a map

$$\hom_R(C_\bullet, R) \otimes \hom_R(D_\bullet, R) \to \hom_R(C_\bullet \otimes D_\bullet, R)$$
$$\varphi \otimes \psi \mapsto \left(\sum_i c_i \otimes d_i \mapsto \sum_i \varphi(c_i) \cdot \psi(d_i)\right).$$

In general, this map is not an isomorphism. But if both $C_\bullet, D_\bullet$ are free, it is an isomorphism. This is precisely the first map in Theorem 18.4, called the *algebric cohomological cross product*

$$\times^{\text{alg}} : H^p(C^\bullet) \otimes H^q(D^\bullet) \to H^{p+q}((C_\bullet \otimes D_\bullet)^\star).$$

Exercise 19.1: (*Dual of Tensor*)

Let M, N be R-modules, and $C_\bullet, D_\bullet$ be chain complexes of free R-modules. Then there exists a natural isomorphism

$$\hom_R(C_\bullet, M) \otimes_R \hom_R(D_\bullet, N) \to \hom_R(C_\bullet \otimes D_\bullet, M \otimes N).$$

This leads to a Künneth theorem in cohomology with coefficients.

Now, let X, Y be fixed spaces. Given a choice of EZ-map, say, $\mathrm{AW}_{X,Y} : S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$, dualizing we have a cochain map

$$\mathrm{AW}^\star_{X,Y} : (S_\bullet(X) \otimes S_\bullet(Y))^\star \to S^\bullet(X \times Y).$$

Definition 19.2: (*Cohomology Cross Product*)

Given two spaces X, Y, the *cohomology cross product* is defined as the composition

$$H^p(X) \otimes H^q(Y) \xrightarrow{\times^{\text{alg}}} H^{p+q}((S_\bullet(X) \otimes S_\bullet(Y))^\star) \xrightarrow[\cong]{\mathrm{AW}^\star_{X,Y}} H^{p+q}(X \times Y),$$

where we consider cohomology with coefficients in a ring R.

Note that the definition is independent of the choice of the EZ-map since any such choice are chain homotopic and hence induces identical isomorphism at the cohomology level.

Definition 19.3: (*Cup Product*)

Given a space X, consider the *diagonal map* $\Delta : X \to X \times X$. The *cup product* on X is defined as the composition

$$\smile : H^p(X) \otimes H^q(X) \xrightarrow{\times} H^{p+q}(X \times X) \xrightarrow{\Delta^\star} H^{p+q}(X),$$

where we consider cohomology in a ring R.

We shall see computations of cup product in spaces, which shows cohomology as a vastly greater tool in distinguishing spaces.

Remark 19.4: (*Cup Product in Homology?*)

In homology, we have the cross product $H_p(X) \otimes H_q(X) \to H_{p+q}(X \times X)$. But the diagonal map induces $\Delta_\star : H_{p+q}(X) \to H_{p+q}(X \times X)$, which goes the *wrong way.* Thus, we cannot define a map $H_p(X) \otimes H_q(X) \to H_{p+q}(X)$.

One might ask: what about a coproduct $H_n(X) \to \sum H_p(X) \otimes H_q(X)$? Even this may fail! Recall that for a PID, the Künneth theorem in singular homology (Theorem 18.10) gives the *split* short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(X) \otimes H_j(X) \to H_n(X \times X) \to \bigoplus_{i+j=n-1} \operatorname{Tor}\left(H_i(X), H_j(X)\right) \to 0.$$

Since the sequence is split, we can define a composition

$$H_n(X) \xrightarrow{\Delta_\star} H_n(X \times X) \to \bigoplus_{i+j=n} H_i(X) \otimes H_j(X),$$

where the red arrow is a *choice,* and moreover, the choice is not natural as the splitting is not natural. Thus, in general we do not have coproduct in homology. Of course, if the Tor term vanishes (e.g, when working over a field) we do have a coproduct.

If (X, μ) is a H-space (i.e, X is a based space with a map $\mu : X \times X \to X$ such that $\mu|_{X \vee X} \simeq \nabla_X : X \vee X \to X$), we do get an induced map

$$H_p(X) \otimes H_q(X) \xrightarrow{\times} H_{p+q}(X \times X) \xrightarrow{\mu_\star} H_{p+q}(X).$$

This product is known as the *Pontryagin product*. This is important, e.g, when X is the loop space of another space; loop space is an H-space with the concatenation product.

We have the following useful relations between cup and cross product.

Proposition 19.5: (*Cup and Cross Product*)

Let $f : X' \to X, g : Y' \to Y$ be continuous map. Then, the following holds.

1. $(f \times g)^\star(\alpha \times \beta) = f^\star\alpha \times g^\star\beta$ for $\alpha \in H^\star(X), \beta \in H^\star(Y)$.
2. $\alpha \times \beta = \pi_X^\star\alpha \smile \pi_Y^\star\beta$ for $\alpha \in H^\star(X), \beta \in H^\star(Y)$.
3. $f^\star(\alpha \smile \beta) = f^\star\alpha \smile f^\star\beta$ for $\alpha, \beta \in H^\bullet(X)$.

Proof : The proofs essentially follow from definitions.

1. Since the short exact sequence in the Künneth theorem for cohomology (Theorem 18.12) is natural, it follows that the comological cross product is also natural. In particular, $(f \times g)^\star(\alpha \times \beta) = f^\star\alpha \times g^\star\beta$.
2. We have,

$$\begin{aligned}\pi_X^\star\alpha \smile \pi_Y^\star\beta &= \Delta_{X\times Y}^\star(\pi_X^\star\alpha \times \pi_Y^\star\beta)\\ &= \Delta_{X\times Y}^\star(\pi_X \times \pi_Y)^\star(\alpha \times \beta)\\ &= ((\pi_X \times \pi_Y) \circ \Delta_{X\times Y})^\star(\alpha \times \beta)\\ &= \mathrm{Id}_{X\times Y}^\star(\alpha \times \beta)\\ &= \alpha \times \beta.\end{aligned}$$

3. We have,

$$\begin{aligned}f^\star(\alpha \smile \beta) &= f^\star\Delta^\star(\alpha \times \beta)\\ &= (\Delta \circ f)^\star(\alpha \times \beta)\\ &= ((f \times f) \circ \Delta)^\star(\alpha \times \beta)\\ &= \Delta^\star(f \times f)^\star(\alpha \times \beta)\\ &= \Delta^\star(f^\star\alpha \times f^\star\beta)\\ &= f^\star\alpha \smile f^\star\beta.\end{aligned}$$

□

Proposition 19.6: (*Properties of the Cup Product*)

Given a space X, the following holds for cohomology with coefficients in a ring R.

1. **(Associativity)** : $a \smile (b \smile c) = (a \smile b) \smile c$ for any $a, b, c \in H^\star(X; R)$.
2. **(Graded Commutativity)** : $a \smile b = (-1)^{|a|\cdot|b|} b \smile a$ for $a \in H^{|a|}(X; R), b \in H^{|b|}(X; R)$.
3. **(Unitality)** : $1 \smile a = a = a \smile 1$ for any $a \in H^\star(X; R)$. Here $1 \in H^0(X; R)$ is the cohomology class induced by the map $\mathbf{1} : S^0(X) \to R$ taking every 0-simplex to $1_R \in R$.

Proof : The proofs are consequence of the properties of the EZ-maps (Theorem 18.6). □

As a consequence, we now have the following.

Definition 19.7: (*Cohomology Ring*)

Given a space X and a ring R, the *cohomology ring* of X is defined as the graded module $H^\star(X;R) = \bigoplus H^i(X;R)$, equipped with the cup product. It is a unital, associative, graded commutative algebra over R.

19.2 Cochain Cup Product

The importance of cup product warrants some explicit computations! Recall that at the level of cochains, using the Alexander-Whitney map, we have the cup product

$$\begin{aligned} \smile\colon S^p(X;R) \otimes S^q(X;R) &\to S^{p+q}(X;R) \\ \varphi \otimes \psi &\mapsto \left(\sigma \mapsto \varphi\left(\sigma|_{[v_0,\dots,v_p]}\right) \cdot \psi\left(\sigma|_{[v_p,\dots,v_{p+q}]}\right)\right). \end{aligned}$$

We shall prove Proposition 19.6 via explicit computation involving this cochain level cup product.

Proposition 19.8: (*Cup Product is a Chain Map*)

We have a chain map $\smile\colon S^\bullet(X;R) \otimes S^\bullet(X;R) \to S^\bullet(X;R)$. Explicitly, for homogenous elements $\varphi, \psi \in S^\bullet(X;R)$, we have

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^{|\varphi|}\varphi \smile \delta\psi.$$

As a consequence, we have the induced cup product

$$\smile\colon H^p(X;R) \otimes H^q(X;R) \to H^{p+q}(X;R).$$

Proof : Let $\sigma : \Delta^{p+q+1} \to X$ be a singular $(p+q+1)$-simplex, and $\varphi \in S^p(X;R), \psi \in S^q(X;R)$. We compute

$$\begin{aligned} \delta(\varphi \smile \psi)(\sigma) &= (\varphi \smile \psi)(\partial\sigma) \\ &= (\varphi \smile \psi)\left(\sum_{i=0}^{p+q+1} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{p+q+1}]}\right) \\ &= \sum_{i=1}^{p+q+1} (-1)^i (\varphi \smile \psi)\left(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{p+q+1}]}\right) \\ &= \sum_{i=0}^{p} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{p+1}]}\right) \cdot \psi\left(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\right) \\ &\qquad + \sum_{i=p+1}^{p+q+1} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,v_p]}\right) \cdot \psi\left(\sigma|_{[v_p,\dots,\hat{v}_i,\dots,v_{p+q+1}]}\right) \\ &= \sum_{i=0}^{p} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{p+1}]}\right) \cdot \psi\left(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\right) \\ &\qquad + (-1)^{p+1} \varphi\left(\sigma|_{[v_0,\dots,v_p]}\right) \cdot \psi\left(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\right) \\ &\qquad + (-1)^{p} \varphi\left(\sigma|_{[v_0,\dots,v_p]}\right) \cdot \psi\left(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\right) \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i=p+1}^{p+q+1} (-1)^i \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \psi\big(\sigma|_{[v_p,\dots,\hat{v}_i,\dots,v_{p+q+1}]}\big) \\
&= \left[\sum_{i=0}^{p+1} (-1)^i \varphi\big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{p+1}]}\big)\right] \cdot \psi\big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\big) \\
&\quad + (-1)^p \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \left[\sum_{i=p}^{p+q+1} (-1)^{i-p} \psi\big(\sigma|_{[v_p,\dots,\hat{v}_i,\dots,v_{p+q+1}]}\big)\right] \\
&= \varphi\left(\sum_{i=0}^{p+1} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{p+1}]}\right) \cdot \psi\big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\big) \\
&\quad + (-1)^p \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \psi\left(\sum_{i=0}^{q+1} (-1)^i \sigma|_{[v_p,\dots,\hat{v}_{p+i},\dots,v_{p+q+1}]}\right) \\
&= \varphi\big(\partial\big(\sigma|_{[v_0,\dots,v_{p+1}]}\big)\big) \cdot \psi\big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\big) \\
&\quad + (-1)^p \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \psi\big(\partial\sigma|_{[v_p,\dots,v_{p+q+1}]}\big) \\
&= \delta\varphi\big(\sigma|_{[v_0,\dots,v_{p+1}]}\big) \cdot \psi\big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\big) \\
&\quad + (-1)^p \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \delta\psi\big(\sigma|_{[v_p,\dots,v_{p+q+1}]}\big) \\
&= (\delta\varphi \smile \psi)(\sigma) + (-1)^p (\varphi \smile \delta\psi)(\sigma) \\
&= (\delta\varphi \smile \psi + (-1)^p \varphi \smile \delta\psi)(\sigma).
\end{aligned}
$$

Since σ was arbitrary, we have the claim.

As for the induced map, if both φ and ψ are cocycles, it follows that $\varphi \smile \psi$ is also a cocycle, and similarly, if they are coboundaries. Hence, we get the induced cup product at the cohomology. □

Next, we check associativity and unitality.

Proposition 19.9: (*Cochain Cup Product is Associative and Unital*)

The cochain level cup product is associative and unital. Consequently, so is the induced cohomological cup product.

Proof : Let $\varphi \in S^p(X;R), \psi \in S^q(X;R), \zeta \in S^r(X;R)$ are cochains, and consider $\sigma : \Delta^{p+q+r} \to X$ be a singular $(p+q+r)$-simplex. Then, we compute

$$
\begin{aligned}
((\varphi \smile \psi) \smile \zeta)(\sigma) &= (\varphi \smile \psi)\big(\sigma|_{[v_0,\dots,v_{p+q}]}\big) \cdot \zeta\big(\sigma|_{[v_{p+q},\dots,v_{p+q+r}]}\big) \\
&= \Big(\varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \psi\big(\sigma|_{[v_p,\dots,v_{p+q}]}\big)\Big) \cdot \zeta\big(\sigma|_{[v_{p+q},\dots,v_{p+q+r}]}\big) \\
&= \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \Big(\psi\big(\sigma|_{[v_p,\dots,v_{p+q}]}\big) \cdot \zeta\big(\sigma|_{[v_{p+q},\dots,v_{p+q+r}]}\big)\Big) \\
&= \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot (\psi \smile \zeta)\big(\sigma|_{[v_p,\dots,v_{p+q+r}]}\big) \\
&= (\varphi \smile (\psi \smile \zeta))(\sigma).
\end{aligned}
$$

Since σ is arbitrary, we have $(\varphi \smile) \smile \zeta = \varphi \smile (\psi \smile \zeta)$. This proves that the cochain level cup product is strictly associative, and so is the cohomological cup product.

Next, consider the cochain $1 : S_0(X) \to R$ which sends every 0-simplex to the unit $1_R \in R$. For a p-cochain $\varphi \in S^p(X;R)$ and a singular p-simplex $\sigma : \Delta^p \to X$, we compute

$$(\varphi \smile 1)(\sigma) = \varphi\left(\sigma|_{[v_0,\dots,v_p]}\right) \cdot 1\left(\sigma|_{[v_p]}\right) = \varphi(\sigma) \cdot 1_R = \varphi(\sigma),$$
$$(1 \smile \varphi)(\sigma) = 1\left(\sigma|_{[v_0]}\right) \cdot \varphi\left(\sigma|_{[v_0,\dots,v_p]}\right) = 1_R \cdot \varphi(\sigma) = \varphi(\sigma).$$

Thus, $1 \smile \varphi = \varphi = \varphi \smile 1$, which shows that 1 is the unit for the cochain level cup product. Hence, we get the induced unit for the cohomological cup product. □

The final claim is about commutativity.

Proposition 19.10: (*Cochain Cup Product is **Not** Commutative*)

Let X be a space and R be commutative ring. Then, the cochain level cup product on $S^\bullet(X;R)$ is not commutative, but the cohomological cup product on $H^\bullet(X;R)$ is graded commutative.

Proof : Let us define a map of degree -1, called the Steenrod cup-1 map,

$$\smile_1 : S^\bullet(X;R) \otimes S^\bullet(X;R) \to S^\bullet(X;R)$$

as follows. Say $\varphi \in S^p(X;R), \psi \in S^q(X;R)$ are cochains, and $\tau : \Delta^{p+q-1} \to X$ be a singular simplex. Define

$$(\varphi \smile_1 \psi)(\tau) = \sum_{0 \le j < p} (-1)^{(p-j)(q+1)} \varphi\left(\tau|_{[v_0,\dots,v_j,v_{j+q},\dots v_{p+q-1}]}\right) \cdot \psi\left(\tau|_{[v_j,\dots,v_{j+q}]}\right).$$

Next, say $\sigma : \Delta^{p+q} \to X$ is a singular $(p+q)$-simplex. We compute

$$\begin{aligned}
&\delta(\varphi \smile_1 \psi)(\sigma) \\
&= (\varphi \smile_1 \psi)(\partial\sigma) \\
&= \sum_{i=0}^{p+q} (-1)^i (\varphi \smile_1 \psi)\left(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{p+q}]}\right) \\
&= \sum_{0 \le j < p} (-1)^{(p-j)(q+1)} \Bigg[\sum_{0 \le i \le j} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{j+1},v_{j+q+1},\dots,v_{p+q}]}\right) \cdot \psi\left(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\right) \\
&\qquad + \sum_{j < i \le j+q} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\right) \cdot \psi\left(\sigma|_{[v_j,\dots,\hat{v}_i,\dots,v_{j+q+1}]}\right) \\
&\qquad + \sum_{j+q < i \le p+q} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,v_j,v_{j+q},\dots,\hat{v}_i,\dots,v_{p+q}]}\right) \cdot \psi\left(\sigma|_{[v_j,\dots,v_{j+q}]}\right) \Bigg]
\end{aligned}$$

Next, we have

$$\begin{aligned}
&\delta\varphi \smile_1 \psi(\sigma) \\
&= \sum_{0 \le j \le p} (-1)^{(p+1-j)(q+1)} \delta\varphi\left(\sigma|_{[v_0,\dots,v_j,v_{j+q},\dots,v_{p+q}]}\right) \cdot \psi\left(\sigma|_{[v_j,\dots,v_{j+q}]}\right) \\
&= \sum_{0 \le j \le p} (-1)^{(p+1-j)(q+1)} \left(\sum_{i=0}^{j} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_j,v_{j+q},\dots,v_{p+q}]}\right) \right)
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \sum_{i=j}^{p} (-1)^{i+1} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q},\dots\hat{v}_{q+i},\dots,v_{p+q}]}\Big)\Bigg) \cdot \psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big) \\
&= (-1)^{(p+1)(q+1)} \left(\varphi\Big(\sigma|_{[v_q,\dots,v_{p+q}]}\Big) + \sum_{i=0}^{p} (-1)^{i+1} \varphi\Big(\sigma|_{[v_0,v_q,\dots,\hat{v}_{q+i},\dots,v_{p+q}]}\Big)\right) \cdot \psi\Big(\sigma|_{[v_0,\dots,v_q]}\Big) \\
&\quad + \sum_{0\le j<p} (-1)^{(p-j)(q+1)} \sum_{i=0}^{j+1} (-1)^{i} \varphi\Big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{j+1},v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big) \\
&\qquad + \sum_{0<j\le p} (-1)^{(p+1-j)(q+1)} \sum_{i=j}^{p} (-1)^{i+1} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q},\dots,\hat{v}_{q+i},\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big) \\
&= (-1)^{(p+1)(q+1)} (\psi \smile \varphi)(\sigma) \\
&\quad + \sum_{0\le j<p} (-1)^{(p-j)(q+1)} \sum_{i=0}^{j} (-1)^{i} \varphi\Big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{j+1},v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big) \\
&\quad + \sum_{0\le j<p} (-1)^{(p-j)(q+1)} (-1)^{j+1} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big) \\
&\quad + \sum_{0\le j<p} (-1)^{(p-j)(q+1)} \sum_{j+q\le i\le p+q} (-1)^{i} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q},\dots,\hat{v}_i,\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big) \\
&\quad + (-1)^{p+q} \varphi\Big(\sigma|_{[v_0,\dots,v_p]}\Big) \cdot \psi\Big(\sigma|_{[v_p,\dots,v_{p+q}]}\Big) \\
&= (-1)^{(p+1)(q+1)} (\psi \smile \varphi)(\sigma) + (-1)^{p+q} (\varphi \smile \psi)(\sigma) \\
&\quad + \sum_{0\le j<p} (-1)^{(p-j)(q+1)} \sum_{i=0}^{j} (-1)^{i} \varphi\Big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{j+1},v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big) \\
&\quad + \sum_{0\le j<p} (-1)^{(p-j)(q+1)} (-1)^{j+1} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big) \\
&\quad + \sum_{0\le j<p} (-1)^{(p-j)(q+1)} (-1)^{j+q} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big) \\
&\quad + \sum_{0\le j<p} (-1)^{(p-j)(q+1)} \sum_{j+q<i\le p+q} (-1)^{i} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q},\dots,\hat{v}_i,\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big) \\
&= (-1)^{p+q} ((\varphi \smile \psi)(\sigma) - (-1)^{pq} (\psi \smile \varphi)(\sigma)) \\
&\quad + \sum_{0\le j<p} (-1)^{(p-j)(q+1)} \sum_{i=0}^{j} (-1)^{i} \varphi\Big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{j+1},v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big) \\
&\quad + \sum_{0\le j<p} (-1)^{(p-j)(q+1)} \sum_{j+q<i\le p+q} (-1)^{i} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q},\dots,\hat{v}_i,\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big) \\
&\quad + \sum_{0\le j<p} (-1)^{(p-j)(q+1)} (-1)^{j+1} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big) \\
&\quad + \sum_{0\le j<p} (-1)^{(p-j)(q+1)} (-1)^{j+q} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big)
\end{aligned}
$$

Finally,

$$\varphi \smile_1 \delta\psi(\sigma)$$

$$= \sum_{0 \le j < p} (-1)^{(p-j)q} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \delta\psi\Big(\sigma|_{[v_j,\dots,v_{j+q+1}]}\Big)$$

$$= \sum_{0 \le j < p} (-1)^{(p-j)q} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \left(\sum_{i=0}^{q+1} (-1)^i \psi\Big(\sigma|_{[v_j,\dots,\hat{v}_{j+i},\dots,v_{j+q+1}]}\Big)\right)$$

$$= \sum_{0 \le j < p} (-1)^{(p-j)q} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \left(\sum_{i=j}^{j+q+1} (-1)^{i-j} \psi\Big(\sigma|_{[v_j,\dots,\hat{v}_i,\dots,v_{j+q+1}]}\Big)\right)$$

$$\begin{aligned}
&= (-1)^p \sum_{0 \le j < p} (-1)^{(p-j)(q+1)} \sum_{j \le i \le j+q} (-1)^i \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_j,\dots,\hat{v}_i,\dots,v_{j+q+1}]}\Big) \\
&\quad + (-1)^p \sum_{0 \le j < p} (-1)^{(p-j)(q+1)} (-1)^j \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_{j+1},\dots,v_{j+q+1}]}\Big) \\
&\quad + (-1)^p \sum_{0 \le j < p} (-1)^{(p-j)(q+1)} (-1)^{j+q+1} \varphi\Big(\sigma|_{[v_0,\dots,v_j,v_{j+q+1},\dots,v_{p+q}]}\Big) \cdot \psi\Big(\sigma|_{[v_j,\dots,v_{j+q}]}\Big)
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
&\delta(\varphi \smile_1 \psi)(\sigma) - (\delta\varphi \smile_1 \psi)(\sigma) - (-1)^p(\varphi \smile_1 \delta\psi)(\sigma) \\
&= (-1)^{p+q+1}(\varphi \smile \psi)(\sigma) - (-1)^{pq}(\psi \smile \varphi)(\sigma).
\end{aligned}$$

In other words,

$$\begin{aligned}
&\varphi \smile \psi - (-1)^{pq}\psi \smile \varphi \\
&= (-1)^{p+q+1}(\delta(\varphi \smile_1 \psi) - \delta\varphi \smile_1 \psi - (-1)^p \varphi \smile_1 \delta\psi).
\end{aligned}$$

This implies that, passing to cohomology, the cup product is graded commutative! □

Remark 19.11: (*Cup-i Product*)

More generally, one can define the cup-i product as a map $\smile_i: S^p(X) \otimes S^q(X) \to S^{p+q-i}(X)$. They were originally defined by Steenrod in order to define the Steenrod square operations on cohomology with $\mathbb{Z}_2$-coefficincents. Generalizing even further, one gets to the notion of E_∞-algebra structure on the cochain complex $S^\bullet(X)$.

Day 20 : 17th April, 2026

`cohomology ring of product – cohomology ring of wedge – cup product computation – sphere – torus – ℝℙ² – klein bottle`

20.1 Cohomology Ring of Product of Spaces

Recall, an *algebra* over a ring R is an R-module A with a map $\mu : A \otimes_R A \to A$. The algebra is *associative* if the diagram commutes

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes \mathrm{Id}_A"] \arrow[d, "\mathrm{Id}_A \otimes \mu"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "\mu"] \\
A \otimes \arrow[r, "\mu"'] & A
\end{tikzcd}$$

The algebra is *unital* if there is an R-module map $\eta : R \to A$ such that the diagram commutes

$$\begin{tikzcd}
R \otimes A \arrow[r, "\eta \otimes \mathrm{Id}_A"] \arrow[dr, "\cong"'] & A \otimes \arrow[d, "\mu"] & A \otimes R \arrow[l, "\mathrm{Id}_A \otimes \eta"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

An R-linear map $f : (A, \mu_A) \to (B, \mu_B)$ is an *algebra map* if the diagram commutes

$$\begin{tikzcd}
A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\circlearrowright"] & B \otimes B \arrow[d, "\mu_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

If both A, B are unital, we moreover require that an algebra map preserves the unit, i.e, the diagram commutes

$$\begin{tikzcd}
& A \arrow[dd, "f"] \\
R \arrow[ur, "\eta_B"] \arrow[dr, "\eta_A"'] & \\
& B
\end{tikzcd}$$

When A is graded, we assume that the multiplication map $\mu : A \otimes A \to A$ and the unit map $\eta : R \to A$ has degree 0, and consequently, any algebra map of graded algebra must be a degree 0 map as well. The *tensor product* of two graded algebras is the graded module $A \otimes B$, equipped with the multiplication given by the composition

$$(A \otimes B) \otimes (A \otimes B) \xrightarrow{\mathrm{Id}_A \otimes \tau \otimes \mathrm{Id}_B} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B,$$

where $\tau : A \otimes B \to B \otimes A$ is the flip map (also known as the *braiding*). Explicitly,

$$\mu(a_1 \otimes b_1, a_2 \otimes b_2) = (-1)^{|b_1| \cdot |a_2|} \mu_A(a_1, a_2) \otimes \mu_B(b_1, b_2),$$

for homogeneous elements $a_1, a_2 \in A, b_1, b_2 \in B$. The flip map induces a sign according to the Koszul sign rule.

We have seen that given a space X and a ring R, the cohomology ring $H^\star(X;R)$ is a graded commutative, unital, associative algebra over R.

Proposition 20.1: (*Cross Product is an Algebra Map*)

The cohomological cross product

$$\times : H^\star(X;R) \otimes H^\star(Y;R) \to H^\star(X \times Y;R)$$

is a map of algebras over R.

Proof : Let $a_i \in H^{p_i}(X;R), b_i \in H^{q_i}(Y;R)$ be given for $i = 1, 2$. We have

$$\begin{aligned}(a_1 \times b_1) \smile (a_2 \times b_2) &= (\pi_X^\star a_1 \smile \pi_Y^\star b_1) \smile (\pi_X^\star a_2 \smile \pi_Y^\star b_2) \\ &= (-1)^{|\pi_Y^\star b_1||\pi_X^\star a_2|} \pi_X^\star a_1 \smile \pi_X^\star a_2 \smile \pi_Y^\star b_1 \smile \pi_Y^\star b_2 \\ &= (-1)^{|b_1||a_2|} \pi_X^\star (a_1 \smile a_2) \smile \pi_Y^\star (b_1 \smile b_2) \\ &= (-1)^{|b_1||a_2|} (a_1 \smile a_2) \times (b_1 \smile b_2).\end{aligned}$$

Note that the cross product $\times$, and the cup product $\smile$ has degree 0, and thus, there is no extra signs appearing while evaluating them! In other words, we have the commutativity of the diagram

$$\begin{tikzcd}
(H^\star(X;R) \otimes H^\star(Y;R)) \otimes (H^\star(X;R) \otimes H^\star(Y;R)) \arrow[r, "\times \otimes \times"] \arrow[d, "\mathrm{Id} \otimes \tau \otimes \mathrm{Id}"'] & H^\star(X \times Y;R) \otimes H^\star(X \times Y;R) \arrow[dd, "\smile"] \\
H^\star(X;R) \otimes H^\star(X;R) \otimes H^\star(Y;R) \otimes H^\star(Y;R) \arrow[d, "\smile \otimes \smile"'] & \\
H^\star(X;R) \otimes H^\star(Y;R) \arrow[r, "\times"'] & H^\star(X \times Y;R)
\end{tikzcd}$$

Thus, we have that $\times$ is a map of algebras. $\square$

As a consequence, we get the following important result.

Theorem 20.2: (*Künneth Agebra Isomorphism*)

Let R be a PID, X, Y be spaces. Suppose that

- either $H_\bullet(X;R)$ or $H_\bullet(Y;R)$ are of finite type, and
- $\mathrm{Tor}\left(H_i(X;R), H_j(X;R)\right) = 0$ for all i, j.

Then, the cohomological cross product $\times : H^\bullet(X;R) \otimes H^\bullet(Y;R) \to H^\bullet(X \times Y;R)$ is an algebra isomorphism of cohomology rings.

Proof : From the hypothesis, it follows that cross product Theorem 18.12 is an isomorphism of R-modules, and by Proposition 20.1 it is a map of R-algebras. Hence, the cross product is an algebra isomorphism. $\square$

20.2 Cohomology Ring of Wedge Sum of Spaces

Suppose (X, x_0) is a based space. Then, the inclusion map $\iota : \{\star\} \hookrightarrow (X, x_0)$ of the basepoint induces a map $\iota^\star : H^\bullet(X) \to H^\bullet(\star)$. The *reduced cohomology* is defined as $\tilde{H}^\bullet(X) = \ker(\iota^\star : H^\bullet(X) \to H^\bullet(\star))$. We immediately have

$$H^k(X) = \begin{cases} \tilde{H}^k(X) \oplus R, & k = 0 \\ \tilde{H}^k(X), & \text{otherwise.} \end{cases}$$

If $A \hookrightarrow X$ is a cofibration (e.g, if (X, A) is a CW pair), just like Theorem 10.14, we have that the quotient map $q : X \to X/A$ induces an isomorphism $\tilde{H}^\bullet(X/A) \to H^\bullet(X, A)$.

Proposition 20.3: (*Cohomology of Wedge Sum*)

Suppose $\{(X_i, x_i)\}_{i \in I}$ be a given collection of based spaces, such that the inclusion $\{x_i\} \hookrightarrow X_i$ is a cofibration. Then, for any ring R-module M, there exists an isomorphism

$$\tilde{H}^\bullet\left(\bigvee_{i\in I} X_i; M\right) = \prod_{i \in I} \tilde{H}^\bullet(X_i; M),$$

induced componentwise by the maps $\iota_j : X_j \hookrightarrow \bigvee_{j \in I} X_j$

Proof : It is easy to see that the inclusion maps $\iota_j : X_j \hookrightarrow \coprod_{i \in I} X_i$ induces chain level isomorphism

$$\sum S_\bullet(\iota_j) : \bigoplus_{j\in I} S_\bullet(X_j; R) \to S_\bullet\left(\coprod_{j \in I} X_j; R\right).$$

Dualizing, we get the isomorphism of cochains

$$\left(S^\bullet(\iota)_j\right)_{j \in I} : S^\bullet\left(\coprod_{j\in I} X_j; M\right) \to \prod_{j\in I} S_\bullet(X_j; M),$$

since $\hom(\_, M)$ takes coproducts to products. Thus, we have isomorphism in cohomology

$$H^\bullet\left(\coprod_{j \in I} X_j; M\right) \cong \prod_{j\in I} H^\bullet(X_j; M).$$

The same isomorphisms work for *relative* cohomology as well (Check!). Then, we have

$$\tilde{H}^\bullet\left(\bigvee_{i\in I} X_i; M\right) = H^\bullet\left(\coprod_{i\in I} X_i, \coprod_{i \in I}\{x_i\}; M\right) = \prod_{i\in I} H^\bullet(X_i, x_i; M) = \prod_{i \in I} \tilde{H}^\bullet(X_i; M).$$

One can see that componentwise the map is induced by $X_j \to (X_j, x_j) \to (\sqcup_{i\in I} X_i, \sqcup_{i \in I}) \to \vee_{i\in I} X_i$, which is precisely $\iota_j : X_j \hookrightarrow \vee_{i \in I} X_j$. This proves the claim. $\square$

Our next goal is to understand the cohomology ring structure of a wedge sum.

Theorem 20.4: (*Cohomology Ring of Wedge*)

Let $(X, x_0), (Y, y_0)$ are based spaces, with the inclusions of the basepoints $\iota_X : \{\star\} \hookrightarrow X, \iota_Y : \{\star\} \hookrightarrow Y$ being cofibrations. Then, there exists an isomorphism of algebras

$$H^\star(X \vee Y) = \{(a, b) \in H^\star(X) \times H^\star(Y) \mid \iota_X^\star(a) = \iota_Y^\star(b)\} \subset H^\star(X) \times H^\star(Y),$$

where the cohomologies are computed with coefficients in some fixed ring.

Proof : Denote

$$\mathcal{R} = \{(a, b) \in H^\star(X) \times H^\star(Y) \mid \iota_X^\star(a) = \iota_Y^\star(b)\}.$$

Let us first obeserve that $\mathcal{R}$ is actually a subalgebra of $H^\star(X) \times H^\star(Y)$, where we have componentwise product. Indeed, for $(a_1, b_1), (a_2, b_2) \in \mathcal{R}$, we have

$$\iota_X^\star(a_1 \smile a_2) = \iota_X^\star(a_1) \smile \iota_X^\star(a_2) = \iota_Y^\star(b_1) \smile \iota_Y^\star(b_2) = \iota_Y^\star(b_1 \smile b_2),$$

which implies $(a_1 \smile a_2, b_1 \smile b_2) \in \mathcal{R}$. Also, $\iota_X^\star(1_X) = 1_{\{\star\}} = \iota_Y^\star(1_Y)$, which implies that the unit $(1_X, 1_Y) \in \mathcal{R}$. Thus, $\mathcal{R}$ is indeed a graded subring of $H^\star(X) \times H^\star(Y)$.

As for the isomorphism, note that $\mathcal{R}_k = H^k(X) \times H^k(Y) = H^k(X \vee Y)$ for $k \neq 0$, since ι_X, ι_Y induces zero map. For $k = 0$, we have

$$\mathcal{R}_0 = R \oplus \ker(\iota_X^\star) \times \ker(\iota_Y^\star) = R \oplus \ker(\iota_X^\star \vee \iota_Y^\star) = R \oplus \ker(\iota_{X \vee Y}^\star) = H^0(X \vee Y).$$

Thus, we have $H^\star(X \vee Y) = \mathcal{R}$, as graded modules. A direct computation shows that this an algebra isomorphism. Indeed, for any $z \in H^k(X \vee Y)$, with $k \neq 0$, we have $z = (\iota_1^\star z, \iota_2^\star z)$, where $\iota_1 : X \hookrightarrow X \vee Y, \iota_2 : Y \hookrightarrow X \vee Y$ are the inclusions. Thus, for $z \in H^k(X \vee Y), w \in H^l(X \vee Y)$, with $k, l \neq 0$, we have

$$z \smile w = (\iota_1^\star(z \smile w), \iota_2^\star(z \smile w)) = (\iota_1^\star z \smile \iota_1^\star w, \iota_2^\star z \smile \iota_2^\star w) = (i_1^\star z, \iota_2^\star z) \cdot (\iota_1^\star w, \iota_2^\star w).$$

As for $k = 0$ or $l = 0$, we only need to check the cup product for the unit $1_{X \vee Y}$. But $1_{X \vee Y} = (1_X, 1_Y)$ is clearly the unit in $\mathcal{R}$. Thus, we have an algebra isomorphism. □

Exercise 20.5: (*Cohomology Ring of Arbitrary Wedge*)

Assume $\{(X_i, x_i)\}_{i \in I}$ is a collection based spaces, such that the basepoint inclusion map $\iota_j : \{\star\} \hookrightarrow X_j$ is a cofibration for all $j \in I$. Given a ring R, prove that there exists an algebra isomorphism

$$H^\star\left(\bigvee_{i \in I} X_i; R\right) = \left\{(a_i)_{i \in I} \in \prod_{i \in I} H^\star(X_i; R) \;\middle|\; \iota_j^\star(a_j) = \iota_k^\star(a_k) \forall j, k\right\}.$$

20.3 Computation of Cup Products

In this section, we shall compute the cup product in a few spaces. In fact, we shall identify the cohomology rings for these spaces. Let us recall a few noations. For a ring R, we have the usual polynomial ring $R[t]$. More generally, we shall write $R[t_1, ..., t_k]$ with $|t_i| = n_i$ to mean that the indeterminates are at degree n_i respectively. Given polynomials $f_1, ..., f_k$, we shall denote the ideal generated by them by $\langle f_1, ..., f_k \rangle$.

20.3.1 Cup Product in Spheres Let us begin with the simplest example of the sphere.

Example 20.6: (*Cup Produt in Sphere*)

Suppose $n > 0$. By the UCT (Theorem 17.1), we have $H^k(S^n; R) = \begin{cases} R, k=0,n \\ 0, \text{ otherwise} \end{cases}$. Then, for degree reasons, all the cup products vanish, except for the unit! Indeed, for any generator $a \in H^n(S^n; R)$ we have $a^2 = a \smile a \in H^{2n}(S^n; R) = 0$, and so $a^2 = 0$. Thus, we have the cohomology ring $H^\bullet(S^n; R) = \frac{R[t]}{\langle t^2 \rangle}$ with $|t| = n$. In fact, it is easy to see that the same remains true for $n = 0$.

Exercise 20.7: (*Cohomology Ring of Product of Spheres*)

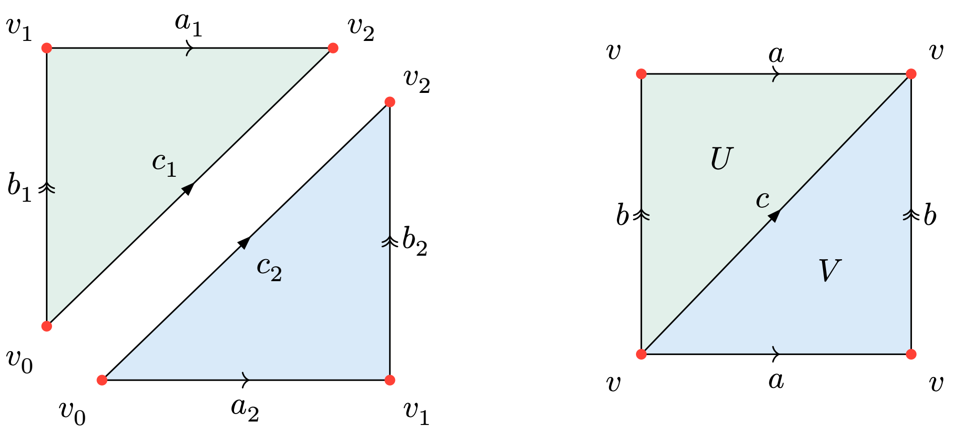
Verify that the cohomology ring of the space $X = \prod_{i=1}^k S^{n_i}$ for $n_i \geq 0$ over a PID R is isomorphic to the algebra

$$\frac{R[t_1, \dots, t_k]}{\langle t_1^2, \dots, t_k^2 \rangle}, |t_1| = n_1, \dots, |t_k| = n_k.$$

Hint : We have an *algebra* isomorphism $R[X] \otimes R[Y] = R[X, Y]$, which preserves degrees of the inderminates.

20.3.2 Cup Product in Torus Of course we can compute the cohomology ring of the torus $\mathbb{T}^2 = S^1 \times S^1$ by applying Theorem 20.2 and get $H^\star(\mathbb{T}^2) = \frac{\mathbb{Z}[X,Y]}{\langle X^2, Y^2 \rangle}$, with $|X| = 1 = |Y|$. Instead, we compute it from the definition!

Let us consider the the torus along with simplices givign rise to cellular decomposition



The 1-simplices a, b generates the first homology $H_1(\mathbb{T}^2)$. Next, consider the two 2-simplices $U, V : \Delta^2 \to \mathbb{T}^2$, oriented such a way that

$$\partial U = a - c + b, \quad \partial V = b - c + a,$$

so that $\partial(V - U) = 0$. Thus, $[V - U]$ represents a homology class in $H_2(\mathbb{T}^2)$. Moreover, By cellular homology, it can be seen that $[V - U]$ is a generator of $H_2(\mathbb{T}^2)$.

By the UCT (Theorem 17.1), we have $H^\bullet(\mathbb{T}^2) = \hom(H_\bullet(\mathbb{T}^2), \mathbb{Z})$, since the ext modules are zero. Then, we have generators $\alpha, \beta \in H^1(\mathbb{T}^2, \mathbb{Z})$ and $\gamma \in H^2(\mathbb{T}^2, \mathbb{Z})$ given as dual to $[a]$, $[b]$, and $[V - U]$ respectively. Let us find cocycle representations for α, β.

Suppose $\alpha = [\varphi]$ for some $\varphi : C_1^{\text{cell}}(\mathbb{T}^2) = \mathbb{Z}\langle a, b, c \rangle \to \mathbb{Z}$. Since $\alpha([a]) = 1, \alpha([b]) = 0$, we must have $\varphi(a) = 1, \varphi(b) = 0$. Since φ needs to be a cocyle, we require $\delta\varphi = 0$. This gives,

$$0 = \delta\varphi(U) = \varphi(\partial U) = \varphi(a - c + b) = \varphi(a) - \varphi(c) + \varphi(b) = 1 - \varphi(c) + 0 \Rightarrow \varphi(c) = 1.$$

Similarly, we have $\psi : C_1^{\text{cell}}(\mathbb{T}^2) \to \mathbb{Z}$ given by $\psi(a) = 0, \psi(b) = 1, \psi(c) = 1$, which is a cocylce representing β. Note that in order to make these into a *singular* cocylce, we need to consider all possible singular 1-simplices. One can do that as follows. Fix paths γ_x joining the vertex v to any point $x \in \mathbb{T}^2$. Then, for any $\sigma : \Delta^1 = [0,1] \to \mathbb{T}^2$, we have the 1-cycle $\tilde{\sigma} = \gamma_{\sigma(0)} \star \sigma \star \overline{\gamma}_{\sigma(1)}$. Set $\varphi(\sigma) := \alpha([\tilde{\sigma}])$. Similarly, we can extend the definition of ψ to all of $S_1(\mathbb{T}^2)$.

Remark 20.8: (*Cellular Cohomology*)

We can instead work with *cellular cohomology*. That is, we consider the cellular chain complex $C_\bullet^{\text{cell}}$, apply the hom-functor $\hom(\_, M)$, and get the cohomology $H^\bullet_{\text{cell}}(X) = H^\bullet(\hom(C^{\text{cell}}, M))$. Recall from Theorem 13.18 that $H_\bullet^{\text{cell}}(X)$ is naturally isomorphic to $H_\bullet(X)$. One can apply the algebraic version of the UCT (Theorem 17.1) to the cellular chain complex $C_\bullet^{\text{cell}}(X)$, and then the naturality of the short exact sequence (and an application of five-lemma) will give us a natural isomorphism $H^\bullet_{\text{cell}}(X) \cong H^\bullet(X)$.

In fact, one can show that the singular chain complex $S_\bullet(X)$ is chain homotopy equivalent to the cellular chain complex $C_\bullet^{\text{cell}}(X)$, which can directly lead to the fact that $H^\bullet_{\text{cell}}(X) = H^\bullet(X)$.

Note that we are actually not defining how to compute cohomology for any arbitrary cellular cochain complex; the spaces that we consider gives rise to nice cellular structure whose cells are also simplices (sometimes these are called *delta complexes*).

Example 20.9: (*Cup Product in the Torus* $\mathbb{T}^2$)

The only interesting cup products appear involving the generators $\alpha, \beta \in H^1(\mathbb{T}^2, \mathbb{Z})$.

- By graded commutativity,

$$\alpha \smile \alpha = (-1)^{|\alpha|\cdot|\alpha|}\alpha \smile \alpha = -\alpha \smile \alpha \Rightarrow 2(\alpha \smile \alpha) = 0 \Rightarrow \alpha \smile \alpha = 0.$$

- By similar argument, we have $\beta \smile \beta = 0$.
- We claim $\alpha \smile \beta = \gamma$. This requires checking $(\alpha \smile \beta)([V - U]) = 1$. Indeed, we have

$$\begin{aligned}(\varphi \smile \psi)(U - V) &= \varphi\big(V|_{[v_0,v_1]}\big) \cdot \psi\big(V|_{[v_1,v_2]}\big) - \varphi\big(U|_{[v_0,v_1]}\big) \cdot \psi\big(U|_{[v_1,v_2]}\big) \\ &= \varphi(a) \cdot \psi(b) - \varphi(b) \cdot \psi(a) = 1.\end{aligned}$$

  Thus, $\alpha \smile \beta = \gamma$ follows.
- By graded commutativity, $\beta \smile \alpha = (-1)^{1\cdot 1}\alpha \smile \beta = -\alpha \smile \beta = -\gamma$.

Hence, we can identify the cohomology ring

$$H^\star(\mathbb{T}^2) = \frac{\mathbb{Z}[X,Y]}{\langle X^2, Y^2\rangle}, \quad |X| = 1 = |Y|,$$

with the understanding that it is a *graded* commutative algebra and thus, $XY = YX$. This also known as the *exterior algebra* $\Lambda_{\mathbb{Z}}(X,Y)$ over $\mathbb{Z}$, with two degree 1 generators.

Exercise 20.10: (*Cohomology Ring of Torus with $\mathbb{Z}_2$ Coefficient*)

Compute the comology ring of the the torus with $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ coefficients. Note: $2x = 0 \not\Rightarrow x = 0$ in $\mathbb{Z}_2$!

20.3.3 Cup Product in Oriented Surfaces Recall, the *oriented* genus-g surface Σ_g is given as the the g-fold connected sum of 2-tori

$$\Sigma_g := \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g\text{-times}}.$$

Equivalently, Σ_g is constructed by considering $2g$-fold wedge of circles $\bigvee_{i=1}^g S^1_{a_i} \vee S^1_{b_i}$, and then attaching a 2-disc by the attaching map $S^1 \to \vee^{2g} S^1$ given by

$$[a_1, b_1] \cdots [a_g, b_g] = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}.$$

By Hurewicz theorem (Theorem 10.1), it follows that

$$H_1(\Sigma_g) = \pi_1(\Sigma_g)^{\mathrm{ab}} = \bigoplus_{i=1}^g \mathbb{Z}\langle a_i \rangle \oplus \mathbb{Z}\langle b_i \rangle = \mathbb{Z}^{2g}.$$

By cellular homology, we also have $H_2(\Sigma_g) = \mathbb{Z}$. Then, by UCT (Theorem 17.1), it follows that $H^\bullet(\Sigma_g) = \hom(H_\bullet(\Sigma_g), \mathbb{Z})$, and thus,

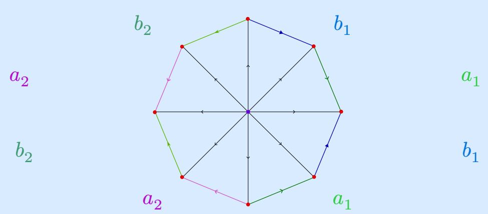
$$H^k(\Sigma_g) = \begin{cases} \mathbb{Z}, & k = 0, 2 \\ \mathbb{Z}^{2g}, & k = 1 \\ 0, & \text{otherwise}. \end{cases}.$$

Again, the only interseting cup products are $\smile: H^1(\Sigma_g) \otimes H^1(\Sigma_g) \to H^2(\Sigma_g)$. Denote, α_i, β_i as the duals of a_i, b_i resepectively, which are generators of $H^1(\Sigma_g)$.

Exercise 20.11:

Compute the cup products $\alpha_i \smile \alpha_j, \alpha_i \smile \beta_j, \beta_i \smile \beta_j$ for all i, j by considering a suitable cellular decomposition of Σ_g.

Hint : Consider the following diagram for Σ_2



Since we have already computed the cohomology ring of $\Sigma_1 = \mathbb{T}^2$, let us try to use it as a base case of an induction.

Proposition 20.12: (*Cohomology Ring of Σ_g*)

The cohomology ring of the oriented genus-g surface is the graded commutative ring

$$H^\star(\Sigma_g) = \frac{\mathbb{Z}[a_1, b_1, ..., a_g, b_g]}{\langle a_i^2 \ \ \forall i, \quad b_i^2 \ \ \forall i, \quad a_i b_j \ \ \forall i \neq j \rangle}, \qquad |a_i| = 1 = |b_j|.$$

Proof : Consider the quotient map $\Sigma_g \to X_i$ which collapses the two 1-cells labeled by a_i and b_i. It is then easy to see that X_i is homotopy equivalent to Σ_{g-1}. Composing with a homotopy equivalence, we then get a map $q_i : \Sigma_g \to \Sigma_{g-1}$. Moreover, this map is *surjective* on H_1 and isomorphism on H_2, which can be verified by cellular homology. Hence, by the naturality of the UCT, we have maps $q_i^\star : H^1(\Sigma_{g-1}) \hookrightarrow H^1(\Sigma_g)$, which maps the generators to the corresponding generators, skipping the i^{th} one. That is,

$$q_i^\star(\alpha_j') = \begin{cases} \alpha_j, & j < i \\ \alpha_{j+1}, & j \geq i \end{cases} \qquad \text{and} \qquad q_i^\star(\beta_j') = \begin{cases} \beta_j, & j < i \\ \beta_{j+1}, & j \geq i. \end{cases}$$

Moreover, $q_i^\star(\gamma') = \gamma$, where γ' and γ are respectively the generators of $H^2(\Sigma_{g-1})$ and $H^2(\Sigma_g)$. Then, for $j_1 \neq i, j_2 \neq i$, we have

$$\alpha_{j_1} \smile \alpha_{j_2} = q_i^\star\left(\alpha_{j_1}'\right) \smile q_i^\star\left(\alpha_{j_2}'\right) = q_i^\star\left(\alpha_{j_1}' \smile \alpha_{j_2}'\right) = q_i^\star(0) = 0,$$

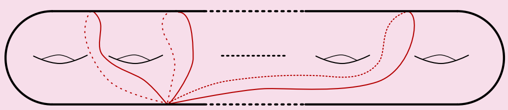
and similarly, $\beta_{j_1} \smile \beta_{j_2} = 0$. If $j_1 \neq j_2$, then $\alpha_{j_1} \smile \beta_{j_2} = 0$ as well. Finally, for any $j \neq i$ we have

$$\alpha_j \smile \beta_j = q_i^\star(\alpha_j' \smile \beta_j') = q_i^\star(\gamma') = \gamma.$$

Since i is arbitrary, we have all the cup products. Inductively, we have the claim. □

Exercise 20.13:

Consider Σ_g, the origented genus-g surface



Observe that collapsing the red curves to a point, we have a quotient map $q : \Sigma_g \to \bigvee_{i=1}^g \mathbb{T}^2$ to g-fold wedge of the torus. Compute the cohomology ring of Σ_g using this map.

Hint : By computing the local degree (Proposition 13.8), it follows that $q_\star(c) = c_1 + \cdots + c_g$, where $H_2(\Sigma_g) = \mathbb{Z}\langle c \rangle$ and $H_2(\vee^g \mathbb{T}^2) = \mathbb{Z}\langle c_1, ..., c_g \rangle$, with suitable choice of the 2-cells representing c_i. Then, by naturality of UCT, we get $q^\star(\gamma_i) = \gamma$, where γ_i, γ are respectively dual of c_i, c.

20.3.4 Cup Product in $\mathbb{RP}^2$ Recall, the homology of the real projective space $\mathbb{RP}^2$ (Example 13.13) is

$$H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/2\mathbb{Z}, & k = 1 \\ 0, & \text{otherwise.} \end{cases}$$

An easy application of the UCT (Theorem 17.11) then gives

$$H^k(\mathbb{RP}^2; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/2\mathbb{Z}, & k = 2 \\ 0, & \text{otherwise.} \end{cases}$$

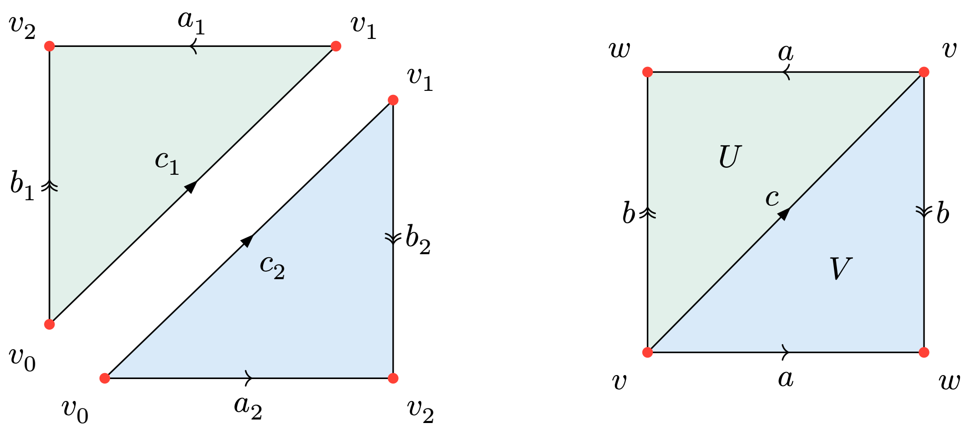
From degree reasons, it is clearl that the cohomology ring structure of $H^\star(\mathbb{RP}^2, \mathbb{Z})$ is not very interesting! Now, another application of UCT gives the cohomology groups with $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ coefficients

$$H^k(\mathbb{RP}^2; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & k = 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

There is a possibility of an interesting cup product

$$\smile : H^1(\mathbb{RP}^2, \mathbb{Z}_2) \otimes H^1(\mathbb{RP}^2, \mathbb{Z}_2) \to H^2(\mathbb{RP}^2, \mathbb{Z}_2).$$

Just like the computation of cup product in torus (Example 20.9), we consider the following cellular decomposition of $\mathbb{RP}^2$.



Let us compute the cellular homology with coefficients in $\mathbb{Z}_2$. We have the complex

$$0 \to \underset{\mathbb{Z}_2\langle U, V\rangle}{C_2^{\text{cell}}(\mathbb{RP}^2; \mathbb{Z}_2)} \to \underset{\mathbb{Z}_2\langle a, b, c\rangle}{C_1^{\text{cell}}(\mathbb{RP}^2; \mathbb{Z}_2)} \to \underset{\mathbb{Z}_2\langle v, w\rangle}{C_0^{\text{cell}}(\mathbb{RP}^2; \mathbb{Z}_2)} \to 0.$$

Note that $\partial b = w - v = \partial a$, and in particular, $H_0(\mathbb{RP}^2, \mathbb{Z}_2)) = \mathbb{Z}_2\langle [v] = [w] \rangle = \mathbb{Z}_2$. Also, $\partial U = a - b + c, \partial V = b - a + c$.Over $\mathbb{Z}_2$, we have $\partial(V - U) = 2(b - a) = 0$. So, $H_2(\mathbb{RP}^2); \mathbb{Z}_2) = \mathbb{Z}_2\langle [V - U] \rangle = \mathbb{Z}_2$ and $H_1(\mathbb{RP}^2; \mathbb{Z}_2) = \mathbb{Z}_2\langle [c] = [b - a] \rangle = \mathbb{Z}_2$.

Example 20.14: (*Chomology Ring of $\mathbb{RP}^2$ with $\mathbb{Z}_2$*)

Since $\mathbb{Z}_2$ is a field, it follows from the UCT (Theorem 17.11) that $H^\bullet(\mathbb{RP}^2;\mathbb{Z}_2) = \hom(H_\bullet(\mathbb{RP}^2;\mathbb{Z}_2),\mathbb{Z}_2)$, as the Ext terms vanish over a field. Hence, the nontrivial term $\alpha \in H^1(\mathbb{RP}^2;\mathbb{Z}_2) = \mathbb{Z}_2$ is represented by a 1-cocycle $\varphi : C_1^{\mathrm{cell}}(\mathbb{RP}^2;\mathbb{Z}_2) \to \mathbb{Z}_2$ such that $\varphi(b-a) = 1$. Since we must take value in $\mathbb{Z}_2$, we can have $\varphi(b) = 1, \varphi(a) = 0$ (or, $\varphi(b) = 0, \varphi(a) = 1$). Let us assume the first case. Then, the cocylce condition gives

$$0 = \delta\varphi(U) = \varphi(\partial U) = \varphi(a-b+c) = \varphi(a) - \varphi(b) + \varphi(c) = 0 - 1 + \varphi(c) \Rightarrow \varphi(c) = 1.$$

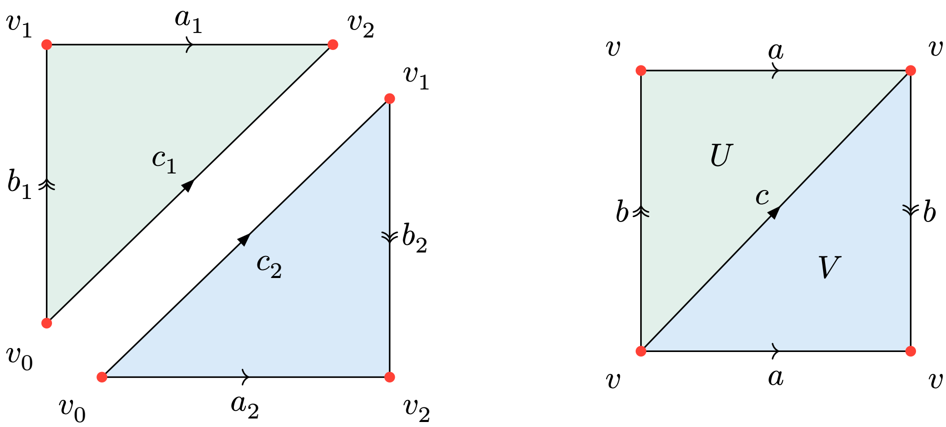
We compute

$$\begin{aligned}(\varphi \smile \varphi)(V-U) &= \varphi\big(V|_{[v_0,v_1]}\big)\cdot\varphi\big(V|_{[v_1,v_2]}\big) - \varphi\big(U|_{[v_0,v_1]}\big)\cdot\varphi\big(U|_{[v_1,v_2]}\big) \\ &= \varphi(c)\cdot\varphi(b) - \varphi(c)\cdot\varphi(a) = 1\cdot 1 - 1\cdot 0 = 1.\end{aligned}$$

Thus, $\alpha \smile \alpha$ is dual to the class $[V-U]$, and hence, is a generator of $H^2(\mathbb{RP}^2;\mathbb{Z}_2)$. Consequently, we have the cohomology ring

$$H^\star(\mathbb{RP}^2,\mathbb{Z}_2) = \frac{\mathbb{Z}_2[X]}{\langle X^3\rangle}, \quad |X| = 1.$$

20.3.5 Cup Product in Klein Bottle The Klein bottle K is given the following cellular decomposition cellular decomposition



An argument similar to that of $\mathbb{RP}^2$ leads us to compute

$$H^k(K;\mathbb{Z}_2) = \begin{cases}\mathbb{Z}_2, & k = 0,2\\ \mathbb{Z}_2^2, & k = 1\\ 0, & \text{otherwise.}\end{cases}$$

The first cohomology is generated by duals of the generators $[a],[b]$ in the homology $H_1(K;\mathbb{Z}_2)$, say, α,β. Let $\alpha = [\varphi], \beta = [\psi]$ for some cellular 1-cocyles $\varphi,\psi : C_1^{\mathrm{cell}}(K;\mathbb{Z}_2) \to \mathbb{Z}_2$. We have, $\varphi(a) = 1, \varphi(b) = 0, \psi(a) = 0, \psi(b) = 1$. Note that $\partial U = a - c + b, \partial V = b - a + c$, and thus $\partial(V-U) = (b-a+c) - (a-c+b) = 2(c-a) = 0$. In particular, $[V-U]$ generates $H_2(K;\mathbb{Z}_2)$, and its dual generates $H^2(K;\mathbb{Z}_2)$. The cocycle condition $\delta\varphi = 0$ gives

$$0 = \delta\varphi(U) = \varphi(\partial U) = \varphi(a-c+b) = \varphi(a) - \varphi(c) + \varphi(b) = 1 - \varphi(c) + 0 \Rightarrow \varphi(c) = 1,$$

and similarly, $\psi(c) = 1$. Then, we compute

$$\begin{aligned}(\varphi \smile \psi)(V-U) &= \varphi\big(V|_{[v_0,v_1]}\big)\cdot\psi\big(V|_{[v_1,v_2]}\big) - \varphi\big(U|_{[v_0,v_1]}\big)\cdot\psi\big(U|_{[v_1,v_2]}\big)\\ &= \varphi(c)\cdot\psi(b) - \varphi(b)\cdot\psi(a) = 1\cdot 1 - 0\cdot 0 = 1.\end{aligned}$$

In other words, $\alpha \smile \beta$ is the generator of $H^2(K;\mathbb{Z}_2)$. A similar computation shows $\beta \smile \beta$ is also a generator, but $\alpha \smile \alpha = 0$. Thus, we have the ring structure

$$H^\star(K;\mathbb{Z}_2) = \frac{\mathbb{Z}_2[X,Y]}{\langle X^2, Y^2+XY\rangle}, \quad |X| = 1 = |Y|.$$

By a change of basis $X \mapsto X+Y, Y \mapsto X$, we can get the isomorphism

$$H^\star(K;\mathbb{Z}_2) = \frac{\mathbb{Z}_2[X,Y]}{\langle X^2+Y^2, XY\rangle}, \quad |X| = 1 = |Y|.$$

20.3.6 Cup Product in Nonorientable Surfaces The nonorientable genus-g surface N_g is given as the g-fold connected sum of $\mathbb{RP}^2$. It can be reaslized as a 2-dimensional CW complex whose 1-skeleton is the wedge of circles $\bigvee_{i=1}^{g} S^1_{a_i}$, and then a two cell is attached along the map $a_1^2 \cdots a_g^2$. By Hurewicz theorem (Theorem 10.1), it follows that

$$H_1(N_g) = \pi_1(N_g)^{\text{ab}} = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2.$$

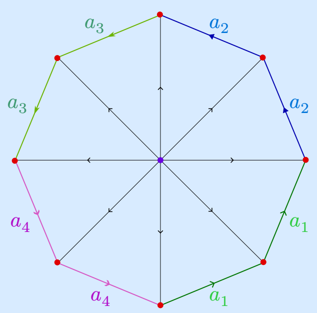
By cellular homology, $H_2(N_g;\mathbb{Z}) = 0$. Thus, cup products in $\mathbb{Z}$ coefficients are not that interesting, and just like the case of $N_1 = \mathbb{RP}^2$ or $N_2 = K$ (Klein bottle), we focus on $\mathbb{Z}_2$ coefficient. By UCT, or by cellular homology with $\mathbb{Z}_2$-coefficients, we have

$$H_k(N_g;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & k = 0,2\\ \mathbb{Z}_2^g, & k = 1\\ 0 & \text{otherwise.}\end{cases}$$

Exercise 20.15:

Compute $H^\bullet(N_g;\mathbb{Z}_2)$ from UCT, and then compute the cup product by a suitable cellular decomposition as in Exercise 20.11.

Hint : Consider the following diagram for N_4



We can also use an inductive argument, and show that

$$H^{\star}\left(N_g;\mathbb{Z}_2\right)=\frac{\mathbb{Z}_2\left[a_1,...,a_g\right]}{\langle a_ia_j \quad \forall i\neq j, \qquad a_i^2+a_j^2 \quad \forall i\neq j\rangle}, \quad |a_i|=1.$$

Day 21 : 19th April, 2026

`Hopf invariant – relative cup product – cohomology ring of projective spaces – orientation`

21.1 Application of Cup Product : Hopf Invariant

Given a map $f : X \to Y$, recall the *mapping cone* is defined as

$$C_f = CX \cup_f Y,$$

where we attach the space Y to the base of the cone $CX = \frac{X \times [0,1]}{X \times \{0\}}$ by identifying $(x, 1) \sim f(x)$. It can be proved that if $f \simeq g : X \to Y$, then C_f is homotopy euquivalent to C_g.

Exercise 21.1:

If $h : f \simeq g : X \to Y$ is a homotopy, then show that

$$\begin{aligned} H : C_f &\to C_g \\ y &\mapsto y \\ [x, t] &\mapsto \begin{cases} h(x, 2t), & t \le \frac{1}{2} \\ h(x, 2t-1), & t \ge \frac{1}{2} \end{cases} \end{aligned}$$

is a homotopy equivalence, with homotopy inverse given by

$$\begin{aligned} G : C_g &\to C_f \\ y &\mapsto y \\ [x, t] &\mapsto \begin{cases} h(x, 1-2t), & t \le \frac{1}{2} \\ h(x, 2t-1), & t \ge \frac{1}{2}. \end{cases} \end{aligned}$$

Assume $n \ge 2$. Given a map $S^{2n-1} \to S^n$, consider the mapping cone C_f. Then, from cellular homology and UCT, it follows that

$$H^k(C_f) = H_k(C_f) = \begin{cases} \mathbb{Z}, & k = 0, n, 2n \\ 0, & \text{otherwise.} \end{cases}$$

Fix some generator $\alpha \in H^n(C_f), \gamma \in H^{2n}(C_f)$. Then, the cup product $\alpha \smile \alpha \in H^{2n}(C_f)$ is an integer multiple of γ. We write

$$\alpha \smile \alpha = h(f)\gamma.$$

The element $h(f) \in \mathbb{Z}$ is called the *Hopf invariant* of the map f, which is well-defined up to a sign (based on the choice of generators), and independent of the homotopy class of the map f. When n is odd, it is clear that $h(f) = 0$, since

$$\alpha \smile \alpha = (-1)^{n \cdot n} \alpha \smile \alpha = -\alpha \smile \alpha \Rightarrow 2\alpha \smile \alpha = 0 \Rightarrow \alpha \smile \alpha = 0.$$

Thus, let us consider maps $f : S^{4n-1} \to S^{2n}$ only.

Remark 21.2: (*Hopf Invariant One*)

One can explicitly produce maps $f : S^{4n-1} \to S^{2n}$ with $h(f) = 1$ for $n = 1, 2, 4$. Adams proved that these are the only possibilities for a map to have Hopf invariant 1 (up to a sign).

Let us now consider the space $S^{2n} \times S^{2n}$. It can be given a CW structure with one 0-cell, two $2n$-cells, and one $4n$-cell. We have the attaching map for the top $4n$-cell is given by $\rho : S^{4n-1} \to S^{2n} \vee S^{2n}$. Consider the composition

$$f : S^{4n-1} \xrightarrow{\rho} S^{2n} \vee S^{2n} \xrightarrow{\nabla} S^{2n},$$

where ∇ is the fold map.

Proposition 21.3:

The Hopf invariant is $h(f) = \pm 2$.

Proof : Consider the induced (quotient) map $q : S^{2n} \times S^{2n} = C_\rho \to X = C_f$. From cellular homology and Kunneth formula, we have generators

$$u_1, u_2 \in H_1(S^{2n} \times S^{2n}), v \in H_2(S^{2n} \times S^{2n}),$$

and

$$\overline{u} \in H_1(S^{2n} \times S^{2n}), \overline{v} \in H_2(S^{2n} \times S^{2n}),$$

such that

$$q_\star(u_1) = \overline{u} = q_\star(u_2), \quad q_\star(v) = \overline{v}.$$

By UCT, it follows that the cohomology is dual to the homology, and by naturality $q^\star(\overline{u}^\star) = u_1^\star + u_2^\star, q^{\star(\overline{v}^\star)} = v^\star$. From Theorem 20.2, we get $u_1^\star \smile u_2^\star = \pm v^\star$, and $u_1^\star \smile u_1^\star = 0 = u_2^\star \smile u_2^\star$. Now,

$$\begin{aligned} q^\star(\overline{u}^\star \smile \overline{u}^\star) &= q^\star(\overline{u}) \smile q^\star(\overline{u}) \\ &= (u_1^\star + u_2^\star) \smile (u_1^\star + u_2^\star) \\ &= \underbrace{u_1^\star \smile u_1^\star}_{0} + \underbrace{u_2^\star \smile u_2^\star}_{0} + u_1^\star \smile u_2^\star + u_2^\star \smile u_1^\star \\ &= u_1^\star \smile u_1^\star + (-1)^{2n \cdot 2n} u_1^\star \smile u_2^\star \\ &= 2u_1^\star \smile u_2^\star \\ &= \pm 2v^\star = \pm 2q^\star(\overline{v}^\star). \end{aligned}$$

Hence, we have $h(f) = \pm 2$. $\square$

As an immediate consequence, we have $\pi_{4n-1}(S^{2n}) \neq 0$. One can moreover prove that the Hopf invariant map

$$\begin{aligned} h : \pi_{4n-1}(S^{2n}) &\to \mathbb{Z} \\ [f] &\mapsto h(f) \end{aligned}$$

is a group homomorphism, with some generators fixed. Consequently, we get that

$$\pi_{4n-1}S^{2n} = \mathbb{Z} \oplus \ker(h).$$

One can show that $\ker(h)$ is always a *finite* Abelian group, e.g, by Serre's technique. One can compute

$$\pi_3 S^2 = \mathbb{Z}, \quad \pi_7 S^4 = \mathbb{Z} \oplus \mathbb{Z}_{12}, \quad \pi_{15}(S^8) = \mathbb{Z} \oplus \mathbb{Z}_{120}.$$

21.2 Relative Cup Product

Suppose (X, A) is a pair. Recall, we have the short exact sequence

$$0 \to S_\bullet(A; R) \to S_\bullet(X; R) \to S_\bullet(X, A; R) \to 0$$

of chain complexes of singular simplices with coefficients in a ring R. Then, given an R-module M, the *relative cohomology* with coefficients in M is computed as

$$H^\bullet(X, A; M) := H^\bullet(\hom(S_\bullet(X, A), M))$$

A relative cochain $\varphi \in S^n(X, A; M)$ can be understood as a cochain on X which vanishes on all p-chains completely contained in A.

Now, suppose (X, A) and (X, B) are two pairs. Then, for $\varphi \in S^p(X, A)$ and $\psi \in S^q(X, B)$, consider the cochain $\varphi \smile \psi$. For any simplex $\sigma \in S_{p+q}(A) + S_{p+q}(B)$ (not a direct sum), i.e, a simplex that is contained in either A or B, we have

$$\varphi \smile \psi(\sigma) = \varphi\Big(\sigma|_{[v_0,\ldots,v_p]}\Big) \cdot \psi\Big(\sigma|_{[v_p,\ldots,v_{p+q}]}\Big) = 0,$$

since either the first term or the second term in the product vanishes! Thus, we have a cochain level cup product

$$\smile : S^p(X, A) \otimes S^q(X, B) \to \frac{S^{p+q}(X)}{S^{p+q}(A) + S^{p+q}(B)}.$$

Passing to cohomology, we get the cup product

$$\smile : H^p(X, A) \otimes H^q(X, B) \to H^{p+q}\left(\frac{S^\bullet(X)}{S^\bullet(A) + S^\bullet(B)}\right).$$

Exercise 21.4:

Verify that given a pair (X, A) we have relative cup products

$$\smile : H^p(X) \otimes H^q(X, A) \to H^{p+q}(X, A)$$
$$\smile : H^p(X, A) \otimes H^q(X) \to H^{p+q}(X, A)$$
$$\smile : H^p(X, A) \otimes H^q(X, A) \to H^{p+q}(X, A)$$

These makes $H^\star(X, A)$ as a graded algebra over the graded ring $H^\star(X)$.

Now, call the pair of subspaces (A, B) *excisive* if the inclusion map

$$S_\bullet(A) + S_\bullet(B) \hookrightarrow S_\bullet(A \cup B)$$

is a chain homotopy equivalence.

Example 21.5: (*Excisive Pairs*)

Here are a few useful examples of excisive pairs.

1. Suppose $A, B \subset X$ are open. Then, $\mathcal{U} = \{A, B\}$ is an open cover of $A \cup B$. By Lemma 9.4, we then have

$$S_\bullet^{\mathcal{U}}(X) = S_\bullet(A) + S_\bullet(B) \hookrightarrow S_\bullet(A \cup B)$$

is a chain homotopy equivalence.

2. If X is a CW complex, and $A, B \subset X$ are subcomplexes we have $S_\bullet(A) + S_\bullet(B) \hookrightarrow S_\bullet(A \cup B)$ is a chain homotopy equivalence.

To see this, note that A and B are deformation retracts of open sets in $A \cup B$. Then, by excision, we have that this map induces isomorphism in homology (i.e, a quasi-isomorphism). But then it must be a chain homotopy equivalence (fact : a quasi-equivalence between bounded below chain complex of free R-modules is a chain homotopy equivalence). One can also use the chain homotopy equivalences $S_\bullet(A) \to S_\bullet(\mathcal{O}(A)), S_\bullet(B) \to S_\bullet(\mathcal{O}(B))$, where $\mathcal{O}(A), \mathcal{O}(B) \underset{\text{open}}{\subset} A \cup B$ deforms onto A, B resepectively, and then use Lemma 9.4.

Given excisive pairs $A, B \subset X$, we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet(A) + S_\bullet(B) \arrow[r] \arrow[d, "\simeq"'] & S_\bullet(X) \arrow[r] \arrow[d, equal] & \frac{S_\bullet(X)}{S_\bullet(A)+S_\bullet(B)} \arrow[r] \arrow[d, dashed, "\simeq"'] & 0 \\
0 \arrow[r] & S_\bullet(A \cup B) \arrow[r] & S_\bullet(X) \arrow[r] & S_\bullet(X, A \cup B) \arrow[r] & 0
\end{tikzcd}$$

Dualizing the induced chain homotopy equivalence, and then passing to the cohomology, we get natural isomorphism

$$H^\bullet(X, A \cup B) \xrightarrow{\cong} H^\bullet\left(\hom\left(\frac{S(X)}{S(A) + S(B)}, R\right)\right).$$

Definition 21.6: (*Relative Cup Product*)

Given a space X and two excisive subspaces $A, B \subset X$ (i.e, $S_\bullet(A) + S_\bullet(B) \hookrightarrow S_\bullet(A \cup B)$ is a chain homotopy equivalence), the *relative cup product* is defined as the composition

$$\smile : H^p(X, A) \otimes H^q(X, B) \to H^{p+q}\left(\frac{S^\bullet(X)}{S^\bullet(A) + S^\bullet(B)}\right) \xleftarrow{\cong} H^\bullet(X, A \cup B).$$

Exercise 21.7:

Suppose $A, B, C \subset X$ are open subsets. Verify the following.

1. The relative cup product is associative, i.e, the diagram commutes

$$\begin{tikzcd}
H^p(X,A) \otimes H^q(X,B) \otimes H^r(X,C) \arrow[r, "\smile \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes \smile"'] & H^{p+q}(X, A\cup B) \otimes H^r(X,C) \arrow[d, "\smile"] \\
H^p(X,A) \otimes H^{q+r}(X, B\cup C) \arrow[r, dash, "\smile"'] & H^{p+q+r}(X, A\cup B\cup C)
\end{tikzcd}$$

2. Relative cup product is graded commutative, i.e,

$$a \smile b = (-1)^{pq} b \smile a, \quad a \in H^p(X,A), b \in H^q(X,B).$$

3. There are unit elements $1_A \in H^0(X,A), 1_B \in H^0(X,B), 1_{A\cup B} \in H^0(X, A\cup B)$, and

$$1_A \smile 1_B = 1_{A\cup B}.$$

Let us describe an interesting application of relative cup product.

Definition 21.8: (*Cup Length*)

Given a space X, the *cup-length* is defined as the largest integer n for which that there are n-many cohomology classes $a_i \in H^{p_i}(X)$, with $p_i \geq 1$ such that $a_1 \smile \cdots \smile a_n \neq 0$.

Sated another way, if the cup-length of a space is n, then any k-fold cup product vanishes for $k < n$. Of course, $1 \smile \ldots \smile 1 = 1 \neq 0$, and thus we only consider elements in degree ≥ 1.

Example 21.9:

For the sphere S^n, the cup-length is 1, and for the torus $\mathbb{T}^2$ it is 2.

Next, let us define another related numerical invariant.

Definition 21.10: (*LS Category*)

The *LS category* (i.e, ***L**usternik–**S**chnirelmann category*) of a space X is the least integer n such that there are n-many open sets $U_i \subset X$ that covers X, i.e, $X = \cup_{i=1}^n U_i$, and U_i is contractible in X, i.e, $U_i \hookrightarrow X$ is null-homotopic.

Example 21.11:

One can see that S^n can be covered by the open sets $U = S^n \setminus \{N\}$ and $V = S^n \setminus \{S\}$, the complement of north and south poles. Clearly, $U, V \hookrightarrow S^n$ are null-homotopic. Thus, LS category of S^n is at most 2. Since S^n is not contractible, we can justify that the LS category is exact 2.

For the torus $\mathbb{T}^2$, from the standard pasting diagram, we can see that it can be covered by 3 open sets which are also contractible in $\mathbb{T}^2$. Thus, the LS category of $\mathbb{T}^2$ is at most 3. It is clearly not 1, since $\mathbb{T}^2$ is not contractible. Can we rule out that $\mathrm{LS\text{-}Cat}(\mathbb{T}^2) \neq 2$?!

We have the following proposition, that lets us get a lower bound on the LS category of a space.

Proposition 21.12: (*Cup Length and LS Category*)

Suppose $X = \cup_{i=1}^n U_i$, where $U_i \subset X$ is open and $U_i \hookrightarrow X$ is null-homotopic. Then, any n-fold cup product $a_1 \smile \dots \smile a_n = 0$, where $a_i \in H^{p_i}(X)$ with $p_i \geq 1$. In particular,

$$\text{LS-Cat }(X) \geq \text{Cup-Length }(X) + 1.$$

<u>**Proof**</u> **:** Since $\iota_j : U_j \hookrightarrow X$ is null-homotopic, from the long exact sequence of the pair (X, U_j)

$$\dots H^{n-1}(U_j) \to H^n(X, U_j) \twoheadrightarrow H^n(X) \underset{0}{\overset{i_j^\star}{\to}} H^n(U_j),$$

we have surjections $\eta_j^\star : H^n(X, U_j) \to H^n(X)$, where $\eta_j : X \hookrightarrow (X, U_j)$ are the inclusions. Let $a_i \in H^{p_i}(X)$ be cohomology classes with $p_i \geq 1$, and $\tilde{a}_i \in H^{p_i}(X, U_j)$ be the corresponding lifts, i.e, we have $\eta_j^\star(\tilde{a}_j) = a_j$. Since the relative cup product is associative, we have

$$\tilde{a}_1 \smile \dots \smile \tilde{a}_n \in H^{\sum p_i}(X, \cup_{i=1}^n U_i) = H^{\sum p_i}(X, X) = 0.$$

But then from the naturality of the cup product, we have

$$a_1 \smile \dots \smile a_k = \eta_1^\star(\tilde{a}_1) \smile \dots \smile \eta_n^\star(\tilde{a}_n) = \eta^\star(\tilde{a}_1 \smile \dots \smile \tilde{a}_n) = \eta^\star(0) = 0,$$

where $\eta : X \hookrightarrow (X, U_1 \cup \dots \cup U_n)$ is the inclusion. Thus, $\text{Cup-Length }(X) \lneq n$.

Now, suppose X has finite LS category (if it is inifinite, there is nothing to prove). Say, $\text{LS-Cat}(X) = n$. Hence, there are n-many open sets $U_j \subset X$ that covers X, such that $U_j \hookrightarrow X$ is null-homotopic. Then, we have

$$\text{Cup-Length}(X) + 1 \leq n = \text{LS-Cat }(X),$$

as required. $\square$

As a corollary, we can now claim that $\text{LS-Cat}(\mathbb{T}^2) = 3$.

Exercise 21.13: (*LS Category of Higher Torus*)

Show that the LS category of the higher torus $\mathbb{T}^n = \underbrace{S^1 \times \dots \times S^1}_{n\text{-times}}$ is $\geq n+1$. In fact, $\text{LS-Cat}(M) \leq \dim M + 1$, where M is a second countable, connected manifold, and thus, $\text{LS-Cat}(\mathbb{T}^n) = n + 1$.

Remark 21.14:

Note that in the literature, LS category is sometimes difined as follows : $\text{cat}(X)$ is the least integer n such that there are $(n+1)$-many open sets $U_i \subset X$ covering X such that $U_i \hookrightarrow X$ is null-homotopic. Thus, $\text{cat}(X) + 1 = \text{LS-Cat}(X)$. This defintion has the advantage that one can write

$$\text{cat}(X) \geq \text{Cup-Length}(X).$$

Moreover, one can prove $\text{cat}(X \times Y) \leq \text{cat }(X) \times \text{cat}(Y)$ holds. This led to the famous *Ganea conjecture* which states that $\text{cat}(X \times S^n) = \text{cat}(X) + 1$. Note $\text{cat}(S^n) = 1$. The conjecture was disproved by Norio Iwase.

21.3 Cohomology Ring of Projective Spaces

Let us denote $\mathbb{P}^n(d)$ to be the projective space of $\mathbb{K}^{n+1}$, where $\mathbb{K}$ is the vector space of dimension d over $\mathbb{R}$. We can only consider $d = 1, 2, 4$, which corresponds to

$$\mathbb{P}^n(1) = \mathbb{RP}^n, \quad \mathbb{P}^n(2) = \mathbb{CP}^n, \quad \mathbb{P}^n(4) = \mathbb{HP}^n.$$

Indeed, the definition of projectivization requires an *associative* algebra structure on $\mathbb{K}$. There is also the projective space $\mathbb{P}^n(4) = \mathbb{OP}^1$ corresponding to the octonionic numbers for $n = 0, 1, 2$, but they do not generalize to higher projective spaces.

Remark 21.15:

Stasheff defined a notion of A_n-space and proved that an A_n-space X admits a projective space $XP(i)$ of higher order associated to it for $0 \leq i \leq n$. They generalize the notion of usual projective spaces.

The cotonions $\mathbb{O}$ has a non-associative multiplication, making it into an H-space, or an A_2-space which is in fact not A_3. Thus, we can only have $\mathbb{OP}^n$ for $n = 0, 1, 2$.

The A_∞-spaces are the loop spaces (more precisely, they are weak homotopy equivalent to a loop space).

When the base field is $\mathbb{R}$, fix the ring $R = \mathbb{Z}/2\mathbb{Z}$, and otherwise fix $R = \mathbb{Z}$. Then, we can compute via UCT that

$$H^k(\mathbb{P}^n(d); R) = \begin{cases} R, & k = 0, d, ..., nd \\ 0, & \text{otherwise.} \end{cases}$$

As a graded module, we can write

$$H^\star(\mathbb{P}^n(d); R) = \frac{R[X]}{\langle X^{n+1} \rangle}, \quad |X| = d.$$

The right hand side is the truncated polynomial algebra.

Theorem 21.16: (*Cohomology Ring of Projective Space*)

The cohomology ring of the projective space $\mathbb{P}^n(d)$ with coefficient ring R is the truncated polynomial ring $\frac{R[X]}{\langle X^{n+1} \rangle}$ with $|X| = d$. Moreover, passing to the limit, we have

$$H^\star(\mathbb{P}^\infty(d)) = R[X], \quad |X| = d.$$

21.4 Orientation of a Manifold and Homology

Let us now focus on a *manifolds.*

Definition 21.17: (*Topological Manifold*)

A *manifold* is a space M which is *locally Euclidean,* i.e, given any $x \in X$ there exists an open neighborhood $x \in U \subset M$ such that U is homeomorphic to some open subset of the Euclidean space $\mathbb{R}^n$ for some n. The tuple (U, φ), where $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is the homeomorphism, is called a *coordinate chart*. Usually, we assume that a manifold is *second countable,* i.e, it admits a basis of countable open sets, and it is *Hausdorff.*

One can show (by invariance of domain) that for each connected component (which are same as path components), the integer n appearing in the definition remains constant. Thus, assuming M is connected, we can define $\dim M$ to be this integer, we say M is an n-fold (or n-dimensional manifold).

Example 21.18:

The real line is a manifold which is T_2 and second countable. The *real line with two origins* is a manifold which is second countable, but not T_2. The *long line* (or, *Sorgenfrey line*) is a manifold which is T_2 but not second countable. The sphere S^n, the projective spaces $\mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n$, their products etc are examples of compact manifolds, which are T_2 and second countable.

Remark 21.19: (*Manifold With Boundary*)

A space M is called a *manifold with boundary* if each $x \in M$ admits an open neighborhood which is homeomorphic to an open subset of the closed upper half plane

$$\mathbb{R}^n_{\geq 0} := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$

for some n. By invariance of domain, one can define $x \in M$ to be an *interior point* if x has a neighborhood homeoromphic to an open set in the open upper half plane

$$\mathbb{R}^n_{>0} := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

A point $x \in M$ which is not an interior point is called a *boundary point*. The collection of boundary point is defined to be ∂M. Every manifold M can be considered a manifold with boundary $\partial M = \emptyset$, but not the other way around.

Exercise 21.20:

Suppose M is an n-fold. Let $x \in M$, and $x \in V \subset M$ any open neighborhood. Show that there exists a open neighborhood $x \in U \subset V$, and a homeomorphism $\varphi : U \to \mathbb{R}^n$ such that $\varphi(x) = 0$.

Fix an n-fold M. Given subsets $K \subset L \subset M$, we have the restriction maps

$$r^L_K : H_\bullet(M, M \setminus L) \to H_\bullet(M, M \setminus K)$$

induced by the inclusions. When $K = \{x\}$ a singleton, we denote r^L_x as the restriction.

Lemma 21.21:

Given any open neighborhood $x \in W \subset M$, there exists a neighborhood $x \in U \subset W$ such that $r^U_y : H_\bullet(M, M \setminus U) \to H_\bullet(M, M \setminus y)$ is an isomorphism for all $y \in U$.

Proof : Get a chart (V, φ) such that $x \in V \subset W$ and $\varphi : V \to \mathbb{R}^n$ is a homeomorphism with $\varphi(x) = 0$ (Exercise 21.20). Set $U = \varphi^{\{-1\}} B^n$, where $B^n = \{v \in \mathbb{R}^n \mid \|v\| < 1\}$ is the open unit ball. For any $y \in U$, consider the diagram

$$
\begin{tikzcd}
H_\bullet(M, M \setminus U) \arrow[d, "r_y^U"'] & H_\bullet(V, V \setminus U) \arrow[l] \arrow[r, "\varphi_\star", "\cong"'] \arrow[d] & H_\bullet(\mathbb{R}^n, \mathbb{R}^n \setminus B^n) \arrow[d] \\
H_\bullet(M, M \setminus \{y\}) & H_\bullet(V, V \setminus \{y\}) \arrow[l] \arrow[r, "\cong", "\varphi_\star"'] & H_\bullet(\mathbb{R}^n, \mathbb{R}^n \setminus \{\varphi(y)\})
\end{tikzcd}
$$

The blue arrows are isomorphism as we are excising out $M \setminus V$. The green arrow is induced by a homotopy equivalence, and hence isomorphism. The commutativity then proves that r_y^U is an isomorphism for all $y \in U$. □

Next, fix a ring R, and consider the following set

$$M_R := \bigsqcup_{x \in M} H_n(M, M \setminus \{x\}; R),$$

along with the projection map $\pi : M_R \to M$. Observe that

$$H_n(M, M \setminus \{x\};) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong H_{n-1}(\mathbb{R}^n \setminus \{0\}) \cong R.$$

We assume $H_n(M, M \setminus \{x\}; R)$ has the discrete topology. We topologize M_R in the following way. For each $x \in M$, using Lemma 21.21, get a chart (U, φ) around x such that r_y^U is an isomorphism for each $y \in U$. Consider the map

$$\begin{aligned} \Phi_{x,U} : U \times H_n(M, M \setminus \{x\}; R) &\to \pi^{-1}(U) \\ (y, a) &\mapsto r_y^U (r_x^U)^{-1}(a). \end{aligned}$$

It is easy to see that $\Phi_{x,U}$ is a fiber-preserving bijection. Give M_R the coarsest topology that makes the collection of maps $\Phi_{x,U}$ continuous. Then, it follows that M_R is a covering space of M, and $\pi : M_R \to M$ is a covering map. This covering space is called the *orientation cover* of M with coefficients in R.

Exercise 21.22:

Check that $\Phi_{x,U}$ actually defines continuous *transition maps.* That is, show that the restricted map

$$\Phi_{y,V}^{-1} \circ \Phi_{x,U} : (U \cap V) \times H_n(M, M \setminus \{x\}; R) \to (U \cap V) \times H_n(M, M \setminus \{y\}; R)$$

is continuous.

Observe that for any U as in Lemma 21.21 and for $z \in H_n(M, M \setminus U)$, the map $y \mapsto r_y^U(z)$ is a local section of M_R. Let us denote the collection of all sections of M_R over some $A \subset M$ by $\Gamma(A; R)$. Since each fiber of M_R is an Abelian group, we can define a group structure in $\Gamma(A; R)$. Moreover, we can define the subgroup $\Gamma_c(A; R) \subset \Gamma(A; R)$ of sections with compact support, i.e, sections that are 0 outside a compact subset of A.

Remark 21.23:

One can more generally define $M_{R,k} := \bigsqcup_{x \in M} H_k(M, M \setminus \{x\}; R)$, and topologize it similarly. It follows that each $M_{G,k}$ is a covering space (with 0 as the fiber for $k \neq n$), and $M_{G,n} = M_G$ for an n-fold M.

Definition 21.24: (*R-Orientation*)

Given a ring R, and a subset $A \subset M$, an *R-orientation of M along A* is a section $s \in \Gamma(A;R)$ such that $s(a) \in H_n(M, M \setminus \{a\}; R) \cong R$ is a generator of R. For $A = M$, if such a section exists, we simply say it is an *R-orientation* of M, and M is then said to be *R-orientable*.

Exercise 21.25: (*$\mathbb{Z}_2$-Orientation*)

Justify that any manifold is $\mathbb{Z}_2$-orientable.

Exercise 21.26:

Justify that if M is R-orientable along some $A \subset M$, then it is R-orientable along any $B \subset A$ as well.

When $R = \mathbb{Z}$, we say an R-orientation in $\Gamma(M;\mathbb{Z})$ is an *orientation* of R. If such an orientation exists, we say M is *orientable*.

Remark 21.27: (*Orientation Sheaf*)

Technically, we are constructing the *sheaf* M_R whose stalk over each point $x \in M$ is the group (or ring) $H_n(M, M \setminus \{x\})$. This is also called the *orientaion sheaf* of M with coefficient R. An R-orientation is thus a global section of the sheaf M_R such that it takes values in generators.

Definition 21.28: (*Orientation Double Cover*)

Given a manifold M, the *orientation double cover* is defined as the subset $\widetilde{M} \subset M_{\mathbb{Z}}$, which consists of all the generators in all the fibers. As each fiber is isomorphic to $\mathbb{Z}$, which admits two generators, namely $\{\pm 1\}$, it follows that $\pi : \widetilde{M} \to M$ is a 2-sheeted cover of M.

We have the following useful result.

Proposition 21.29: (*Orientability and Orientation Double Cover*)

Given a manifold M, the following are equivalent.

1. M is orientable.
2. M is orientable along any compact set $K \subset M$.
3. The orientation covering $\widetilde{M}$ is trivial.
4. The covering $M_{\mathbb{Z}} \to M$ is trivial

Proof : $1 \Rightarrow 2$: If M is orientable, then it is orientable along any subset, and in particular, any compact subset.

$2 \Rightarrow 3$: Suppose M is orientable along any compact set. Since triviality can be verified over each connected component, we may assume M is connected. Since $\widetilde{M}$ is a double cover of M, it is trivial if and only if it is *not* connected. So, we have a path $\gamma : [0,1] \to \widetilde{M}$ joining two distinct points in the fiber over some point $x \in M$. Composing with π, we get a compact, connected set $K = \pi(\gamma([0,1])) \subset M$. It follows that $\widetilde{M}|_K$ is nontrivial, as γ is a path joining two points in a fiber. But then M is not orientable along K, a contradiction.

$3 \Rightarrow 4$: Suppose $\sigma : M \to \widetilde{M}$ is a section. Then, we have

$$\begin{aligned} \rho : M \times \mathbb{Z} &\to M_{\mathbb{Z}} \\ (x,k) &\mapsto k\rho(x) \end{aligned}$$

is a trivialization of $\mathbb{Z}$. Indeed, it follows that $M_{\mathbb{Z}} = \widetilde{M} \times_{\mathbb{Z}_2} \mathbb{Z}$ is the Borel construction.

$4 \Rightarrow 1$: If $M_{\mathbb{Z}}$ is trivial, then we can choose a section with values in $\widetilde{M}$, i.e, we can choose an orientaion on M. $\square$

Remark 21.30: (*Orientation Double Cover and Associated Bundle*)

If R is an arbitrary ring, we have the action of $\mathbb{Z}_2$ on R given by multiplication by -1. This leads to the following map

$$\begin{aligned} \widetilde{M} \times R &\to M_{\mathbb{Z}} \otimes R \\ (z,r) &\mapsto z \otimes r. \end{aligned}$$

Applying the UCT (Theorem 17.10) fiberwise, we get $M_{\mathbb{Z}} \otimes R \cong M_R$ (as the Tor vanishes). Hence, we have the induced isomorphism $\widetilde{M} \times_{\mathbb{Z}_2} R \to M_R$. Moreover, sections of $\Gamma(M;R)$ corresponds to $\mathbb{Z}_2$-equivariant maps $\lambda : \widetilde{M} \to R$.

Remark 21.31: (*Stiefel-Whitney Class*)

Since $\widetilde{M}$ is a 2-sheeted cover of M, we have a group homomorphism

$$\pi_1(M,x) \to \widetilde{M}_x = \mathbb{Z}_2.$$

As $\mathbb{Z}_2$ is Abelian, this map induces

$$\pi_1(M,x)^{\mathrm{ab}} \to \mathbb{Z}_2.$$

By the Hurewicz theorem, we then have a map

$$\omega : H_1(M;\mathbb{Z}) = \pi_1(M,x)^{\mathrm{ab}} \to \mathbb{Z}_2.$$

As $\mathbb{Z}_2$ is a field, by the UCT (Theorem 17.11), it follows that $H^1(M;\mathbb{Z}_2) = \hom(H_1(M;\mathbb{Z}),\mathbb{Z}_2)$. Thus, we have a cohomology class $\omega \in H^1(M;\mathbb{Z}_2)$. This class is called the *first Stiefel-Whitney class*, usually denoted as $\omega_1(M)$. From the definition, it follows that M is orientable if and only if $\omega_1(M) = 0$. For disconnected M, we work on each components separately.

Day 22 : 24th April, 2026

`section of orientation sheaf — fundamental class — cap product — duality theorems — Poincaré duality`

22.1 Fundamental Class

Suppose M is a manifold, and $A \subset M$ is a closed subset. We consider homology with coefficients in a fixed Abelian group G (which could as well be a ring). Note that by UCT (Theorem 17.10), we have $M_G = M_{\mathbb{Z}} \otimes G$. For each $\alpha \in H_n(M, M \setminus A; G)$, we have the following map

$$\begin{aligned} \mathscr{J}^A(\alpha) : A &\to A; G \\ x &\mapsto r_x^A(\alpha). \end{aligned}$$

One can see that $\mathscr{J}^A(\alpha)$ is a section of M_G over A. In fact, it has compact support in A. Indeed, choose a cycle $c \in S_n(M; G)$ that represents α. Then, there exists a compact set $K \subset M$ such that image of singular simplices in c is contained in K, i.e, $c \in S_n(K)$. For any $x \in A \setminus K$, we then have the composition

$$\begin{tikzcd}
 & & S_n(M, M \setminus A; G) \arrow[dr] & \\
S_n(K; G) \arrow[r] \arrow[urr, bend left] \arrow[rr, bend right, "0"'] & S_n(M; G) \arrow[r] \arrow[ur] & S_n(M, K; G) \arrow[r] & S_n(M, M \setminus \{x\}; G)
\end{tikzcd}$$

which maps c to 0. Passing to homology, this shows hat(t) $r_x^A(\alpha) = 0$. Thus, $\mathscr{J}^A(\alpha)$ has support in $K \cap A$, which is a compact set as A is closed. Thus, $\mathscr{J}^A(\alpha) \in \Gamma_c(A; G)$ We have the following.

Theorem 22.1: (*Sections of Orientation Cover with Compact Support*)

Let M be a Hausdorff, second countable n-dimensional manifold, and $A \subset M$ be closed. Fix a group G (or an R-module) as coefficents for the homology. Then, the following holds

1. $H_i(M, M \setminus A) = 0$ for $i > n$.
2. $\mathscr{J}^A : H_n(M, M \setminus A) \to \Gamma_c(A; G)$ is an isomorphism.

Proof : The proof technique is like an *induction* over the complexity over all possible closed sets of M. For any closed set $A \subset M$, let us denote the statements

- $\mathscr{Z}(A) : H_i(M, M \setminus A) = 0$ for $i > n$.
- $\mathscr{I}(A) : \mathscr{J}^A : H_n(M, M \setminus A) \to \Gamma_c(A; G)$ is an isomorphism.

The proof now breaks down in to the following steps.

Step 1. We claim that if $\mathscr{Z}(A), \mathscr{Z}(B)$ and $\mathscr{Z}(A \cap B)$ holds, then so does $\mathscr{Z}(A \cup B)$. Similarly, if $\mathscr{I}(A), \mathscr{I}(B)$ and $\mathscr{I}(A \cap B)$ holds, then so does $\mathscr{I}(A \cup B)$.

We use the relative Mayer-Vietoris sequence (Theorem 5.6) for the *proper* triad $(M \setminus (A \cap B); M \setminus A, M \setminus B)$ (Check!). We have the diagram

$$\begin{tikzcd}
H_{n+1}(M, M \setminus (A \cap B)) \arrow[r, "\cong"] \arrow[d] & 0 \arrow[d] \\
H_n(M, M \setminus (A \cup B)) \arrow[r, "\mathcal{J}^{A \cup B}"] \arrow[d] & \Gamma_c(A \cup B; G) \arrow[d] \\
H_n(M, M \setminus A) \oplus H_n(M, M \setminus B) \arrow[r, "\mathcal{J}^A \oplus \mathcal{J}^B", "\cong"'] \arrow[d] & \Gamma_c(A; G) \oplus \Gamma_c(B; G) \arrow[d] \\
H_n(M, M \setminus (A \cap B)) \arrow[r, "\mathcal{J}^{A \cap B}", "\cong"'] & \Gamma_c(A \cap B; G)
\end{tikzcd}$$

The left hand column is part of the relative Mayer-Vietoris sequence. The right hand side is also a long exact sequence induced by the (usual) Mayer-Vietoris sequence applied to each stalk (i.e, each fiber). Note that we can define the sheaves $M_{G,k} = \sqcup_{x \in M} H_k(M, M \setminus \{x\}; G)$ for any k, and then, $\Gamma_c(M_{G,n+1}) = 0$ as each stalk is 0. The commutativity follows from the naturality of the Mayer-Vietoris sequence. Then, by the 5-lemma, we have $\mathcal{J}^{A \cup B}$ is an isomorphism, proving $\mathcal{J}(A \cup)$

For any $i > n$, we have the relative Mayer-Vietoris sequence

$$\cdots \to \underset{0}{H_{i+1}(M, M \setminus (A \cap B))} \to H_i(M, M \setminus (A \cup B)) \to \underset{0}{H_i(M, M \setminus A)} \oplus \underset{0}{H_i(M, M \setminus B)} \to \cdots$$

which proves that $H_i(M, M \setminus (A \cup B)) = 0$. This proves $\mathcal{Z}(A \cup B)$.

Step 2. Call a closed set *good* if $A \subset U$ for some chart $\varphi : U \to \mathbb{R}^n$ such that $\varphi(A) \subset \mathbb{R}^n$ is convex. We claim that $\mathcal{Z}(A)$ and $\mathcal{J}(A)$ holds for any good compact sets. Note, M is Hausdorff and thus, compact sets are closed.

Choose a chart $\varphi : U \to \mathbb{R}^n$ such that $\varphi(A)$ is convex. Fix some $x \in A$, and assume that $\varphi(x) = 0$, and $0 \in \varphi(A) \subset \mathring{D}^n$, which can be done by some affine transform and then radially shrinking suitably. We consider the following diagram

$$\begin{tikzcd}
H_i(D^n, S^{n-1}) \arrow[r] \arrow[d, equal] & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi(A)) \arrow[d] & H_i(U, U \setminus A) \arrow[l, "\varphi_\star"', "\cong"] \arrow[r] \arrow[d] & H_i(M, M \setminus A) \arrow[d, "r_x^A"] \\
H_i(D^n, S^{n-1}) \arrow[r] & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & H_i(U, U \setminus \{x\}) \arrow[l, "\varphi_\star"', "\cong"] \arrow[r] & H_i(M, M \setminus \{x\})
\end{tikzcd}$$

The commutativity is clear from the space level maps. The green arrows are homotopy equivalences, note that the standard radial homotopy pusshes the *convex* set $\varphi(A) \subset \mathring{D}^n$ to the boundary S^{n-1}. The blue arrows are excision isomorphisms, obtained from excising $M \setminus U$. Thus, it follows that r_x^A is an isomorphism.

Now, for $i > n$, we have $H_i(M, M \setminus \{x\}) = H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = H_{i-1}(\mathbb{R}^n \setminus \{0\}) = 0$, which shows $H_i(M, M \setminus A) = 0$, proving $\mathcal{Z}(A)$.

Next, note that a section of a covering space of a connected set (in this case A) is determined by the value at a single point. Thus, the isomorphis r_x^A shows that $\mathcal{J}^A$ is a bijection. This justifies $\mathcal{J}(A)$.

Step 3. Next, we show that both $\mathcal{Z}(A)$ and $\mathcal{J}(A)$ holds when $A \subset U$, where $\varphi : U \to \mathbb{R}^n$ is a chart, and $A = \cup_{i=1}^k K_i$ where K_i is a good compact set with respect to the chart map φ.

Recall that the intersection of compact sets of $\mathbb{R}^n$ is again compact. By Step 2, we have $\mathcal{Z}(K_i)$ holds for each i. Inductively assume that $\mathcal{Z}(K_1 \cup ... \cup K_{n-1})$ holds. Since $(K_1 \cup ... \cup K_{n-1}) \cap K_n$ is again a good compact set, we have $\mathcal{Z}((K_1 \cup ... \cup K_{n-1}) \cap K_n)$ holds. But then by Step 1, we have $\mathcal{Z}(K_1 \cup ... \cup K_n) = \mathcal{Z}(A)$ holds. Similar argument shows that $\mathcal{J}(A)$ holds as well.

Step 4. Next, we claim that both $\mathcal{Z}(A)$ and $\mathcal{J}(A)$ holds when $A \subset U$, where $\varphi : U \to \mathbb{R}^n$ is a chart, and A is compact.

Firstly, fix any open neighborhood $A \subset W \subset U$. Cover the image of $\varphi(A)$ by the interior of finitely many balls, which are small enough to be contained in $\varphi(W)$. Then, the inverse images of the *closed* balls are good compact sets, and their union, say, V is of the same type as Step 3. Moreover, V is actually a neighborhood of A (i.e, A is contained in the interior of A) $A \subset V \subset W$, where W is arbitrary. In other words, we have arbitrarily small compact neighborhoods V of A.

The collection of all such $A \subset V \subset U$ forms a directed system. Passing to the colimit, we have the diagram

$$\begin{tikzcd}
\operatorname{colim}_V H_i(M, M \setminus V) \arrow[r, "\Phi"] \arrow[d, "\operatorname{colim}_V \mathscr{J}^V"'] & H_i(M, M \setminus A) \arrow[d, "\mathscr{J}^A"'] \\
\operatorname{colim}_V \Gamma_c(M_{G,i}|_V) \arrow[r, "\Psi"'] & \Gamma_c(M_{G,i}|_A)
\end{tikzcd}$$

The horizontal maps Φ and Ψ are induced by the restriction to K, in particular, the Φ is induced by the collection r_K^V. Commutativity follows from the universal property of the colimit.

- We check that Φ is induced from an isomorphism at the chain level. Indeed, any chain has a compact support (i.e, it has a compact image). Thus, if there is a chain that is completely contained in $M \setminus K$, then there is a sufficiently small compact neighborhood $A \subset V$ such that the chain is contained in $M \setminus V$. Then, any such chain lifts to the colimit $\operatorname{colim}_V S_\bullet(M, M \setminus V)$, proving the surjectivity. Injectivity, is easy to verify : an element in the colimit is represented by a chain in $S_\bullet(M, M \setminus V)$ for some $A \subset V$, if this chain becomes 0 in $S_\bullet(M, M \setminus A)$, then it must have been 0 to begin with. Now, a general fact is that homology commutes with colimit, and hence we get Φ is an isomorphism.
- The fact that Ψ is an isomorphism is a general fact about sheaf of sections of covering spaces. Essentially, surjectivity follows from the fact that any section on the compact set A can be extended to an arbitrarily small neighborhood of A. For injectivity, one can show that if two sections defined on some neighborhoods V_1, V_2 of A matches on A, then they in fact matches on some smaller neighborhood $A \subset V \subset V_1 \cap V_2$.

For any $i > n$, we have $\operatorname{colim} \ H_i(M, M \setminus V) = 0$, as $\mathcal{Z}(V)$ holds. As Φ is an isomorphism, we have $H_i(M, M \setminus A) = 0$, proving $\mathcal{Z}(A)$. Finally, let us take $i = n$, which immediately gives that $\operatorname{colim}\mathscr{J}^V$ is an isomorphism by $\mathcal{J}(V)$. Then, $\mathscr{J}^A$ is isomorphism, proving $\mathcal{J}(A)$.

Step 5. We now claim that $\mathcal{Z}(A)$ and $\mathcal{J}(A)$ holds for *arbitrary* compact sets $A \subset M$.

Indeed, we can cover A by finitely many compact sets, each of which is contained some coordinate chat. Then, $\mathcal{Z}(A)$ and $\mathcal{I}(A)$ follows inductively from step 1.

Step 6. Next, suppose $A = \cup_{i=1}^{\infty} K_i$, where K_i is compact, and for each $i \neq j$ there are disjoint open neighborhoods of K_i and K_j. Then, we show that $\mathcal{Z}(A)$ and $\mathcal{I}(A)$ holds.

It follows from the additivity of the homology theory, that $\mathcal{J}^A = \sum \mathcal{K}^{K_i}$, and similarly $H_i(M, M \setminus A) = \oplus H_i(M, M \setminus K_i)$. Since $\mathcal{Z}(K_i)$ and $\mathcal{I}(K_i)$ holds, we have $\mathcal{J}^A$ is an isomorphism and $H_i(M, M \setminus A) = 0$ for $i > n$. This shows $\mathcal{Z}(A)$ and $\mathcal{I}(A)$.

Step 7. Finally, let A be an arbitrary closed set of M. Since M is locally compact, T_2, and second countable, we have compact sets K_i such that $M = \cup_{i=1}^{\infty} K_i$ and $K_i \subset \mathring{K}_{i+1}$. Set $A_i = A \cap \left(K_i \setminus \mathring{K}_{i-1}\right)$, with $K_0 = \emptyset$. Let

$$B = \bigcup_{i \text{ is even}} A_i, \quad C = \bigcup_{i \text{ is odd}} A_i.$$

By step 6, $\mathcal{Z}, \mathcal{I}$ holds for B, C and $B \cap C$. By step 1, we conclude that $\mathcal{Z}(A)$ and $\mathcal{I}(A)$ holds as well.

This finishes the proof. □

As a consequence of Theorem 22.1, we get the following.

Theorem 22.2: (*Orientation along Closed, Connected Set*)

Suppose $A \subset M$ is a closed, connected subset of an n-fold M. Let R be a ring, and G be an R-module.

1. Suppos A is noncompact. then $H_n(M, M \setminus A; G) = 0$.
2. Suppose A is compact.
   a. If M is R-orientable along A, then $H_n(M, M \setminus A; G) = G$. Moreover, $r_x^A : H_n(M, M \setminus A; G) \to H_n(M, M \setminus \{x\}; G)$ is an isomorphism for each $x \in M$.
   b. If $R = \mathbb{Z}$, and M is not orientable along A, then
   $$H_n(M, M \setminus A; G) = {}_2G := \{g \in G \mid 2g = 0\}.$$

Proof : Suppose A is noncompact. Since A is connected, any secion in $s \in \Gamma(A; G)$ is completely determined by its value at a fixed point $x \in A$. If $s(x) \neq 0$, then $s \neq 0$ on A. In particular, it support is A, which is noncompact. In other words, there are no nonzero section with compact support on A, i.e, $\Gamma_c(A; G) = 0$. By Theorem 22.1, we have $H_n(M, M \setminus A; G) = 0$.

Next, suppose A is compact. Then, any section on A has compact support, i.e, $H_n(M, M \setminus A; G) = \Gamma_c(A; G) = \Gamma(A; G)$. Recall, a section is determined by its value at a point. We have a diagram

$$\begin{tikzcd}
H_n(M, M \setminus A; G) \arrow[r, "\mathcal{J}^A", "\cong"'] \arrow[d, "r_x^A"'] & \Gamma(A; G) \arrow[d, "b"] \\
H_n(M, M \setminus \{x\}; G) \arrow[r, "\cong", "\mathcal{J}^x"'] & \Gamma(\{x\}; G)
\end{tikzcd}$$

Here, b is the restriction map. Since A is connected, any section is determined by its value at the point x, and in particualar, b is injective.

- Suppose M is R-orientable along A. Then, it is G-orientable along A, in particular, there exists $s \in \Gamma(A; G)$ such that $b(s) = s(x)$ is a generator of $H_n(M, M \setminus x; G) = G$. Thus, b is surjective, and hence, an isomorphism. This implies, r_x^A is an isomorphism.
- Recall, $M_G = \widetilde{M} \times_{\mathbb{Z}_2} G$, where $\widetilde{M} \subset M_{\mathbb{Z}}$ is the orientation double cover, and $\mathbb{Z}_2$ acts on G by multiplication by -1. Thus, a section of $\Gamma(A; G)$ corresponds to a map $\lambda : \widetilde{M} \to G$ such that $\lambda \circ t = -\lambda$, where t is the generator of $\mathbb{Z}_2$. Since M is not orientable along A, it follows that $\widetilde{M}|_A$ is connected, and thus, λ must be constant. Thus, image of b is precisely ${}_2G$. The commutativity shows that $H_n(M, M \setminus A; G) = {}_2G$.

This concludes the proof. □

For a compact, connected n-fold M, let us restate Theorem 22.2.

Theorem 22.3: (*Orientaion and Compact, Connected Manifold*)

Suppose M is a compact, connected n-fold. Then, precisely one of the following holds true.

1. M is orientable, $H_n(M) = \mathbb{Z}$, and for each $x \in M$ the map $r_x^A : H_n(M) \to H_n(M, M \setminus \{x\})$ is an isomorphism.
2. M is non-orietable and $H_n(M) = 0$.

Proof : Since M is compact and connected, we can take $A = M$ in Theorem 22.2. Note that $H_n(M) = H_n(M, \emptyset) = H_n(M, M \setminus M)$. When M is not orientable, we get $H_n(M) = {}_2\mathbb{Z} = \{n \in \mathbb{Z} \mid 2n = 0\} = 0$. This concludes the proof. □

In particular, over integers, orientaility (resp. nonorientability) is equivalent to $H_n(M) = \mathbb{Z}$ (resp. 0). A consequence of Theorem 22.3 is the following important definition.

Definition 22.4: (*Fundamental Class*)

Suppose M is a compact, connected n-fold, which R-orientable for some ring R. Then, a *fundamental class* (or orientation class) for M is a generator $[M]_R$ of $H_n(M; R) = R$.

Note that for each $x \in M$ the map $r_x^A : H_n(M; R) \to H_n(M, M \setminus \{x\}; R)$ maps a fundamental class $[M]$ to a generator of $H_n(M, M \setminus \{x\}; R) = R$. When $R = \mathbb{Z}$, we simply denote a fundamental class as $[M]$.

Let us also point out another important consequence of Theorem 22.2.

Proposition 22.5: (*Torsion in $H_{n-1}(M)$*)

Suppose M is an n-fold, and $A \subset M$ is a closed connected subset. Denote $\mathcal{T} \subset H_{n-1}(M, M \setminus A; \mathbb{Z})$ as the torsion subgroup, i.e,

$$\mathcal{T} = \{a \in H_{n-1}(M, M \setminus A) \mid na = 0 \ \text{ for some } \ n \in \mathbb{Z}\}.$$

Then, we have the following dichotomy.

- $\mathcal{T} = \mathbb{Z}_2$ if A is compact and M is nonorientable along A.
- $\mathcal{T} = 0$ otherwise.

In particular, if M is a compact, connected, orientable n-fold, then $H_{n-1}(M)$ is a free Abelian group.

Proof : Let $q \in \mathbb{N}$ be a positive integer. If A is noncompact, then from the relative version of the UCT (Theorem 17.10), we have the short exact sequence

$$0 \to \underbrace{H_n(M, M \setminus A; \mathbb{Z}) \otimes \mathbb{Z}_q}_{0} \to \underbrace{H_n\big(M, M \setminus A; \mathbb{Z}_q\big)}_{0} \to \operatorname{Tor}\big(H_{n-1}(M, M \setminus A; \mathbb{Z}), \mathbb{Z}_q\big) \to 0.$$

That the groups are 0 follows from Theorem 22.2. But then the Tor term vanishes. In particular, the group $H_{n-1}(M, M \setminus A; \mathbb{Z})$ has no q-torsion. As q is arbitrary, we get $\mathcal{T} = 0$ for A noncompact. If M is orientable along a *compact* set A, again applying Theorem 22.2 to the short exact sequence we have

$$0 \to \underbrace{H_n(M, M \setminus A; \mathbb{Z}) \otimes \mathbb{Z}_q}_{\mathbb{Z}} \to \underbrace{H_n\big(M, M \setminus A; \mathbb{Z}_q\big)}_{\mathbb{Z}_q} \to \operatorname{Tor}\big(H_{n-1}(M, M \setminus A; \mathbb{Z}), \mathbb{Z}_q\big) \to 0.$$

That is,

$$0 \to \mathbb{Z}_q \to \mathbb{Z}_q \to \operatorname{Tor}\big(H_{n-1}(M, M \setminus A; \mathbb{Z}), \mathbb{Z}_q\big) - 0.$$

As the sequence splits, it follows that the Tor term vanishes. As q is arbitrary, we again get $\mathcal{T} = 0$.

Now, suppose A is compact, and M is nonorientable along A. For q odd, we have ${}_2\mathbb{Z}_q = \big\{g \in \mathbb{Z}_q \mid 2q = 0\big\} = 0$, as 2 is invertible in $\mathbb{Z}_q$. Hence, by Theorem 22.2, we have $H_n\big(M, M \setminus A; \mathbb{Z}_q\big) = 0$ for q odd. Thus, from the UCT sequence, we have $\mathcal{T}$ has no *odd* torsion. Moreover, with $G = \mathbb{Z}_4$, we have ${}_2\mathbb{Z}_4 = \{x \in \mathbb{Z}_4 \mid 2x = 0\} = \mathbb{Z}_2$, and hence the sequence

$$0 \to \underbrace{H_n(M, M \setminus A; \mathbb{Z}) \otimes \mathbb{Z}_4}_{0} \to \underbrace{H_n(M, M \setminus A; \mathbb{Z}_4)}_{\mathbb{Z}_2} \to \operatorname{Tor}(H_{n-1}(M, M \setminus A; \mathbb{Z}), \mathbb{Z}_4) \to 0.$$

This shows that

$$\mathbb{Z}_2 = \operatorname{Tor}(H_{n-1}(M, M \setminus A; \mathbb{Z}), \mathbb{Z}_4) = \operatorname{Tor}(\mathcal{T}, \mathbb{Z}_4) = \{x \in \mathcal{T} \mid 4x = 0\}.$$

Since $\mathcal{T}$ has no odd torsion, it follows that any element has finite order, which is a power of 2. But ${}_4\mathcal{T}$ implies that there can only exists a unique element in $\mathcal{T}$, which then must have order 2. Thus, $\mathcal{T} = \mathbb{Z}_2$, as claimed. $\square$

22.2 Cap Product

Modifying the cochain level cup product, we get the *cap product*. Given a ring R, define

$$\frown : S^k(X; R) \otimes S_{k+l}(X; R) \to S_l(X; R)$$

by the following formula. For k-cochain $\varphi : S_k(X) \to R$, and a singular simplex $\sigma : \Delta^{k+l} \to X$, define

$$\varphi \frown \sigma := \varphi\big(\sigma|_{[v_0,\dots,v_k]}\big)\sigma|_{[v_k,\dots,v_{k+l}]},$$

and then extend R-linearly. In fact, we define $\frown: S^k(X;R) \otimes S_n(X;R) \to S_{n-k}(X;R)$ by setting it 0 whenever $n < k$.

Proposition 22.6: (*Properties of Cap Product*)

Let $\varphi \in S^p(X;R), \psi \in S^q(X;R)$ be cochains, and $\sigma : \Delta^n \to X$ be a simplex. The cohain level cap product satisfies the following.

1. $\partial(\varphi \frown \sigma) = (-1)^k(\varphi \frown \partial\sigma - \delta\varphi \frown \sigma)$.
2. $(\varphi \smile \psi) \frown \sigma = \varphi \frown (\psi \frown \sigma)$.
3. $(\varphi \smile \psi)(\sigma) = \psi(\varphi \frown \sigma)$, for $n = p + q$.
4. $1 \frown \sigma = \sigma$, where $1 : S_0(X) \to R$ is the constant map 1_R.
5. For any $f : Y \to X$, and simplex $\tau : \Delta^n \to Y$, we have

$$f_\#(f^\#\varphi \frown \tau) = \varphi \frown f_\#\tau,$$

where $f^\# : S^\bullet(X) \to S^\bullet(Y)$ and $f_\# : S_\bullet(Y) \to S_\bullet(X)$ are the respective induced map.

Consequently, cap product induces the cohomological cap product

$$\frown: H^k(X;R) \otimes H_n(X;R) \to H_{n-k}(X;R)$$

which makes $H_\star(X;R)$ into a graded $H^\star(X;R)$-module.

Proof : The proves are via explicit computation.

1. Let us assume $n \geq k$. In fact, for $n = k$, the LHS and RHS are zero. So we can take $n \geq k+1$. We compute

$$\begin{aligned}
\varphi \frown \partial\sigma &= \varphi \frown \left(\sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots v_n]}\right) \\
&= \sum_{i=0}^{n} (-1)^i \varphi \frown \sigma|_{[v_0,\dots,\hat{v}_i,\dots v_n]} \\
&= \sum_{i=0}^{k} (-1)^i \varphi\big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]}\big)\sigma|_{[v_{k+1},\dots,v_n]} + \sum_{i=k+1}^{n} (-1)^i \varphi\big(\sigma|_{[v_0,\dots,v_k]}\big)\sigma|_{[v_{k+1},\dots,\hat{v}_i\dots,v_n]} \\
&= \varphi\left(\sum_{i=0}^{k+1} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]}\right)\sigma|_{[v_{k+1},\dots,v_n]} - (-1)^{k+1}\varphi\big(\sigma|_{[v_0,\dots,v_k]}\big)\sigma|_{[v_{k+1},\dots,v_n]} \\
&\qquad + (-1)^k \varphi\big(\sigma|_{[v_0,\dots,v_k]}\big)\left(\sum_{i=k+1}^{n} (-1)^{i-k}\sigma|_{[v_{k+1},\dots,\hat{v}_i\dots,v_n]}\right) \\
&= \varphi\big(\partial\sigma|_{[v_0,\dots,v_{k+1}]}\big)\sigma|_{[v_{k+1},\dots,v_n]} \\
&= \delta\varphi\big(\sigma|_{[v_0,\dots,v_{k+1}]}\big)\sigma|_{[v_{k+1},\dots,v_n]} + (-1)^k\varphi\big(\sigma|_{[v_0,\dots,v_k]}\big)\left(\sum_{i=0}^{n} (-1)^i\sigma|_{[v_k,\dots,\hat{v}_{k+i},\dots,v_n]}\right) \\
&= \delta\varphi \frown \sigma + (-1)^k\varphi\big(\sigma|_{[v_0,\dots,v_k]}\big)\big(\partial\sigma|_{[v_k,\dots,v_n]}\big)
\end{aligned}$$

$$
\begin{aligned}
&= \delta\varphi \frown \sigma + (-1)^k \partial\Big(\varphi\big(\sigma|_{[v_0,\dots,v_k]}\big)\sigma|_{[v_k,\dots,v_n]}\Big) \\
&= \delta\varphi \frown \sigma + (-1)^k \partial(\varphi \frown \sigma)
\end{aligned}
$$

Thus, we have, $\partial(\varphi \frown \sigma) = (-1)^k(\varphi \frown \partial\sigma - \delta\varphi \frown \sigma)$

2. Again, let us assume $n \geq p+q$. We compute

$$
\begin{aligned}
(\varphi \smile \psi) \frown \sigma &= (\varphi \smile \psi)\big(\sigma|_{[v_0,\dots,v_{p+q}]}\big)\sigma|_{[v_{p+q},\dots,v_n]} \\
&= \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big)\psi\big(\sigma|_{[v_p,\dots,v_{p+q}]}\big)\sigma|_{[v_{p+q},\dots,v_n]} \\
&= \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big)\big(\psi \frown \sigma|_{[v_p,\dots,v_n]}\big) \\
&= (\varphi \frown (\psi \frown \sigma)).
\end{aligned}
$$

3. We compute

$$
\begin{aligned}
(\varphi \smile \psi)(\sigma) &= \varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \psi\big(\sigma|_{[v_p,\dots,v_n]}\big) \\
&= \psi\Big(\varphi\big(\sigma|_{[v_0,\dots,v_p]}\big) \cdot \sigma|_{[v_p,\dots,v_n]}\Big) \\
&= \psi(\varphi \frown \sigma)
\end{aligned}
$$

1. We have

$$
1 \frown \sigma = 1\big(\sigma|_{[v_0]}\big)\sigma|_{[v_0,\dots,v_n]} = 1_R \cdot \sigma = \sigma.
$$

2. For any $f : Y \to X$, we have

$$
\begin{aligned}
f_\#(f^\#\varphi \frown \sigma) &= f_\#\Big(f^\#\varphi\big(\sigma|_{[v_0,\dots,v_p]}\big)\sigma|_{[v_p,\dots,v_n]}\Big) \\
&= \varphi\big(f \circ \sigma|_{[v_0,\dots,v_p]}\big)\big(f \circ \sigma|_{[v_p,\dots,v_n]}\big) \\
&= \varphi \frown f_\#\sigma.
\end{aligned}
$$

This concludes the proof. ☐

Exercise 22.7:

Define the cochain level cap product as

$$
\varphi \frown \sigma = (-1)^{pq}\varphi\big(\sigma|_{[v_0,\dots,v_p]}\big)\sigma|_{\left[v_{[v_p,\dots,v_{p+q}]}\right]}, \quad \varphi \in S^p(X;R), \sigma \in S_{p+q}(X;R).
$$

Verify that $\partial(\varphi \frown \sigma) = \delta\varphi \frown \sigma + (-1)^{|\varphi|}\varphi \frown \partial\sigma$.

The cap product is actually the adjoint of the cup product. To see this, note that we have the *Kronecker pairing*

$$
\langle , \rangle H^k(X;R) \otimes H_k(X;R) \to R
$$

induced by the map

$$
\langle , \rangle : S^k(X;R) \otimes S_k(X;R) \to R
$$

$$\varphi \otimes \sigma \mapsto \varphi(\sigma).$$

By definition, $\delta\varphi(\sigma) = \varphi(\delta\sigma)$, which means $\langle \delta\varphi, \sigma\rangle = \langle \varphi, \partial\sigma\rangle$, which justifies that the (co)chain level pairing gives rise to the Kronecker pairing (see also Remark 17.4). Then, from Proposition 22.6, we have the following.

Proposition 22.8: (*Cap Product is Adjoint of Cup Product*)

For any $\alpha \in H^p(X;R), \beta \in H^q(X;R), z \in H_{p+q}(X;R)$, we have

$$\langle \alpha \smile \beta, z\rangle = \langle \beta, \alpha \frown z\rangle.$$

That is, $\alpha \smile \_$ is adjoint to $\alpha \frown \_$ with respect to the Kronecker pairing.

Let us also define the notion of *relative cap product*. Given a pair (X, A) with $\iota : A \hookrightarrow X$ the inclusion, we have following diagram

$$\begin{tikzcd}
0 \arrow[r] & S^k(X)\otimes S_n(A) \arrow[r, "\mathrm{Id}\otimes\iota_\#"] \arrow[d, "\iota^\#\otimes\mathrm{Id}"'] & S^k(X)\otimes S_n(X) \arrow[r] \arrow[dd, "\frown"'] & S^k(X)\otimes S_n(X,A) \arrow[r] \arrow[dd, dashed, "\frown"] & 0 \\
& S^k(A)\otimes S_n(A) \arrow[d, "\frown"] & & & \\
0 \arrow[r] & S_{n-k}(A) \arrow[r, "\iota_\#"'] & S_{n-k}(X) \arrow[r] & S_{n-k}(X,A) \arrow[r] & 0
\end{tikzcd}$$

Since we are tensoring a *split* exact sequence $0 \to S_\bullet(A) \to S_\bullet(X) \to S_\bullet(X,A) \to 0$ by $S^k(X)$, it follows that both the rows are exact. The commutativity is a consequence of Proposition 22.6. Hence, we have the induced relative cap product

$$\frown : S^k(X)\otimes S_n(X,A) \to S_{n-k}(X,A).$$

One can verify that these pass to cohomology, giving the relative cap product

$$\frown : H^k(X;R)\otimes H_n(X,A;R) \to H_{n-k}(X,A;R),$$

which makes $H_\star(X,A;R)$ into a graded module over $H^\star(X;R)$.

On the other hand, we have the following situation. Given a ring R, we can define

$$\frown : S^k(X,A;R)\otimes S_n(X,A) \to S_{n-k}(X)$$

as follows. For relative k-cochain $\varphi : S_k(X,A) \to R$ and an n-simplex $\sigma : \Delta^n \to X$ not contained in A, we have

$$\varphi \frown \sigma = \varphi\Big(\sigma|_{[v_0,\dots,v_k]} + S_k(A)\Big)\cdot \sigma|_{[v_k,\dots,v_n]}.$$

One can check that this gives the relative cap product

$$\frown : H^k(X,A;R)\otimes H_n(X,A;R) \to H_{n-k}(X;R).$$

Exercise 22.9:

Verify that given a pair of maps $f : (X, A) \to (Y, B)$, we have $f_\star(f^\star\alpha \frown z) = \alpha \frown f_\star(z)$ for $\alpha \in H^\star(X, A), z \in H_\star(X, A)$. Also, check that $\langle \alpha \smile \beta, z\rangle = \langle \alpha, \beta \frown z\rangle$ for $\alpha \in H^p(X; R), \beta \in H^q(X; R), z \in H_{p+q}(X; R)$.

More generally, for an excisve pair (A, B), i.e, for $S_\bullet(A) + S_\bullet(B) \hookrightarrow S_\bullet(A \cup B)$ is a chain homotopy equivalence, we have a relative cap product

$$\frown: H^k(X, A; R) \otimes H_n(X, A \cup B; R) \to H_{n-k}(X, B; R).$$

22.3 Cohomology with Compact Support

Given a space X, the collection of all compact subsets of X forms a directed system via inclusion. Given compact sets $K \subset L \subset X$, we have a map $r^K_L : H^\bullet(X, X \setminus K) \to H^\bullet(X, X \setminus L)$ given by the inclusions. Passing to the colimit (or direct limit), we define the *cohomology with compact support*

$$H^\bullet_c(X) := \underset{\substack{K \subset X \\ \text{compact}}}{\operatorname{colim}} H^\bullet(X, X \setminus K).$$

If X is compact, it follows that $H^\bullet_c(X) = H^\bullet(X)$.

Exercise 22.10:

Denote $S^n_c(X)$ to be the module generated by those n-cochains φ for which there exists a compact $K \subset X$ such that φ vanisheh for any singular simplex *not* completely contained in K. Verify that $S^\bullet_c(X)$ is a cochain complex with the usual boundary map, and $H^\bullet(S^\bullet_c(X)) = H^\bullet_c(X)$.

Let us point out that in general, a map $f : X \to Y$ does not induce a map in the cohomology with compact support. Indeed, for any compact set $K \subset X$, we have $f(K) \subset Y$ is compact, but we don't necessarily get $f(X \setminus K) \subset Y \setminus f(K)$.

Exercise 22.11: ($H^\bullet_c$ *is a Functor for Proper Maps*)

Recall, a map $f : X \to Y$ is called *proper* if $f^{-1}(L) \subset X$ is compact for any $L \subset Y$ compact. Verify that given a proper map $f : X \to Y$, there exists a well-defined map $H^\bullet_c(f) : H^\bullet_c(Y) \to H^\bullet_c(X)$. In particular, $H^\bullet_c$ is a contravariant functor when restricted to the class of proper maps.

Hint : Given a proper map, for any compact set $L \subset Y$, we have the composition

$$f^{\star,L} : H^\bullet(Y, Y \setminus L) \to H^\bullet(X, X \setminus f^{-1}(L)) \to \operatorname{colim} H^\bullet(X, X \setminus K).$$

Verify that for $L_1 \subset L_2 \subset Y$, we have $f^{\star,L_2} \circ r^{L_1}_{L_2} = f^{\star,L_1}$, so that we can pass to colimit.

Let us note that for an *open* subset $U \subset X$, where X is T_2, we have a canonical map $H^\bullet_c(U) \to H^\bullet_c(X)$. Indeed, for any compact set $K \subset U$, note that $(U, U \setminus K) \hookrightarrow (X, X \setminus K)$ induces an excistion isomorphism $H^\bullet(X, X \setminus K) \to H^\bullet(U, U \setminus K)$. Naturality of excision isomorphism shows that the maps

$$H^\bullet(U, U \setminus K) \to H^\bullet(X, X \setminus K) \to H^\bullet_c(X)$$

are compatible with restrictions. Hence, passing to the colimit we get a map $H_c^\bullet(U) \to H_c^\bullet(X)$.

Example 22.12: (*Compactly Supported Cohomology of* $\mathbb{R}^n$)

Take $U = \mathbb{R}^n \subset S^n$ as the complement of a point $x \in S^n$. For any $K \subset U$ compact, there exists a contractible compact neighborhood $x \in L \subset S^n$ such that $K \cap L = \emptyset$. Indeed, we can take L to be a small closed disc centered at x and disjoint from K. Since L is contractible, we have $H^\bullet(S^n, S^n \setminus L) \to \tilde{H}^\bullet(S^n)$ is an isomorphism. Hence, passing to colimit, we get an isomorphism $H_c^\bullet(U) = \tilde{H}^\bullet(S^n)$. In particular,

$$H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & k = n \\ 0, & \text{otherwise.} \end{cases}$$

The above example shows that the compactly supported singular cohomology may not coincide with the usual one if the space is noncompact.

22.4 Poincaré Duality

Suppose M is an R-orientable n-fold. Fix a section $\zeta \in \Gamma(M_R)$ representing the orientation. For any compact set $K \subset M$, we have the restriction $\zeta|_K \in \Gamma(M_R|_K) = \Gamma_c(M_R|_K)$. By Theorem 22.1, we have the isomorphism $\mathcal{J}^K : H_n(M, M \setminus K; R) \to \Gamma_c(M_R|_K)$. Denote $\zeta_K \in H^n(M, M \setminus K; R)$ as the element whic maps to $\zeta|_K$ under $\mathcal{J}^K$. Recall that $\mathcal{J}^K(\zeta_K) = r_x^K(\zeta|_K) \in H_n(M, M \setminus \{x\}; R)$ for any $x \in K$. Using the *relative* cap product, we have a map

$$\begin{aligned} \_ \frown \zeta_K : H^q(M, M \setminus K; R) &\to H_{n-q}(M; R) \\ \varphi &\mapsto \varphi \frown \zeta_K. \end{aligned}$$

Note that here $(M \setminus K, \emptyset)$ is obviously excisive. For $K \subset L \subset M$ compact subsets, we have the inclusion $\iota : (M, M \setminus L) \hookrightarrow (M, M \setminus K)$. Note that $r_L^K = \iota^\star$, and $\zeta_K = \iota_\star \zeta_L$. Then, from Exercise 22.9, we have

$$r_L^K \varphi \frown \zeta_L = \mathrm{Id}_\star(i^\star \varphi \frown \zeta_L) = \varphi \frown i_\star \zeta_L = \varphi \frown \zeta_K.$$

Thus, passing to colimit, we get a map

$$\mathcal{P} : H_c^q(M; R) \to H_{n-q}(M; R).$$

This map $\mathcal{P}$ is known as the *Poincaré duality* map. In particular, if M is compact, we have

$$\mathcal{P} : H^q(M; R) \xrightarrow{\frown [M]_R} H_{n-q}(M; R),$$

where $[M]_R$ is the fundamental class of M.

Theorem 22.13: (*Poincaré Duality*)

Suppose M is an R-orientable n-fold for some ring R, with a chosen orientation $\zeta \in H_n(M; R)$. Then, the Poincaré duality map $\mathcal{P} : H_c^q(M; R) \to H_{n-q}(M; R)$ associated to ζ is an isomorphism.

Let us point out an important consequence.

Proposition 22.14: (*Non-degenerate Pairing*)

Suppose M is an R-orientable compact, connected n-fold for some PID R. Assume that $H_\bullet(M;R)$ is finitely generated. Then, there exists a non-degenerate pairing

$$\langle , \rangle : \frac{H^p(M;R)}{\text{Torsion}} \otimes \frac{H^{n-p}(M;R)}{\text{Torsion}} \to R$$

induced by the cup product. If R is a field, the pairing is $H^p(M;R) \otimes H^{n-p}(M;R) \to R$

Proof : Fix an R-orientaion $\zeta \in H_n(M;R)$. Then, using the Kronecker pairing, we have a pairing

$$\langle , \rangle : H^p(M;R) \otimes H^{n-q}(M;R) \to R$$
$$\alpha \otimes \beta \mapsto \langle \alpha \smile \beta, \zeta \rangle$$

Say $\alpha \in H^p(M;R)$ is a torsion element, i.e, $r\alpha = 0$ for some $0 \neq r \in R$. Then, for any $\beta \in H^{n-p}(M;R)$ we have

$$r(\alpha \smile \beta) = (r\alpha) \smile \beta = 0.$$

Since R is an integral domain, we must have $\alpha \smile \beta = 0$. This shows that the above pairing descends to a pairing modulo the torsion elements.

Let us now prove the nondegeneracy. Given any finitely generated R-module N, we have $\mathrm{Ext}_{R(N,R)}$ is torsion, i.e, every element is annihilated by some nonzero element of R. This follows from the structure theorem of finitely generated modules over a PID R and basic properties of Ext_R. From the UCT (Theorem 17.11), we have the split exact sequence

$$0 \to \mathrm{Ext}(H_{k-1}(M;R), R) \to H^k(M;R) \to \hom_R(H_k(M;R), R) \to 0,$$

where the $\hom$ is free. Thus, $\mathrm{Ext}(H_{k-1}(M;R), R)$ is precisely the torsion submodule of $H^k(M;R)$, and the free part is represented by the dual of $H_k(M;R)$. In particular, for any $[\alpha] \in \frac{H^p(M;R)}{\text{Torsion}}$, there is a unique $a \in H_k(M;R)$ such that $\alpha(a) = 1$. By Poincaré duality, there exists some $\beta \in H^{n-p}(M;R)$ such that $\beta \frown \zeta = a$. Now, we have

$$\langle \alpha, \beta \rangle = (-1)^{|\alpha||\beta|}\langle \beta \smile \alpha, \zeta \rangle = (-1)^{|\alpha||\beta|}\langle \alpha, \beta \frown \zeta \rangle = (-1)^{|\alpha||\beta|}\langle \alpha, a \rangle = (-1)^{|\alpha||\beta|} \neq 0.$$

Thus, the pairing is nondegenerate in the first variable. By the graded commutativity of the cup product, it then follows that the pairing is nondegenerate in both variables.

If R is a field, then the torsion submodule is 0, and so is the $\mathrm{Ext}(H_\bullet(M;R), R)$. In particular, $H^\bullet(M;R) = \hom(H_\bullet(M;R), R)$ holds. Then, the nondegeneracy follows. □

The existence of this nondegenerate pairing leads to quick computations of some cohomology rings. As an example, let us give a proof of Theorem 21.16.

Proposition 22.15: (*Cohomology Rings of Projective Spaces*)

We have the cohomology rings

1. $H^\star(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[X]/(X^{n+1})$ with $|X| = 2$.
2. $H^\star(\mathbb{HP}^n;\mathbb{Z}) = \mathbb{Z}[X]/(X^{n+1})$ with $|X| = 4$.
3. $H^\star(\mathbb{OP}^n;\mathbb{Z}) = \mathbb{Z}[X]/(X^{n+1})$ with $|X| = 8$, and $n = 0, 1, 2$.
4. $H^\star(\mathbb{RP}^n;\mathbb{Z}) = \mathbb{Z}_2[X]/(X^{n+1})$ with $|X| = 1$.

Proof : Using cellular (co)homology and UCT, we already have the isomorphism of graded modules in each of these cases.

Let us consider $\mathbb{CP}^n$. It is orientable; fix the fundamental class $\zeta_n \in H_{2n}(\mathbb{CP}^n;\mathbb{Z})$. Inductively, assume that $H^\star(\mathbb{CP}^{n-1};\mathbb{Z}) = \mathbb{Z}[X]/(X^n)$ with $|X| = 2$. By cellular homology, we have $\iota : \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ induces isomorphism in homology with degree $< 2n$ (Example 12.5), Then, by UCT (Theorem 17.11), we have $i^\star$ is isomorphism in cohomology in degree $< 2n$.

Suppose $\alpha \in H^2(\mathbb{CP}^n;\mathbb{Z})$ is a generator. Then $\alpha' = \iota^\star\alpha \in H^2(\mathbb{CP}^{n-1};\mathbb{Z})$ is a generator. By induction, $(\alpha')^{n-1}$ is a generator. On the other hand, $(\alpha')^{n-1} = \iota^\star(\alpha^n)$. Since $\iota^\star$ is an isomorphism in degree $< 2n$, we must have α^n is a generator of $H^{2n-2}(\mathbb{CP}^n;\mathbb{Z})$.

By Theorem 11.14, we have some $\beta \in H^{2n-2}(\mathbb{CP}^n;\mathbb{Z})$ such that $\langle \alpha, \beta\rangle = 1$. We write $\beta = m\alpha^n$ for some integer m. Then,

$$1 = \langle \alpha, \beta\rangle = \langle \alpha, m\alpha^{n-1}\rangle = m\langle \alpha, \alpha^{n-1}\rangle,$$

which forces $m = \pm 1$. Consequently, we have

$$1 = \pm\langle \alpha, \alpha^{n-1}\rangle = \pm\langle \alpha \smile \alpha^{n-1}, \zeta_n\rangle = \pm\alpha^n(\zeta_n).$$

Consequently, α^n must be a generator of $H^{2n}(\mathbb{CP}^n;\mathbb{Z})$. This proves the claim for $\mathbb{CP}^n$.

Similar argument works for $\mathbb{HP}^n$, and for $\mathbb{OP}^n$ (for $n = 0, 1, 2$). For $\mathbb{RP}^n$, we work over the field $\mathbb{Z}_2$, where we again have the nondegenrate pairing since any manifold is $\mathbb{Z}_2$-orientable. □