

Algebraic Topology II (KSM4E02)

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homotopy invariance of singular homology – barycentric subdivision

8.1 Homotopy Invariance of Singular Homology

Our goal is to show that homotopic maps induce the same homomorphism of singular homology groups. Given a space X , consider the maps

$$\begin{aligned}\xi^X : X &\rightarrow X \times [0, 1] & \eta^X : X &\rightarrow X \times [0, 1] \\ x &\mapsto (x, 0). & x &\mapsto (x, 1).\end{aligned}$$

They are clearly natural in X , and hence, induce natural chain maps

$$\xi_\bullet^X, \eta_\bullet^X : S_\bullet(X) \rightarrow S_\bullet(X \times [0, 1]).$$

In particular, $\xi_\bullet, \eta_\bullet : S_\bullet(_) \Rightarrow S_\bullet(_ \times [0, 1])$ are natural transformations.

Proposition 8.1:

The chain maps $\xi_\bullet, \eta_\bullet$ are naturally chain homotopic.

Proof : We need to construct maps $s_n^X : S_n(X) \rightarrow S_{n+1}(X \times [0, 1])$, such that it is a chain homotopy:

$$\eta_n^X - \xi_n^X = \partial s_n^X + s_{n-1}^X \partial.$$

Moreover, we need it to be natural: for any $f : X \rightarrow Y$, we require

$$S_{n+1}(f \times \text{Id}) \circ s_n^X = s_n^Y \circ S_n(f).$$

We construct s_n^X inductively.

- For a 0-simplex $\sigma : \Delta^0 \rightarrow X$, let us define $s_0^X(\sigma) : \Delta^1 = \Delta^0 \times [0, 1] \rightarrow X \times [0, 1]$ by

$$s_0^X(\sigma)(t) = (\sigma(0), t).$$

It follows that

$$\partial(s_0^X(\sigma)) = s_0^X(\sigma)d_0 - s_0^X(\sigma)d_1 = \eta_0^X \sigma - \xi_0^X \sigma.$$

Naturality is apparent from the definition.

- Inductively assume that we have constructed s_k^X for $k < n$, for all space X . We construct s_n^X . The identity map $\text{Id} : \Delta^n \rightarrow \Delta^n$ is a singular n -simplex, denote $\iota_n \in S_n(\Delta^n)$ to be the corresponding element. In order to define s_n^X , we require $s_n^{\Delta^n}$ to satisfy

$$\partial s_n^{\Delta^n}(\iota_n) = \eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial \iota_n),$$

where the right-hand-side is defined by induction. We have the following diagram

$$\begin{array}{ccccc} S_n(\Delta^n) & \xrightarrow{\partial} & S_{n-1}(\Delta^n) & \xrightarrow{\partial} & S_{n-2}(\Delta^n) \\ \eta_n^{\Delta^n} \downarrow \xi_n^{\Delta^n} & & \eta_{n-1}^{\Delta^n} \downarrow \xi_{n-1}^{\Delta^n} & & \\ S_n(\Delta^n \times I) & \xrightarrow{\partial} & S_{n-1}(\Delta^n \times I) & & \end{array}$$

(Blue arrows in the original image indicate chain homotopies \$s_{n-1}^{\Delta^n}\$ and \$s_{n-2}^{\Delta^n}\$ from \$S_n(\Delta^n)\$ to \$S_{n-1}(\Delta^n \times I)\$ and from \$S_{n-1}(\Delta^n)\$ to \$S_{n-2}(\Delta^n \times I)\$ respectively.)

Observe that

$$\begin{aligned} & \partial(\eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial \iota_n)) \\ &= \eta_{n-1}^{\Delta^n}(\partial \iota_n) - \xi_{n-1}^{\Delta^n}(\partial \iota_n) - \partial(s_{n-1}^{\Delta^n}(\partial \iota_n)), \quad \text{since } \xi, \eta \text{ are chain maps} \\ &= \eta_{n-1}^{\Delta^n}(\partial \iota_n) - \xi_{n-1}^{\Delta^n}(\partial \iota_n) - (\eta_{n-1}^{\Delta^n}(\partial \iota_n) - \xi_{n-1}^{\Delta^n}(\partial \iota_n) - s_{n-2}^X(\partial \partial \iota_n)), \\ & \quad \text{as } s_{n-1}^{\Delta^n}, s_{n-2}^{\Delta^n} \text{ are part of the chain homotopy} \\ &= 0, \quad \text{as } \partial \partial \iota_n = 0. \end{aligned}$$

In other words, the RHS is an n -cycle. Since $\Delta^n \times [0, 1]$ is contractible, by [Theorem 7.11](#), the RHS is a boundary. In particular, we can choose some singular $(n+1)$ -chain $a \in S_{n+1}(\Delta^n \times I)$ such that

$$\partial a = \eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial \iota_n).$$

Set, $s_n^{\Delta^n}(\iota_n) = a$. Then, for any space X and any singular n -simplex $\sigma : \Delta^n \rightarrow X$, set

$$s_n^X(\sigma) = S_{n+1}(\sigma \times \text{Id})(a).$$

Let us verify the required conditions. We have the diagram

We compute

$$\begin{aligned} \partial s_n^X(\sigma) &= \partial S_{n+1}(\sigma \times \text{Id})(a) = S_n(\sigma \times \text{Id})(\partial a) \\ &= S_n(\sigma \times \text{Id})(\eta_n^{\Delta^n}(\iota_n) - \xi_n^{\Delta^n}(\iota_n) - s_{n-1}^{\Delta^n}(\partial \iota_n)) \\ &= \eta_n^X S_n(\sigma)(\iota_n) - \xi_n^X S_n(\sigma)(\iota_n) - s_{n-1}^X S_{n-1}(\sigma)(\partial \iota_n), \\ & \quad \text{as } \xi^X, \eta^X \text{ are natural transformations, and } s_{n-1} \text{ is natural} \\ &= \eta_n^X S_n(\sigma)(\iota_n) - \xi_n^X S_n(\sigma)(\iota_n) - s_{n-1}^X \partial S_n(\sigma)(\iota_n) \end{aligned}$$

$$= \eta_n^X \sigma - \xi_n^X \sigma - s_{n-1}^X \partial \sigma.$$

Thus, s_n^X satisfies the chain homotopy condition. Next, consider a continuous map $f : X \rightarrow Y$. We have the diagram

$$\begin{array}{ccccc}
 & \sigma & & f \circ \sigma & \\
 & \swarrow s_n^X & S_n(X) \xrightarrow{S_n(f)} & S_n(Y) & \searrow s_n^Y \\
 S_{n+1}(X \times I) & \xrightarrow{S_n(f \times \text{Id})} & S_{n+1}(Y \times I) & & \\
 \uparrow S_{n+1}(\sigma \times \text{Id}) & & \uparrow S_{n+1}((f \circ \sigma) \times \text{Id}) & & \\
 & S_{n+1}(\Delta^n \times I) & & & \\
 & a & & &
 \end{array}$$

We compute

$$\begin{aligned}
 S_{n+1}(f \times \text{Id})s_n^X(\sigma) &= S_{n+1}(f \times \text{Id})S_{n+1}(\sigma \times \text{Id})(a) \\
 &= S_{n+1}((f \circ \sigma) \times \text{Id})(a) \\
 &= s_n^Y(f \circ \sigma) \\
 &= s_n^Y S_n(f)(\sigma).
 \end{aligned}$$

This proves naturality of s_n

Hence, inductively, we have a natural chain homotopy $s_\bullet : \xi_\bullet \simeq \eta_\bullet$. □

Note that the chain homotopy obtained in [Proposition 8.1](#) is not unique since we made a choice at each induction step, but the homotopy is still natural.

Theorem 8.2: (Singular Homology is Homotopy Invariant)

Suppose $f, g : (X, A) \rightarrow (Y, B)$ are homotopic maps. Then, $H_n(f) = H_n(g) : H_n(X, A) \rightarrow H_n(Y, B)$ for all n .

Proof : Suppose $h : (X, A) \times I \rightarrow (Y, B)$ is a homotopy $f \simeq g$. We have the commutative diagram

$$\begin{array}{ccccc}
 A & \xrightarrow[\xi^A]{\eta^A} & A \times [0, 1] & \xrightarrow{h|_{A \times [0, 1]}} & B \\
 \downarrow \iota & & \downarrow \iota \times \text{Id} & & \downarrow \\
 X & \xrightarrow[\xi^X]{\eta^X} & X \times [0, 1] & \xrightarrow{h} & Y
 \end{array}$$

where, $\xi^X(x) = (x, 0)$, $\eta^X(x) = (x, 1)$, $\xi^A(a) = (a, 0)$, $\eta^A(a) = (a, 1)$. By [Proposition 8.1](#), we have a natural chain homotopy $s_\bullet^X : S_\bullet(\eta^X) \simeq S_\bullet(\xi^X)$ and $s_\bullet^A : S_\bullet(\eta^A) \simeq S_\bullet(\xi^A)$. Naturality implies

$$S_{n+1}(\iota \times \text{Id}) \circ s_n^A = s_n^X \circ S_n(\iota).$$

Hence, we can define $s_n^{X,A} : S_n(X, A) \rightarrow S_{n+1}(X \times [0, 1], A \times [0, 1])$, which is clearly a chain homotopy $S_\bullet(\eta^{X,A}) \simeq S_\bullet(\xi^{X,A})$. Then,

$$\begin{aligned} S_n(g) - S_n(f) &= S_n(h \circ \eta^{X,A}) - S_n(h \circ \xi^{X,A}) = S_n(h)(S_n(\eta^{X,A}) - S_n(\xi^{X,A})) \\ &= S_n(h) \circ (\partial s_n^{X,A} + s_{n-1}^{X,A} \partial) = \partial(S_{n+1}(h)s_n^{X,A}) + (S_n(h)s_{n-1}^{X,A})\partial. \end{aligned}$$

Thus, $\zeta_n = S_{n+1}(h)s_n^{X,A} : S_n(X) \rightarrow S_{n+1}(Y)$ is a chain homotopy $S_\bullet(g) \simeq S_\bullet(f)$. Hence, by [Proposition 7.8](#), we have $H_n(f) = H_n(g)$. $\square$

8.2 Barycentric Subdivision

Our next goal is to show that singular homology satisfy the excision axiom. For that we first need to construct *barycentric subdivision* of simplices.

Let $D \subset \mathbb{R}^n$ be a convex set. Given points $v_0, \dots, v_p \in D$, the *affine singular p -simplex* is defined as

$$\begin{aligned} \sigma &= s[v_0, \dots, v_p] : \Delta^p \rightarrow D \\ \sum_{i=0}^p \lambda_i e_i &\mapsto \sum_{i=0}^p \lambda_i v_i, \end{aligned}$$

where e_i are the standard unit vectors of $\mathbb{R}^{p+1}$. Note that convexity of D implies that σ is well-defined. The *barycenter* of σ is defined as

$$\sigma^\beta := \frac{1}{p+1} \sum_{i=0}^p v_i.$$

In particular, we shall be interested in the identity map $\iota_p : \Delta^p \rightarrow \Delta^p$ and the corresponding barycenter ι_p^β .

Now, D being convex, is contractible. In fact, for each $v \in D$, we have a contracting homotopy

$$\begin{aligned} H_v : D \times [0, 1] &\rightarrow D \\ (u, t) &\mapsto (1-t)u + tv. \end{aligned}$$

Using the cone construction from earlier, we then get a chain homotopy $S_\bullet(D) \rightarrow S_{\bullet+1}(D)$ ([Proposition 7.10](#)). Let us denote this chain homotopy as

$$\begin{aligned} S_p(D) &\rightarrow S_{p+1}(D) \\ \sigma &\mapsto v \cdot \sigma. \end{aligned}$$

In particular, the affine simplex $\sigma = [v_0, \dots, v_p]$ is mapped to $v \cdot \sigma := [v, v_0, \dots, v_p]$. Recall from [Proposition 7.10](#) that

$$\partial(v \cdot \sigma) = \begin{cases} \sigma - v \cdot \partial\sigma, & p > 0, \\ \sigma - \varepsilon(\sigma)v, & p = 0, \end{cases}$$

where $\varepsilon : S_0(D) \rightarrow \mathbb{Z}$ is given by $\varepsilon(\sum n_\sigma \sigma) = \sum n_\sigma$.

Let us now inductively define an operator $\mathcal{B}_p = \mathcal{B}_p(X) : S_p(X) \rightarrow S_p(X)$ for any space X , called the *barycentric subdivision*. For any $\sigma : \Delta^p \rightarrow X$ define,

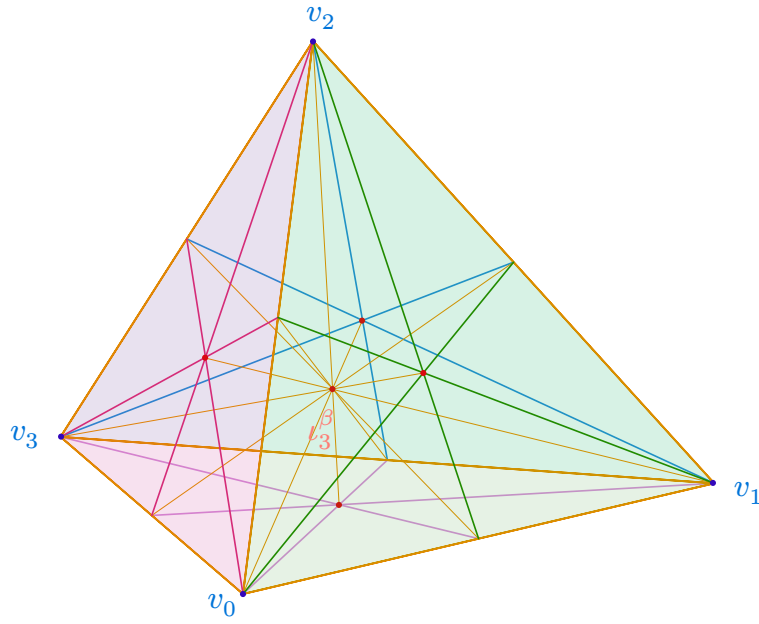
$$\mathcal{B}_p^X(\sigma) = S_p(\sigma)\mathcal{B}_p^{\Delta^p}(\iota_p),$$

where

$$\mathcal{B}_p^{\Delta^p}(\iota_p) = \begin{cases} i_0, & p = 0 \\ \iota_p^\beta \cdot \mathcal{B}_{p-1}^{\Delta^p}(\partial \iota_p), & p > 0. \end{cases}$$

Clearly this is well-defined.

Let us visualize the barycentric subdivision of ι_3 , the identity simplex of Δ^3 .



Barycentric sub-division of Δ^3

On each fact, we have the corresponding barycenter. For the 1-simplex $[0, 1]$, the barycenter is the just midpoint. The sub-division operator is taking the *cone* over the simplices of the barycentric sub-division of the lower-dimensional face. You can see that there are 24 (check!) new Δ^3 after subdividing it once.