

Algebraic Topology II (KSM4E02)

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Day 7 : 8th February, 2026

long exact sequence from short exact sequence of chain complexes – relative singular homology – long exact sequence of singular homology – chain homotopy invariance – singular homology of contractible space

7.1 Long Exact Sequence in Homology

So far we have been working with only Abelian groups, which are nothing but $\mathbb{Z}$ -modules. More generally, we can fix a commutative ring R , and work with R -modules. Generalizing even further, we can consider any arbitrary *Abelian category* $\mathcal{A}$, where 0-maps and taking quotients makes sense (see [Definition 6.27](#)). Then, we have the chain complex of objects from $\mathcal{A}$, denoted as $\text{Ch}(\mathcal{A})$. The definition of n^{th} -homology group makes sense for $\text{Ch}(\mathcal{A})$ as well. Homological algebra is the study of the (co)homology of (co)chain complex over some Abelian category.

Theorem 7.1: (Long Exact Sequence of Homology of Chain Complex)

Given a short exact sequence $0 \rightarrow A_{\bullet} \xrightarrow{\iota_{\bullet}} B_{\bullet} \xrightarrow{j_{\bullet}} C_{\bullet} \rightarrow 0$ of chain complexes, there exists a long exact sequence of homology groups :

$$\cdots \rightarrow H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{\iota_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \cdots,$$

which is natural with respect to maps of short exact sequences of chain complexes.

Proof : We consider part of the diagram

$$\begin{array}{ccccccc}
 & \ker \partial_n^A & & \ker \partial_n^B & & \ker \partial_n^C & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & A_n & \xrightarrow{\iota_n} & B_n & \xrightarrow{j_n} & C_n \longrightarrow 0 \\
 & & \downarrow \partial_n^A & & \downarrow \partial_n^B & & \downarrow \partial_n^C \\
 0 & \longrightarrow & A_{n-1} & \xrightarrow{\iota_{n-1}} & B_{n-1} & \xrightarrow{j_{n-1}} & C_{n-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{coker } \partial_n^A & & \text{coker } \partial_n^B & & \text{coker } \partial_n^C
 \end{array}$$

Since both the rows are exact, applying the snake lemma ([Lemma 6.29](#)), we have the exact sequence

$$0 \rightarrow \ker \partial_n^A \rightarrow \ker \partial_n^B \rightarrow \ker \partial_n^C \rightarrow \text{coker } \partial_n^A \rightarrow \text{coker } \partial_n^B \rightarrow \text{coker } \partial_n^C \rightarrow 0.$$

This holds for each n . In particular, we consider the following diagram

$$\begin{array}{ccccccc}
\text{coker } \partial_{n+1}^A & \longrightarrow & \text{coker } \partial_{n+1}^B & \longrightarrow & \text{coker } \partial_{n+1}^C & \longrightarrow & 0 \\
\delta_n^A \downarrow & & \delta_n^B \downarrow & & \delta_n^C \downarrow & & \\
0 & \longrightarrow & \ker \partial_{n-1}^A & \longrightarrow & \ker \partial_{n-1}^B & \longrightarrow & \ker \partial_{n-1}^C
\end{array}$$

By above, the rows are exact. The maps $\delta_n^A, \delta_n^B, \delta_n^C$ are induced from $\partial_n^A, \partial_n^B, \partial_n^C$ respectively. Thus, the diagram is commutative. Observe that $\ker \delta_n^A = H_n(A)$ and $\text{coker } \delta_n^A = H_{n-1}(A)$, and similarly for δ_n^B, δ_n^C . Indeed, we have the diagram which might be of help

$$\begin{array}{ccccccc}
A_{n+1} & \xrightarrow{\partial_{n+1}^A} & A_n & \xrightarrow{\partial_n^A} & A_{n-1} & \xrightarrow{\partial_{n-1}^A} & A_{n-2} \\
& & \downarrow & \nearrow \tilde{\partial}_n^A & \uparrow & & \\
H_n(A) & \hookrightarrow & \text{coker } \partial_{n+1}^A & \xrightarrow{\delta_n^A} & \ker \partial_{n-1}^A & \twoheadrightarrow & H_{n-1}(A)
\end{array}$$

Hence, again applying the snake lemma, we have an exact sequence

$$H_n(A) \rightarrow H_n(B) \rightarrow H_n(C) \xrightarrow{\partial} H_{n-1}(A) \rightarrow H_{n-1}(B) \rightarrow H_{n-1}(C).$$

Chasing the diagram carefully, shows that the connecting maps are induced by $\iota_\bullet$ and $j_\bullet$. Pasting these exact sequences together, we get the long exact sequence in homology. Since snake lemma produces *natural* exact sequences, again a diagram chase shows that the long exact sequence is natural. $\square$

Remark 7.2: (The Boundary Map)

Let us describe the boundary map explicitly. Consider part of the diagram

$$\begin{array}{ccccc}
& & B_n & \xrightarrow{j_n} & C_n \\
& & \downarrow \partial_n^B & \searrow y & \downarrow x \\
A_{n-1} & \xrightarrow{\iota_n} & B_{n-1} & \xrightarrow{\partial_n^B} & 0 \\
& & \downarrow z & \searrow \partial_n^B(y) & \downarrow 0
\end{array}$$

Suppose $\alpha \in H_n(C)$ is represented as $\alpha = [x]$ for some $x \in C_n$, with $\partial(x) = 0$. Then, it follows that $\partial(\alpha) = [z]$. The well-definedness is a consequence of the snake lemma. Thus, we can 'define' $\partial([x]) = [\iota_n^{-1} \partial_n^B j_n^{-1}(x)]$.

7.2 Relative Singular Homology and Long Exact Sequence

Let us now extend our definition of singular homology to pairs of spaces (X, A) with $A \subset X$. Given such a pair (X, A) , observe that any singular n -simplex $\sigma : \Delta^n \rightarrow A$ is naturally an n -simplex in X via the inclusion map $\iota : A \hookrightarrow X$. Thus, we have an *injective* map $S_\bullet(\iota) : S_\bullet(A) \hookrightarrow S_\bullet(X)$. Let us define the cokernel

$$S_n(X, A) := \frac{S_n(X)}{S_n(A)}.$$

Observe that $S_n(X, A)$ is an Abelian group freely generated by singular n -simplexes $\sigma : \Delta^n \rightarrow X$ such that $\text{im}(\sigma)$ is not *completely* contained in A . We have the well-defined *relative* boundary map

$$\begin{aligned}\partial_n^{X,A} : S_n(X, A) &\longrightarrow S_{n-1}(X, A) \\ \left[\sum a_\sigma \sigma \right] &\longmapsto \sum a_\sigma [\partial_n^X(\sigma)].\end{aligned}$$

In particular, if we have an n -chain in X whose boundary is completely contained in A , then its relative boundary is zero. It follows that $S_\bullet(X, A)$ is a chain complex, which is called the *relative singular chain complex*.

Exercise 7.3: (*Relative Singular Chain Complex is Functorial*)

Verify that $S_\bullet : \text{TopPair} \rightarrow \text{Ch}$ is a functor, and we can identify $S_n(X) = S_n(X, \emptyset)$ (which justifies the same notation). Moreover, $j : X = (X, \emptyset) \rightarrow (X, A)$ induces the quotient map $S_\bullet(X) \rightarrow S_\bullet(X, A)$.

The homology groups

$$H_n(X, A) := H_n(S_n(X, A))$$

are called the *relative singular homology groups* of the pair (X, A) . By [Exercise 6.7](#) and [Exercise 7.3](#), it follows that the relative singular homology groups are functors as well. In particular, given $f : (X, A) \rightarrow (Y, B)$, we have $H_n(f) = H_n(S_\bullet(f))$.

We have a short exact sequence of chain complexes

$$0 \rightarrow S_\bullet(A) \xrightarrow{\iota_\bullet} S_\bullet(X) \xrightarrow{j_\bullet} S_\bullet(X, A) \rightarrow 0,$$

where $\iota : A \hookrightarrow X$ and $j : X \hookrightarrow (X, A)$ are the space level maps.

Theorem 7.4: (*Long Exact Sequence of Singular Homology*)

There exists a natural long exact sequence

$$\cdots \rightarrow H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \cdots \rightarrow H_0(A) \xrightarrow{\iota_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \rightarrow 0.$$

Proof : The proof is immediate from [Theorem 7.1](#). □

[Theorem 7.4](#) justifies that singular homology satisfies the exactness axiom.

7.3 Chain Homotopy Invariance

Let us recall the notion of *chain maps* $f_\bullet : C_\bullet \rightarrow D_\bullet$ ([Definition 1.13](#)), which is given by a collection of maps $f_n : C_n \rightarrow D_n$ such that $f_{n-1} \circ \partial_n^C = \partial_n^D \circ f_n$ holds.

Given a chain complex $C_\bullet$, the *shift (or translation)* of $C_\bullet$ by degree k is the chain complex $C[k]_\bullet$ defined as $C[k]_n = C_{n+k}$ and $\partial_n^{C[k]} = (-1)^k \partial_{n+k}^C$. We shall see later why the sign $(-1)^k$ appears in the definition. A *degree k chain map* $C_\bullet \rightarrow D_\bullet$ is a chain map $C_\bullet \rightarrow D[k]_\bullet$. In particular, a chain map $C_\bullet \rightarrow D_\bullet$ has degree 0.

Exercise 7.5: (*Shift Functor*)

Let $\Sigma^k : \text{Ch} \rightarrow \text{Ch}$ be the functor that shifts a chain complex by degree k . Check that $\Sigma^k \circ \Sigma^l = \Sigma^{k+l}$ and $\Sigma^0 = \text{Id}$. In particular, Σ^k is an *isomorphism* with inverse Σ^{-k} .

Definition 7.6: (Chain Map and Chain Homotopy)

Given two chain maps $f_\bullet, g_\bullet : C_\bullet \rightarrow D_\bullet$, a **chain homotopy** between them is a degree 1 map $h_\bullet : C_\bullet \rightarrow D_{\bullet+1}$ such that $f - g = \partial \circ h + h \circ \partial$ holds. That is, we have the following diagram

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}^C} & C_n & \xrightarrow{\partial_n^C} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\
 & & \downarrow g_{n+1} & & \downarrow g_n & & \downarrow g_{n-1} & & \\
 \cdots & \longrightarrow & D_{n+1} & \xrightarrow{\partial_{n+1}^D} & D_n & \xrightarrow{\partial_n^D} & D_{n-1} & \longrightarrow & \cdots
 \end{array}$$

h_n (blue arrow from C_{n+1} to D_n)
 h_{n-1} (blue arrow from C_n to D_{n-1})

where, we have $f_n - g_n = h_{n-1} \circ \partial_n^C + \partial_{n+1}^D \circ h_n$

Exercise 7.7: (Chain Homotopy is an Equivalence Relation)

Check that the chain homotopy is an equivalence relation on the collection of all chain maps $C_\bullet \rightarrow D_\bullet$.

Proposition 7.8: (Homology of Chain Homotopic Maps)

Suppose $f_\bullet, g_\bullet : C_\bullet \rightarrow D_\bullet$ are homotopic chain maps. Then, they induce the same map between homology groups.

Proof : Suppose $f - g = \partial h + h \partial$ for a chain homotopy $h : C_\bullet \rightarrow D_{\bullet+1}$. Since homology is an additive functor (Exercise 6.7), we only need to show that $H_n(\partial h + h \partial) = 0$. For a class $[\alpha] \in H_n(C)$, we have $\partial(\alpha) = 0$. Recall, $[\alpha] = \alpha + \text{im } \partial$. Then, it follows

$$H_n(\partial h + h \partial)[\alpha] = [(\partial h + h \partial)(\alpha)] = [\partial h(\alpha)] = 0.$$

Thus, $H_n(f) = H_n(g)$ follows for all n . □

7.4 Singular Homology of Contractible Space

Before proving the homotopy invariance for singular homology, let us start with a contractible space, say, X . Fix a point $x_0 \in X$. Let us define a chain map $\varepsilon_\bullet : S_\bullet(X) \rightarrow S_\bullet(X)$ by setting $\varepsilon_n = 0$ for $n \neq 0$, and

$$\varepsilon_0\left(\sum a_\sigma \sigma\right) = \left(\sum a_\sigma\right) \sigma_0,$$

where $\sigma_0 : \Delta^0 \rightarrow X$ is the 0-simplex that maps to x_0 .

Proposition 7.9:

$\varepsilon_\bullet : S_\bullet(X) \rightarrow S_\bullet(X)$ is a chain map.

Proof : The only nontrivial part is the commutativity of the diagram

$$\begin{array}{ccc}
S_1(X) & \xrightarrow{d_1} & S_0(X) \\
\varepsilon_1 = 0 \downarrow & & \downarrow \varepsilon_0 \\
S_1(X) & \xrightarrow{d_1} & S_0(X)
\end{array}$$

Suppose $\sigma : \Delta^1 \rightarrow X$ is a singular 1-simplex. Then,

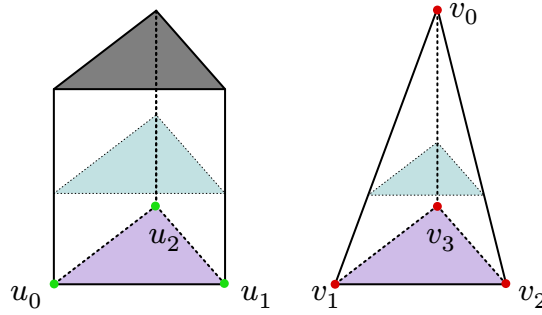
$$\varepsilon_0 d_1(\sigma) = \varepsilon_0(\sigma \circ d_0 - \sigma \circ d_1) = (1 - 1)\sigma_0 = 0.$$

Thus, $\varepsilon_0 d_1 = 0 = d_1 0$. This proves that $\varepsilon_\bullet$ is a chain map. $\square$

Now, suppose $h : X \times [0, 1] \rightarrow X$ is a homotopy from Id_X to the constant map $c_{x_0} : X \rightarrow X$ which maps everything to a point $x_0 \in X$. Using h , we define a chain homotopy between $\varepsilon_\bullet$ and the identity chain map. Firstly, consider the map

$$\begin{aligned}
q_n : \Delta^n \times [0, 1] &\longrightarrow \Delta^{n+1} \\
((t_0, \dots, t_n), s) &\longmapsto (s, (1-s)t_0, \dots, (1-s)t_n)
\end{aligned}$$

Since $s + \sum_{i=0}^n (1-s)t_i = s + (1-s) = 1$, the map is well-defined, and clearly, it is continuous. As q_n is a surjective map between compact T^2 spaces, we have q_n is a quotient map.



The map $q_2 : \Delta^2 \times [0, 1] \rightarrow \Delta^3$

In fact, it follows that q_n identifies $\Delta^n \times \{1\}$ to the vertex $v_0 = (1, 0, \dots, 0) \in \Delta^{n+1}$, and keeps every other point as it is. Note that q_0 is the identity map $[0, 1] \rightarrow [0, 1]$. Now, given a singular n -simplex $\sigma : \Delta^n \rightarrow X$, consider the diagram

$$\begin{array}{ccccc}
\Delta^n \times [0, 1] & \xrightarrow{\sigma \times \text{Id}} & X \times [0, 1] & \xrightarrow{h} & X \\
\downarrow q_n & & & & \uparrow \\
\Delta^{n+1} & \xrightarrow{s_n(\sigma)} & & &
\end{array}$$

Since $h(X \times \{1\}) = \{x_0\}$, it follows from the universal property of a quotient map that there exists a unique continuous map $s_n(\sigma) : \Delta^{n+1} \rightarrow X$ such that

$$s_n(\sigma) \circ q_n = h \circ (\sigma \times \text{Id}_{[0,1]}).$$

Extending linearly, we have a map $s_n : S_n(X) \rightarrow S_{n+1}(X)$.

Now, observe that for $0 < i \leq n+1$ we have the diagram

$$\begin{array}{ccccccc}
\Delta^{n-1} \times [0, 1] & \xrightarrow{d_{i-1}^n \times \text{Id}} & \Delta^n \times [0, 1] & \xrightarrow{\sigma \times \text{Id}} & X \times [0, 1] & \xrightarrow{h} & X \\
q_{n-1} \downarrow & & \downarrow q_n & & & & \uparrow \\
\Delta^n & \xrightarrow{d_i^{n+1}} & \Delta^{n+1} & \xrightarrow{s_n(\sigma)} & & & \\
& & & \nearrow & & & \\
& & & s_{n-1}(\sigma \circ d_{i-1}^n) & & &
\end{array}$$

Hence, $s_n(\sigma)d_i^{n+1} = s_{n-1}(\sigma d_{i-1}^n)$ holds for $0 < i \leq n+1$. Clearly, $s_n(\sigma)d_0^{n+1} = \sigma$.

Proposition 7.10:

$s_\bullet : S_\bullet(X) \rightarrow S_{\bullet+1}(X)$ is a chain homotopy between Id and $\varepsilon_\bullet$.

Proof : For a 0-simplex $\sigma : \Delta^0 \rightarrow X$, we have

$$\partial(s\sigma) = (s\sigma)d_0 - (s\sigma)d_1 = \sigma - \sigma_0,$$

since $h(\sigma(0), 1) = x_0$. For a singular n -simplex $\sigma : \Delta^n \rightarrow X$ with $n \geq 1$, we compute

$$\begin{aligned}
\partial(s\sigma) &= \sum_{i=0}^{n+1} (-1)^i (s\sigma)d_i^{n+1} = (s\sigma)d_0^{n+1} - \sum_{i=1}^{n+1} (-1)^{i-1} (s\sigma)d_i^{n+1} \\
&= \sigma - \sum_{i=1}^{n+1} (-1)^{i-1} s(\sigma d_{i-1}^n) \\
&= \sigma - s\left(\sum_{i=0}^n (-1)^i \sigma d_i^n\right), \text{ by replacing } i \leftrightarrow i-1, \text{ and by linearity of } s \\
&= \sigma - s(\partial\sigma) \\
&\Rightarrow \underbrace{\sigma - \varepsilon_n(\sigma)}_0 = \partial(s\sigma) + s(\partial\sigma).
\end{aligned}$$

Thus, we have $\text{Id} - \varepsilon = \partial s + s\partial$. In other words, $s_\bullet$ is a chain homotopy between Id and $\varepsilon_\bullet$. □

Theorem 7.11: (Singular Homology of Contractible Space)

Let X be a contractible space. Then, $H_n(X) = 0$ for all $n \neq 0$.

Proof : Suppose $h : X \times [0, 1] \rightarrow X$ is a homotopy that takes Id_X to the constant map c_{x_0} for some point $x_0 \in X$. Then, as above, construct the chain map $\varepsilon_\bullet : S_\bullet(X) \rightarrow S_\bullet(X)$. Since by [Proposition 7.10](#) Id is chain homotopic to ε , an application of [Proposition 7.8](#) it follows that they give the same map in homology. But homology is an additive functor. Hence, $\text{Id} = H_n(\text{Id}) = H_n(\varepsilon) = 0$ for $n \neq 0$. The only way identity map can be the zero map if the object is zero. Thus, $H_n(X) = 0$ for $n \neq 0$. □

Exercise 7.12: (0th-Homology and Path Component)

Show that $H_0(X)$ is freely generated by the path components of X . In particular, $H_0(X) = \mathbb{Z}$ for a path connected space.