

Course notes for
Algebraic Topology II (KSM4E02)

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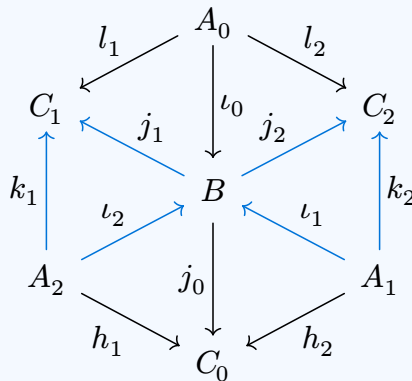
Mayer-Vietoris sequence – suspension isomorphism – homology of spheres – relative Mayer-Vietoris sequence – singular chains – singular homology

5.1 Mayer-Vietoris Sequence of a Proper Triad

Before describing the sequence, let us observe the following hexagonal lemma.

Lemma 5.1: (*Hexagonal Lemma*)

Suppose, we have a diagram of groups, where each triangle commutes.



Assume that $\ker(\iota_\alpha) = \text{im}(j_\alpha)$ for $\alpha = 1, 2$, $j_0 \circ \iota_0 = 0$, and k_1, k_2 are isomorphisms. Then, the left and right sides of the hexagon differs by a side, i.e, $h_1 \circ k_1^{-1} \circ l_2 = -h_2 \circ k_2^{-1} \circ l_2$.

Proof : We can apply [Lemma 4.10](#) to the [blue](#) part of the diagram. In particular, for any $a \in A_0$, we can uniquely write

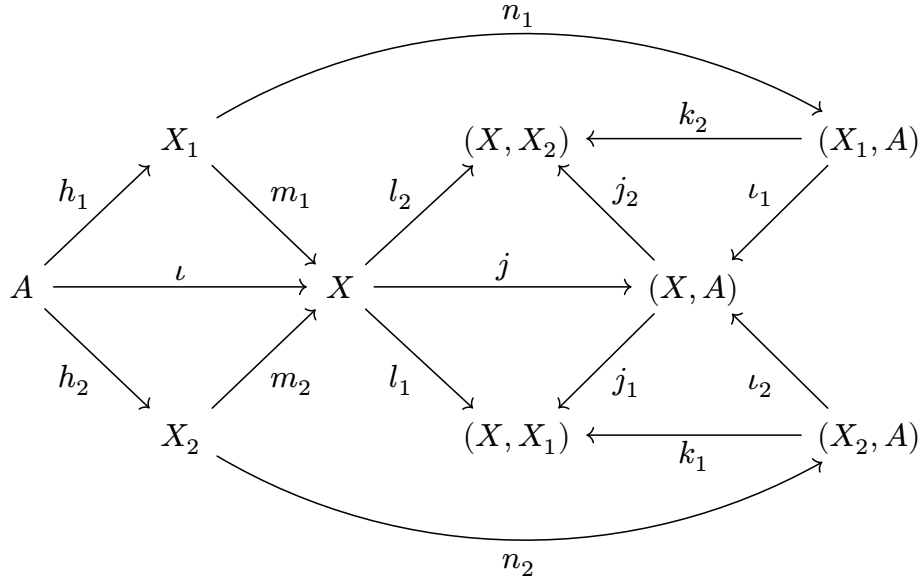
$$\iota_0(a) = \iota_1(k_2^{-1}j_2\iota_0(a)) + \iota_2(k_1^{-1}j_1\iota_0(a)).$$

Applying j_0 , and using $j_0 \circ \iota_0 = 0$, we have

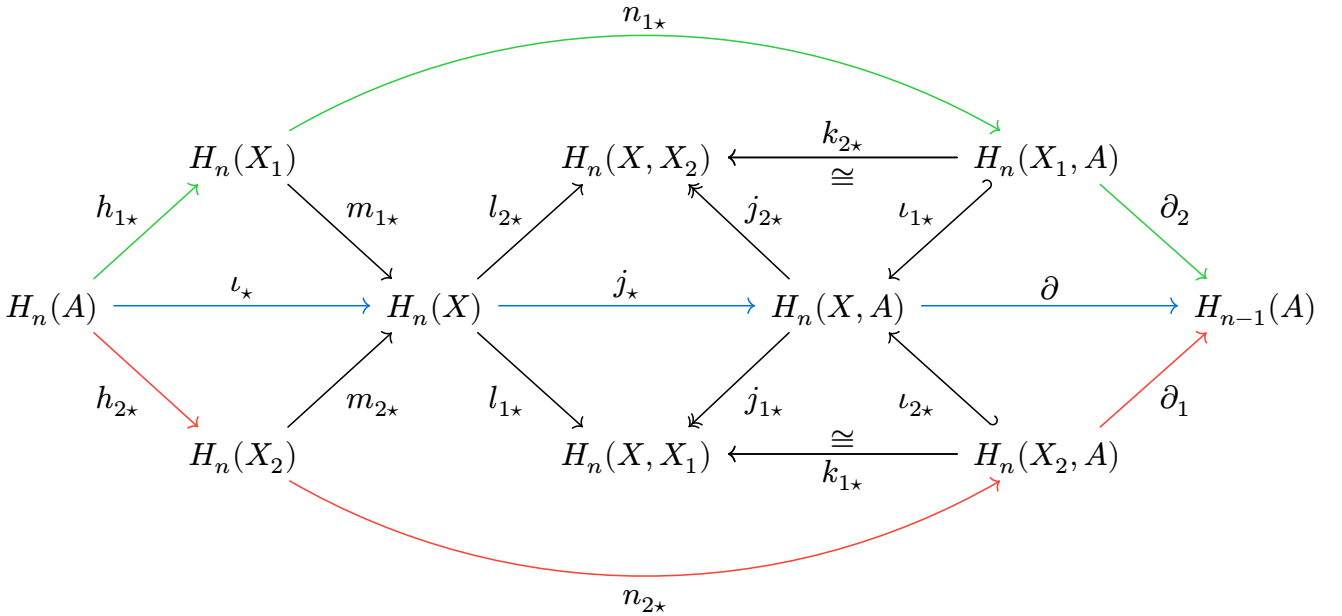
$$0 = j_0\iota_0(a) = h_2k_2^{-1}j_2\iota_0(a) + h_1k_1^{-1}j_1\iota_0(a) = h_2k_2^{-1}l_2(a) + h_1k_1^{-1}l_1(a).$$

As $a \in A_0$ is arbitrary, we have the claim. □

Let us consider a proper triad $(X; X_1, X_2)$ with $X = X_1 \cup X_2$, and set $A = X_1 \cap X_2$. At the space level, we have the following commuting diagram.



Passing to homology, we have the following commuting diagram.



As the triad is proper, we have k_{1*}, k_{2*} are isomorphisms, which gives the diagonal short exact sequences by [Lemma 4.10](#). The commutativity involving the boundary maps follows from the naturality of the homology long exact sequence. The colored arrows are part of long exact sequence of (X_1, A) , (X_2, A) , and (X, A) . We define a sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{\psi} H_n(X_1) \oplus H_n(X_2) \xrightarrow{\varphi} H_n(X) \xrightarrow{\Delta} H_{n-1}(A) \longrightarrow \cdots$$

where the maps are as follows:

$$\begin{aligned} \psi(u) &= (h_{1*}(u), -h_{2*}(u)), & u &\in H_n(A) \\ \varphi(v_1, v_2) &= m_{1*}(v_1) + m_{2*}(v_2), & v_1 &\in H_n(X_1), v_2 \in H_n(X_2) \\ \Delta(w) &= -\partial_1 k_{1*}^{-1} l_{1*}(w) = \partial_2 k_{2*}^{-1} l_{2*}(w), & w &\in H_n(X). \end{aligned}$$

The definition of Δ is a consequence of [Lemma 5.1](#).

Theorem 5.2: (Mayer-Vietoris Sequence for Proper Triad)

Given a proper triad $(X; X_1, X_2)$ with $X = X_1 \cup X_2$ and $A = X_1 \cap X_2$, we have a long exact sequence,

$$\cdots \longrightarrow H_n(A) \xrightarrow{\psi} H_n(X_1) \oplus H_n(X_2) \xrightarrow{\varphi} H_n(X) \xrightarrow{\Delta} H_{n-1}(A) \longrightarrow \cdots$$

known as the *Mayer-Vietoris sequence*, which is natural with respect to morphisms of triads.

Proof : As usual, the proof requires explicitly checking the exactness at each point.

1. $\ker \varphi \supset \text{im } \psi$: For any $u \in H_n(A)$, we have

$$\varphi\psi(u) = \varphi(h_{1\star}(u), -h_{2\star}(u)) = m_{1\star}h_{1\star}(u) - m_{2\star}h_{2\star}(u) = \iota_\star(u) - \iota_\star(u) = 0.$$

Thus, $\varphi \circ \psi = 0 \Rightarrow \ker \varphi \supset \text{im } \psi$.

2. $\ker \varphi \subset \text{im } \psi$: Suppose $\varphi(v) = 0$ for some $v = (v_1, v_2) \in H_n(X_1) \oplus H_n(X_2)$. Then,

$$0 = j_\star \varphi(v) = j_\star m_{1\star}(v_1) + j_\star m_{2\star}(v_2) = \iota_{1\star}n_{1\star}(v_1) + \iota_{2\star}n_{2\star}(v_2)$$

Now, by Lemma 4.10, we have $\iota_{1\star}, \iota_{2\star}$ are monic, and $\text{im}(\iota_{1\star}) \cap \text{im}(\iota_{2\star}) = 0$. Hence, $n_{1\star}(v_1) = 0 = n_{2\star}(v_2)$. By exactness of the long exact sequence of (X_1, A) and (X_2, A) respectively, there exists $u_1, u_2 \in H_n(A)$ such that $v_1 = h_{1\star}(u_1), v_2 = h_{2\star}(u_2)$. In other words,

$$0 = \varphi(v) = m_{1\star}h_{1\star}(u_1) + m_{2\star}h_{2\star}(u_2) = \iota_\star(u_1) + \iota_\star(u_2) = \iota_\star(u_1 + u_2).$$

Now, $u_1 + u_2 \in \ker(\iota_\star) = \text{im } \partial$. Thus, there is some $w \in H_{n+1}(X, A)$ such that $\partial(w) = u_1 + u_2$. Again by Lemma 4.10, there are (unique) $w_1 \in H_{n+1}(X_1, A), w_2 \in H_{n+1}(X_2, A)$ such that $w = \iota_{1\star}(w_1) + \iota_{2\star}(w_2)$. Then,

$$u_1 + u_2 = \partial(w) = \partial\iota_{1\star}(w_1) + \partial\iota_{2\star}(w_2) = \partial_1(w_1) + \partial_2(w_2).$$

Set, $u = u_1 - \partial_2(w_2) = -(u_2 - \partial_1(w_1))$. We then have,

$$h_{1\star}(u) = h_{1\star}(u_1) - \cancel{h_{1\star}\partial_2(w_2)} = v_1, h_{2\star}(u) = -(h_{2\star}(u_2) - \cancel{h_{2\star}\partial_1(w_1)}) = -v_2.$$

Hence, $\psi(u) = (h_{1\star}(u), -h_{2\star}(u)) = (v_1, v_2) = v$. This proves the claim.

3. $\ker \Delta \supset \text{im } \varphi$: For $v = (v_1, v_2) \in H_n(X_1) \oplus H_n(X_2)$, we have

$$\Delta\varphi(v) = \Delta(m_{1\star}(v_1) + m_{2\star}(v_2)) = -\partial_1 k_{1\star}^{-1} l_{1\star} m_{1\star}(v_1) + \partial_2 k_{2\star}^{-1} l_{2\star} m_{2\star}(v_2) = 0.$$

Thus, $\Delta \circ \varphi = 0 \Rightarrow \ker \Delta \supset \text{im } \varphi$.

4. $\ker \Delta \subset \text{im } \varphi$: Suppose for some $w \in H_n(X)$, we have $\Delta(w) = 0$. Thus, $\partial_1 k_{1\star}^{-1} l_{1\star}(w) = 0 = \partial_2 k_{2\star}^{-1} l_{2\star}(w)$. By exactness, for $\alpha = 1, 2$, there is $v_\alpha \in H_n(X_\alpha)$ such that $n_{\alpha\star}(v_\alpha) = k_{\alpha\star}^{-1} l_{\alpha\star}(w)$. By Lemma 4.10, we can (uniquely) write

$$\begin{aligned} j_\star(w) &= i_{1\star} k_{2\star}^{-1} j_{2\star} j_\star(w) + i_{2\star} k_{1\star}^{-1} j_{1\star} j_\star(w) \\ &= i_{1\star} k_{2\star}^{-1} l_{2\star}(w) + \iota_{2\star} k_{1\star}^{-1} l_{1\star}(w) \\ &= \iota_{1\star} n_{1\star}(v_1) + \iota_{2\star} n_{2\star}(v_2) \\ &= j_\star m_{1\star}(v_1) + j_\star m_{2\star}(v_2). \end{aligned}$$

Thus, $w - m_{1\star}(v_1) - m_{2\star}(v_2) \in \ker(j_\star) = \text{im}(\iota_\star)$. Hence, there exists some $u \in H_n(A)$ such that $\iota_\star(u) = w - m_{1\star}(v_1) - m_{2\star}(v_2)$. Set, $v'_1 = v_1 + h_{1\star}(u), v'_2 = v_2$. Then

$$\varphi(v'_1, v'_2) = m_{1\star}(v_1 + h_{1\star}(u)) + m_{2\star}(v_2)$$

$$= m_{1*}(v_1) + (w - m_{1*}(v_1) - m_{2*}(v_2)) + m_{2*}(v_2) = w.$$

This proves the claim.

5. $\ker \psi \supset \text{im } \Delta$: For $w \in H_n(X)$, we have

$$\psi \Delta(w) = (h_{1*} \partial_2 k_{2*}^{-1}) l_{2*}(w), h_{2*} \partial_1 k_{1*}^{-1} l_{1*}(w) = (0, 0) = 0.$$

Thus, $\psi \circ \Delta = 0 \Rightarrow \ker \psi \supset \text{im } \Delta$.

6. $\ker \psi \subset \text{im } \Delta$: Suppose for some $u \in H_n(A)$ we have $\psi(u) = 0$. Thus, $h_{1*}(u) = 0 = h_{2*}(u)$. By exactness, for $\alpha = 1, 2$ there are $x_\alpha \in H_{n+1}(X_\alpha, A)$ such that $\partial_1(x_2) = u$ and $\partial_2(x_1) = -u$. Then,

$$\partial \iota_{1*}(x_1) + \partial \iota_{2*}(x_2) = \partial_2(x_1) + \partial_1(x_2) = -u + u = 0 \Rightarrow \iota_{1*}(x_1) + \iota_{2*}(x_2) \in \ker(\partial) = \text{im}(j_*).$$

Thus, there exists $w \in H_{n+1}(X)$ such that $j_*(w) = -\iota_{1*}(x_1) - \iota_{2*}(x_2)$. Then,

$$\begin{aligned} \Delta(w) &= -\partial_1 k_{1*}^{-1} l_{1*}(w) = -\partial_1 k_{1*}^{-1} j_{1*} j_*(w) \\ &= \partial_1 k_{1*}^{-1} j_{1*} (\iota_{1*}(x_1) + \iota_{2*}(x_2)) \\ &= 0 + \partial_1 k_{1*}^{-1} j_{1*} \iota_{2*}(x_2) \\ &= \partial_1 k_{1*}^{-1} k_{1*}(x_2) = \partial_1(x_2) = u. \end{aligned}$$

This proves the claim.

Thus, the Mayer-Vietoris sequence is exact. One can *easily* prove the naturality of the sequence with respect to maps of triads (Check!). □

5.2 The Suspension Isomorphism

Given a space, recall the *(unreduced) cone* on X is

$$CX = \frac{X \times [0, 1]}{X \times \{0\}},$$

and the *(unreduced) suspension* of X is

$$\Sigma X = \frac{CX}{X \times \{1\}}.$$

It is well known that $CX \simeq \star$, i.e, the cone is always contractible. Consequently we have the following.

Proposition 5.3: *(Cone on a Space Is Homologically Trivial)*

Given a space X , we have CX is homologically trivial, i.e, $\tilde{H}_n(CX) = 0$ for all n .

Let us now prove a fundamental property of (ordinary) homology theory.

Theorem 5.4: (Suspension Isomorphism)

Given a space X , there exists a natural isomorphism $\tilde{H}_n(\Sigma X) \cong \tilde{H}_{n-1}(X)$.

Proof : Consider $A := q(X \times [0, \frac{3}{4}])$, $B := q(X \times [\frac{1}{4}, 1]) \subset \Sigma X$, where $q : X \times [0, 1] \rightarrow \Sigma X$ is the quotient map. It is easy to see that $(\Sigma X; A, B)$ is a proper triad. Now,

$$\Sigma X = A \cup B, A \cong CX \simeq \star, B \cong CX \simeq \star, A \cap B \cong X \times \left[\frac{1}{4}, \frac{3}{4}\right] \simeq X.$$

From the reduced version of the Mayer-Vietoris sequence, we immediately see that $\Delta : \tilde{H}_n(\Sigma X) \rightarrow \tilde{H}_{n-1}(A \cap B)$ is an isomorphism. Since $A \cap B$ deformation retracts onto X , we have the claim. $\square$

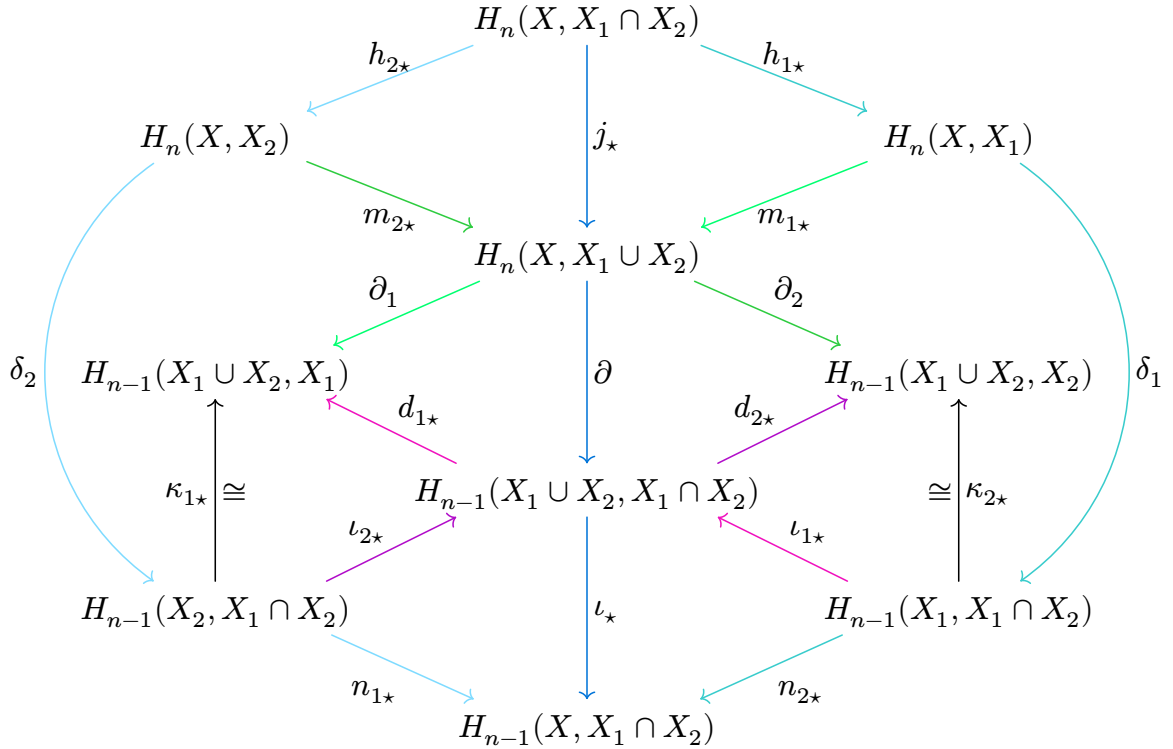
Corollary 5.5: (Reduced Homology of Sphere)

The reduced homology of the n -sphere is $\tilde{H}_k(S^n) = \begin{cases} 0, & k \neq n \\ H_0(\star), & k = n \end{cases}$.

Proof : We apply Theorem 5.4. Since $S^n = \Sigma S^{n-1}$, by induction, we only need compute the homology of S^0 , which is just two points. By Proposition 3.1, we have $H_n(S^0) = H_n(\star) \oplus H_n(\star)$. Then, by the dimension axiom, it follows that $\tilde{H}_k(S^0) = \begin{cases} 0, & k \neq 0 \\ H_0(\star), & k = 0 \end{cases}$. $\square$

5.3 Relative Mayer-Vietoris Sequence

Let us now consider an arbitrary proper triad $(X; X_1, X_2)$. We have the following diagram.



The colored arrows are part of the corresponding long exact sequences of different triples. The isomorphisms $\kappa_{1\star}, \kappa_{2\star}$ follows from the proper triad $(X_1 \cup X_2; X_1, X_2)$. In particular, we can apply Lemma 5.1. Let us define a sequence

$$\cdots \rightarrow H_n(X, X_1 \cap X_2) \xrightarrow{\psi} H_n(X, X_1) \oplus H_n(X, X_2) \xrightarrow{\varphi} H_n(X, X_1 \cup X_2) \xrightarrow{\Delta} H_{n-1}(X, X_1 \cap X_2) \rightarrow \cdots$$

where the maps are as follows:

$$\begin{aligned}
 \psi(u) &= (h_{1\star}(u), -h_{2\star}(u)), & u &\in H_n(X, X_1 \cap X_2) \\
 \varphi(v_1, v_2) &= m_{1\star}(v_1) + m_{2\star}(v_2), & v_1 &\in H_n(X, X_1), v_2 \in H_n(X, X_2) \\
 \Delta(w) &= -n_{1\star}k_{1\star}^{-1}\partial_1(w) = n_{2\star}k_{2\star}^{-1}\partial_2(w), & w &\in H_n(X, X_1 \cup X_2).
 \end{aligned}$$

Theorem 5.6: (*Relative Mayer-Vietoris Sequence*)

Given a proper triad $(X; X_1, X_2)$, there exists a long exact sequence

$$\cdots \longrightarrow H_n(X, X_1 \cap X_2) \xrightarrow{\psi} H_n(X, X_1) \oplus H_n(X, X_2) \xrightarrow{\varphi} H_n(X, X_1 \cup X_2) \xrightarrow{\Delta} H_{n-1}(X, X_1 \cap X_2) \longrightarrow \cdots$$

known as the *relative Mayer-Vietoris sequence*, which is moreover natural with respect to morphism of triads.