Course notes for

Algebraic Topology II (KSM4E02)

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`Künneth theorem for chain complex — Eilenberg-Zilber theorem — Künneth theorem for spaces — homological cross product`

18.1 Künneth Theorems in Homological Algebra

We now prove the Künneth theorems, which helps us compute the (co)homology of tensor product of chain complexes.

Theorem 18.1: (*Künneth Theorem for Homology*)

Let R be a PID, and $C_\bullet$ be a chain complex of free R-modules. Then, for any chain complex $D_\bullet$ of R-modules, we have a natural short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(C_\bullet) \otimes H_j(D_\bullet) \to H_n((C \otimes D)_\bullet) \to \bigoplus_{i+j=n-1} \operatorname{Tor}\big(H_i(C_\bullet), H_j(D_\bullet)\big) \to 0.$$

Moreover, if $D_\bullet$ consists of free R-modules, then the short exact sequence splits (not necessarily naturally).

<u>**Proof**</u> **:** Let us consider the graded modules $Z(C_\bullet)$, $B(C_\bullet)$ and $H(C_\bullet)$ of respectively cycles, boundaries and homology of $C_\bullet$ as chain complexes with *trivial* boundary maps. Since R is a PID, and $C_\bullet$ is a complex of *free* R-modules, we have $Z(C_\bullet)$ and $B(C_\bullet)$ are also complexes of free R-modules. Now, we have the chain complex $\big((Z(C) \otimes D)_\bullet, \partial_\bullet^{Z(C)\otimes D}\big)$, where the boundary is given as

$$\partial_n^{Z(C)\otimes D} = \sum_{i+j=n} \mathrm{Id}_{Z_i(C_\bullet)} \otimes \partial_j^D.$$

Since $Z_i(C_\bullet)$ is free, we have $\operatorname{Tor}(Z_i(C_\bullet), \_) = 0$, and hence it follows that

$$\ker\big(\mathrm{Id} \otimes \partial_j^D : Z_i(C_\bullet) \otimes D_j \to Z_i(C_\bullet) \otimes D_{j-1}\big) = Z_i(C_\bullet) \otimes Z_j(D_\bullet),$$

and

$$\operatorname{im}\big(\mathrm{Id} \otimes \partial_{j+1}^D : Z_i(C_\bullet) \otimes D_{j+1} \to Z_i(C_\bullet) \otimes D_j\big) = Z_i(C_\bullet) \otimes B_j(D_\bullet).$$

In particular, since (co)kernels for graded modules are computed degreewise, we have the equalities

$$(Z(C_\bullet) \otimes Z(D_\bullet))_n = \ker\Big(\mathrm{Id} \otimes \partial^D : (Z(C_\bullet) \otimes D_\bullet)_n \to (Z(C_\bullet) \otimes D_\bullet)_{n-1}\Big),$$

and

$$\left(Z(C_\bullet) \otimes B(D_\bullet)\right)_n = \operatorname{im}\Big(\mathrm{Id} \otimes \partial^D : \left(Z(C_\bullet) \otimes D_\bullet\right)_{n+1} \to \left(Z(C_\bullet) \otimes D_\bullet\right)_n\Big).$$

Moreover, tensoring the short exact sequence $0 \to B_j(D_\bullet) \to Z_j(D_\bullet) \to H_j(D_\bullet) \to 0$ by $Z_i(C_\bullet)$ gives a short exact sequence, and consequently, by the first isomorphism theorem,

$$\frac{Z_i(C_\bullet) \otimes Z_j(D_\bullet)}{Z_i(C_\bullet) \otimes B_j(D_\bullet)} \cong Z_i(C_\bullet) \otimes H_j(D_\bullet).$$

This implies that

$$H(Z(C_\bullet) \otimes D_\bullet) = Z(C_\bullet) \otimes H(D_\bullet),$$

as graded modules. By a similar argument, since $B(C_\bullet)$ is a free (graded) module, we have

$$H(B(C_\bullet) \otimes D_\bullet) = B(C_\bullet) \otimes H(D_\bullet)$$

as graded modules. We now have a *degreewise free* resolution

$$0 \to B(C_\bullet) \to Z(C_\bullet) \to H(C_\bullet) \to 0.$$

Tensoring by $H(D_\bullet)$, and taking homology, we have the exact sequence

$$0 \to \operatorname{Tor}(H(C_\bullet), H(D_\bullet)) \to B(C_\bullet) \otimes H(D_\bullet) \to Z(C_\bullet) \otimes H(D_\bullet) \to H(C_\bullet) \otimes H(D_\bullet) \to 0.$$

On the other hand, we have the *split* short exact sequence of chain complex of free R-modules

$$0 \to Z(C_\bullet) \to C_\bullet \to B(C_\bullet)[-1] \to 0.$$

Tensoring by $D_\bullet$ gives the (split) short exact sequence

$$0 \to Z(C_\bullet) \otimes D_\bullet \to C_\bullet \otimes D_\bullet \to (B(C_\bullet) \otimes D_\bullet)[-1] \to 0.$$

Passing to the long exact sequence in homology, and identifying the (co)kernel of the maps $\iota \otimes \mathrm{Id} : B(C_\bullet) \otimes D_\bullet \to Z(C_\bullet) \otimes D_\bullet$, we get the short exact sequence

$$0 \to H(C_\bullet) \otimes H(D_\bullet) \to H(C_\bullet \otimes D_\bullet) \to \operatorname{Tor}(H(C_\bullet), H(D_\bullet))[-1] \to 0.$$

Since Tor is additive, we get the required short exact sequence.

Finally, assume $D_\bullet$ is a chain complex of free R-modules. Then, we can *choose* retractions $r_\bullet : C_\bullet \to Z_n(C_\bullet)$ and $s_\bullet : D_\bullet \to Z_n(D_\bullet)$ in the respective split exact sequences. This induces a map $(C_\bullet \otimes D_\bullet)_n \to H(C_\bullet) \otimes H(D_\bullet)$ given by $c \otimes d \mapsto [r(c)] \otimes [s(d)]$. One can verify that this map sends boundaries in $C_\bullet \times .D_\bullet$ to 0, and hence, induces a map $\rho : H_n(C_\bullet \otimes D_\bullet) \to H(C_\bullet) \otimes H(D_\bullet)$. Clearly, ρ is a retraction in the short exact sequence, which then splits. As $r_\bullet, s_\bullet$ are choices, it follows that the splitting may not be natural. □

In the above theorem we have freely used homological algebraic machinary developed for R-modules to graded R-modules by degreewise consideration. This naive approach does not work when we concisder chain complexes with nontrivial differentials!

Remark 18.2: (*Künneth Theorems for Spaces*)

Can we directly use Theorem 18.1 for spaces to compute homology of product?! Suppose X, Y are two spaces, and denote $C_\bullet = S_\bullet(X), D_\bullet = S_\bullet(Y)$ to be the singular chain complexes (or the cellular ones, if we are working with CW complexes). Then, Künneth theorem is applicable for $C_\bullet, D_\bullet$, and we are able to compute $H(C_\bullet \otimes D_\bullet) = H(S_\bullet(X) \otimes S_\bullet(Y))$. On the other hand, the homology of the product $X \times Y$ is given as $H(S_\bullet(X \times Y))$. The challenge is then to relate $S_\bullet(X) \otimes S_\bullet(Y)$ with $S_\bullet(X \times Y)$! We shall see later (Theorem 18.6) that these two chain complexes are in fact quasi-isomorphic.

Exercise 18.3: (*Algebraic Cross Product*)

Verify that the first map in Theorem 18.1 is given component wise as

$$\begin{aligned} \times_{\text{alg}} : H_p(C_\bullet) \otimes H_q(D_\bullet) &\to H_{p+q}(C_\bullet \otimes D_\bullet) \\ [\varphi] \otimes [\psi] &\mapsto [\varphi \otimes \psi]. \end{aligned}$$

This is usually known as the *algebraic cross product* in homology.

Similar to Theorem 17.9, we have the following.

Theorem 18.4: (*Künneth Theorem for Cohomology*)

Let R be a PID, and $C_\bullet, D_\bullet$ be cochain complexes of free R-modules. Assume that either $H(C_\bullet)$ or $H(D_\bullet)$ is of finite type. Then, for the dual complex $C^\bullet = \hom(C_\bullet, R), D^\bullet = \hom(D_\bullet, R)$ there exists a natural short exact sequence

$$0 \to \bigoplus_{i+j=n} H^i(C^\bullet) \otimes H^j(D^\bullet) \to H^n(C^\bullet \otimes D^\bullet) \to \bigoplus_{i+j=n+1} \operatorname{Tor}(H^i(C^\bullet), H^j(D^\bullet)) \to 0,$$

which split.

18.2 Künneth Theorem for Spaces

Fix a PID R, and denote by $\mathrm{Ch}_{\geq 0}$ the category of chain complexes of R-modules, with 0 in the strictly negative degrees. We then have two functors

$$\begin{aligned} \mathcal{F} : \mathrm{Top} \times \mathrm{Top} &\to \mathrm{Ch}_{\geq 0} \\ (X, Y) &\to S_\bullet(X) \otimes S_\bullet(Y), \end{aligned} \qquad \begin{aligned} \mathcal{G} : \mathrm{Top} \times \mathrm{Top} &\to \mathrm{Ch}_{\geq 0} \\ (X, Y) &\to S_\bullet(X \times Y). \end{aligned}$$

Here, $S_\bullet(\_)$ is the singular chain complex functor with R-coefficients. It is easy to see that

$$S_0(X) \otimes S_0(Y) = R\langle X \rangle \otimes R\langle X \rangle \cong R\langle X \times Y \rangle = S_0(X \times Y).$$

Indeed, 0-simplicies are nothing but points, and hence, sending the 0-simplices $\sigma_x \in S_0(X), \sigma_y \in S_0(Y)$ to the simplex $\sigma_{x,y} \in S_0(X \times Y)$ defines a natural isomorphism, with inverse given by $\sigma_{x,y} \mapsto \sigma_x \otimes \sigma_y$. But this strategy does not generalize immediately.

Definition 18.5: (*Eilenberg-Zilber Maps*)

A pair of natural transformations

$$\mathfrak{P} : S_\bullet(-) \otimes S_\bullet(-) \Rightarrow S_\bullet(- \times -) \qquad \text{and} \qquad \mathfrak{Q} : S_\bullet(- \times -) \Rightarrow S_\bullet(-) \otimes S_\bullet(-)$$

are called *Eilenberg-Zilber maps* (or *EZ-maps*) if they extend the canonical isomorphism at the 0^{th} level.

By an application of the *acyclic model theorem,* one gets the following.

Theorem 18.6: (*Eilenberg-Zilber Theorem*)

EZ-maps exist. Moreover, the following holds.

- For any pair $(\mathfrak{P}, \mathfrak{Q})$ of EZ-maps, we have $\mathfrak{P} \circ \mathfrak{Q}$ and $\mathfrak{Q} \circ \mathfrak{P}$ are naturally chain homotopic to identity. In particular, $\mathfrak{P}_{X,Y}$ and $\mathfrak{Q}_{X,Y}$ are natural chain equivalences.
- Given two pairs $(\mathfrak{P}, \mathfrak{Q})$ and $(\mathfrak{P}', \mathfrak{Q}')$ of EZ-maps, we have $\mathfrak{P}, \mathfrak{P}'$ (and similarly, $\mathfrak{Q}, \mathfrak{Q}'$) are naturally chain homotopic.
- An EZ map $\mathfrak{P}$ is
  - *associative* up to natural homotopy, i.e, the following diagram commutes up to natural chain homotopy:

$$\begin{tikzcd} S_\bullet(X) \otimes S_\bullet(Y) \otimes S_\bullet(Z) \arrow[r, "\mathfrak{P}_{X,Y} \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes \mathfrak{P}_{Y,Z}"'] & S_\bullet(X \times Y) \otimes S_\bullet(Z) \arrow[d, "\mathfrak{P}_{X \times Y, Z}"] \\ S_\bullet(X) \otimes S_\bullet(Y \times Z) \arrow[r, "\mathfrak{P}_{X, Y \times Z}"'] & S_\bullet(X \times Y \times Z) \end{tikzcd}$$

  - *commutative* up to natural homotopy, i.e, the following diagram commutes up to natural chain homotopy:

$$\begin{tikzcd} S_\bullet(X) \otimes S_\bullet(Y) \arrow[r, "\tau_{X,Y}"] \arrow[d, "\mathfrak{P}_{X,Y}"'] & S_\bullet(Y) \otimes S_\bullet(X) \arrow[d, "\mathfrak{P}_{Y,X}"] \\ S_\bullet(X \times Y) \arrow[r, "S_\bullet(t_{X,Y})"'] & S_\bullet(Y \times X) \end{tikzcd}$$

- An EZ map $\mathfrak{Q}$ is
  - *coassociative* up to natural homotopy, i.e, the following diagram commutes up to natural chain homotopy:

$$\begin{tikzcd} S_\bullet(X \times Y \times Z) \arrow[r, "\mathfrak{Q}_{X \times Y, Z} \otimes \mathrm{Id}"] \arrow[d, "\mathfrak{Q}_{X, Y \times Z}"'] & S_\bullet(X \times Y) \otimes S_\bullet(Z) \arrow[d, "\mathfrak{Q}_{X,Y} \otimes \mathrm{Id}"] \\ S_\bullet(X) \otimes S_\bullet(Y \times Z) \arrow[r, "\mathrm{Id} \otimes \mathfrak{Q}_{Y,Z}"'] & S_\bullet(X) \otimes S_\bullet(Y) \otimes S_\bullet(Z) \end{tikzcd}$$

  - *cocommutative* up to natural homotopy, i.e, the following diagram commutes up to natural chain homotopy:

$$\begin{tikzcd}
S_\bullet(X \times Y) \arrow[r, "S_\bullet(t_{X,Y})"] \arrow[d, "\mathfrak{Q}_{X,Y}"'] & S_\bullet(Y \times X) \arrow[d, "\mathfrak{Q}_{Y,X}"] \\
S_\bullet(X) \otimes S_\bullet(Y) \arrow[r, "\tau_{X,Y}"'] & S_\bullet(Y) \otimes S_\bullet(X)
\end{tikzcd}$$

Here, $t_{X,Y} : X \times Y \to Y \times X$ is natural flip map, and $\tau_{X,Y} : S_\bullet(X) \otimes S_\bullet(Y) \to S_\bullet(Y) \otimes S_\bullet(X)$ is given by $\tau_{X,Y}(\alpha \otimes \beta) = (-1)^{|\alpha||\beta|}\beta \otimes \alpha$.

Although Theorem 18.6 is existential in nature, there are explicit constructions of EZ-maps, which were found out later by Eilenberg and MacLane.

Definition 18.7: (*Alexander-Whitney Map*)

Given spaces X, Y, the *Alexander-Whitney maps*

$$\mathrm{AW}_{X,Y} : S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$$

are defined on an n-simplex $\sigma : \Delta^n \to X \times Y$ by the formula

$$\mathrm{AW}(\sigma) = \sum_{p+q=n} \pi_X \circ \sigma \circ \lambda_p^n \otimes \pi_Y \circ \sigma \circ \rho_q^n,$$

where $\lambda_p^n : \Delta^p \to \Delta^n$ and $\rho_q^n : \Delta^q \to \Delta^n$ are defined linearly as $\lambda_p^n(e_i) = e_i$ for $0 \leq i \leq p$, and $\rho_q^n(e_j) = e_{n-q+j}$ for $0 \leq j \leq q$. Intuitively, λ_p^n embeds Δ^p as the *front* p-face of Δ^n, and ρ_q^n embeds Δ^q as the *back* q-face of Δ^n.

One can verify that AW is a natural chain map, extending the natural isomorphism at the 0^{th} level, i.e, AW is an EZ map. Moreover, AW is *strictly* coassociative.

Definition 18.8: (*Eilenberg-MacLane Map*)

Given spaces X, Y the (explicit) *Eilenberg-Zilber maps* (or *Eilenberg-MacLane formula*)

$$\mathrm{EM}_{X,Y} : S_\bullet(X) \otimes S_\bullet(Y) \to S_\bullet(X \times Y)$$

are defined via *shuffle products*. Explicitly, treat $\Delta^n \subset \mathbb{R}^n$ as the subset $\Delta^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \leq x_1 \leq \cdots \leq x_n \leq 1\}$. Recall, a (p, q)-shuffle σ is a bijection $\sigma : \{1, \dots, p+q\} \to \{1, \dots, p+q\}$ such that $\sigma(1) \leq \cdots \leq \sigma(p)$ and $\sigma(p+1) \leq \cdots \leq \sigma(p+q)$; thus $\mathrm{Sh}(p, q) \subset S_{p+q}$ is a subset of permutaions and in particular the signum function $\mathrm{sgn}(\sigma)$ makes sense. Given a (p, q)-shuffle σ, consider the linear embedding $\ell_\sigma : \Delta^{p+q} \to \Delta^p \times \Delta^q$ given by

$$\ell_\sigma(x_1, \dots, x_p, x_{p+1}, \dots, x_{p+q}) = \Big(\big(x_{\sigma(1)}, \dots, x_{\sigma(p)}\big), \big(x_{\sigma(p+1)}, \dots, x_{\sigma(p+q)}\big)\Big).$$

Then, for $\alpha : \Delta^p \to X, \beta : \Delta^q \to Y$ one can define

$$\mathrm{EM}(\alpha \otimes \beta) = \sum_{\sigma \in \mathrm{Sh}(p,q)} \mathrm{sgn}(\sigma)(\alpha \times \beta) \circ \ell_\sigma.$$

Again, it can be verified that EM is a natural chain map, extending the natural isomorphism at the 0^{th} level, making it an EZ-map! Moreover, EM is strictly coassociative. Note that we chaged the definition of simplicies, but they are naturally homeomorphic to Definition 6.1, which lets us translate the definition of EM back to the usual singular chain complexes where the formula becomes even more complicated. There are other combinatorial ways to define such maps, e.g, by identifying a simplex with paths in an integer grids.

Example 18.9: (*EZ-map from* AW)

The composition

$$\begin{tikzcd}
S_\bullet(X \times Y) \arrow[rr, dashed, "\mathrm{EZ}_{X,Y}"] \arrow[d, "S_\bullet(\Delta_{X\times Y})"'] & & S_\bullet(X) \otimes S_\bullet(Y) \\
S_\bullet(X \times Y \times X \times Y) \arrow[rr, "\mathrm{AW}_{X\times Y, X\times Y}"'] & & S_\bullet(X \times Y) \otimes S_\bullet(X \times Y) \arrow[u, "S_\bullet(\pi_X) \otimes S_\bullet(\pi_Y)"']
\end{tikzcd}$$

defines a strictly associative EZ-map as well. In fact, for any EZ-map $S_\bullet(- \times -) \Rightarrow S_\bullet(-) \otimes S_\bullet(-)$ the above composition gives a EZ-map, which may not be strictly coassociative.

As a consequence of Theorem 18.6, we can now get the Künneth theorem for spaces.

Theorem 18.10: (*Künneth Theorem for Singular Homology*)

Let R be a PID, and X, Y be spaces. Then, there exists a natural short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(X;R) \otimes H_j(Y;R) \to H_n(X \times Y; R) \to \bigoplus_{i+j=n-1} \mathrm{Tor}\big(H_i(X;R), H_j(Y;R)\big) \to 0,$$

which is split, but not naturally.

Proof : Proof is immediate from the Künneth theorem in homological algebra (Theorem 18.1) and the Eilenberg-Zilber theorem (Theorem 18.6). Indeed, naturality of the short exact sequence lets us replace $S_\bullet(X) \otimes S_\bullet(Y)$ by the naturally chain equivalent complex $S_\bullet(X \times Y)$. □

The first map in the Künneth theorem gives a product stucture on singular homology.

Definition 18.11: (*Homology Cross Product*)

Given two spaces X, Y, the *homology cross product* for homology with coefficients in some ring is defined as the composition

$$\times : H_p(X) \otimes H_q(Y) \to H_{p+q}(S_\bullet(X) \otimes S_\bullet(Y)) \xrightarrow{\cong} H_{p+q}(X \times Y),$$

where the natural isomorphism is via an EZ-map.

Explicitly, given $\sigma : \Delta^p \to X, \tau : \Delta^q \to Y$, and a choice of an EZ-map $\mathfrak{Q}$, we have the element $\mathfrak{Q}_{X,Y}(\sigma \otimes \tau) \in S_{p+q}(X \times Y)$, which induces the cross product in homology.

In order to get the Künneth theorem in singular cohomology, we need to take some extra assumptions.

Theorem 18.12: (*Künneth Theorem for Singular Cohomology*)

Let R be a PID, and X, Y be spaces. Suppose that either $H_\bullet(X;R)$ or $H_\bullet(Y;R)$ are of finite type. Then, there exists a natural short exact sequence

$$0 \to \bigoplus_{i+j=n} H^i(X;R) \otimes H^j(Y;R) \to H^n(X \times Y;R) \to \bigoplus_{i+j=n+1} \mathrm{Tor}(H^i(X;R), H^j(Y;R)) \to 0,$$

which splits, but not naturally.

Proof : The proof follows from the algebraic Künneth theorem for cohomology (Theorem 18.4), and the Eilenber-Zilber theorem (Theorem 18.6). □

Remark 18.13: (*Relative Künneth Theorems*)

There are appropriate versions of Künneth theorems for *relative* singular (co)homology as well.