Course notes for

Algebraic Topology II (KSM4E02)

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`Tor via flat resolution – right derived functors – Ext functors – injective Abelian groups – injective` R`-modules – universal coefficient theorems`

16.1 A Digression : Left Derived Functor by Flat Resolution

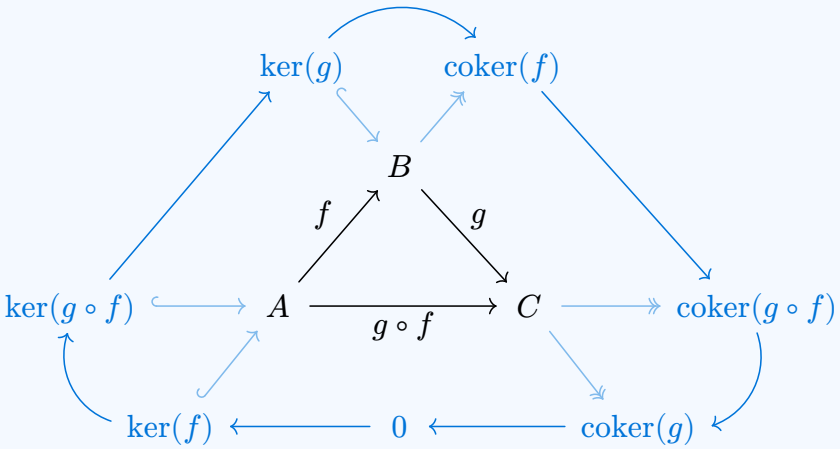
Let us show that Tor can also be computed using flat resolution. We need an algebraic lemma first.

Lemma 16.1: (*Snake Lemma à la Cartier-Weil*)

Let $f : A \to B$ and $g : B \to C$ be maps in an Abelian category $\mathcal{A}$. Then, there exists an exact sequence

$$0 \to \ker(f) \to \ker(g \circ f) \to \ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(g \circ f) \to \operatorname{coker}(g) \to 0.$$

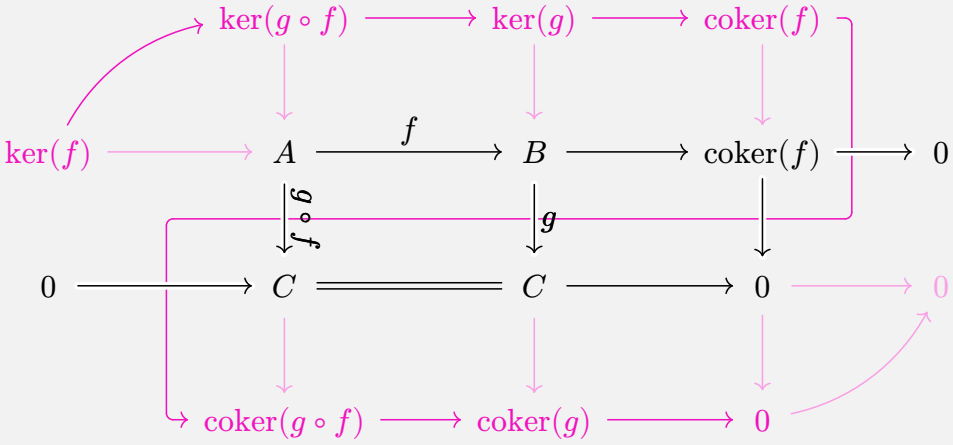
Visualized another way, we have



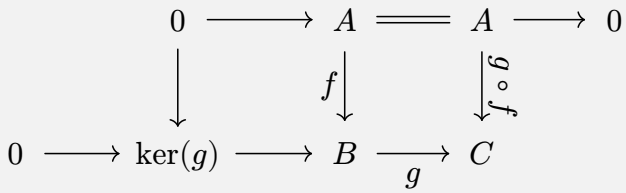
Proof : Consider the diagram of exact sequences

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \longrightarrow & \operatorname{coker}(f) & \longrightarrow & 0 \\ \downarrow{\scriptstyle g\circ f} & & \downarrow{\scriptstyle g} & & \downarrow & & \\ 0 & \longrightarrow & C & = & C & \longrightarrow & 0 \end{array}$$

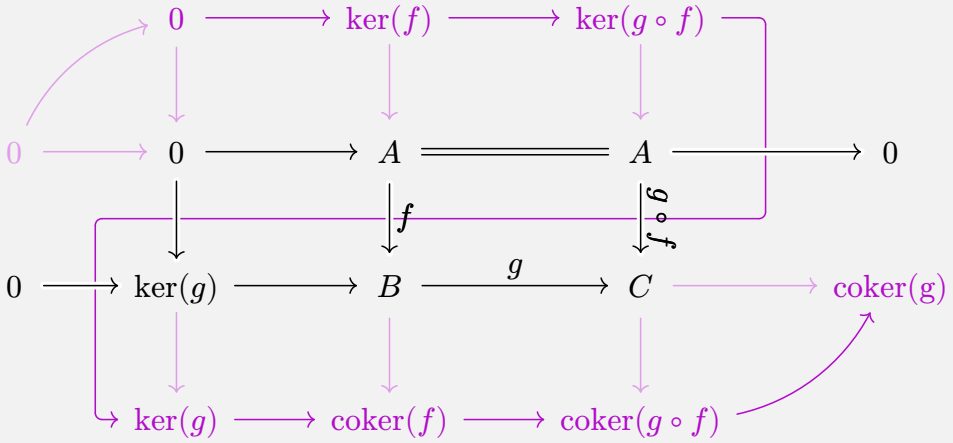
Applying Lemma 6.29, we get the exact sequence



Similarly, consider the diagram of exact sequence,



Again, applying Lemma 6.29, we get the exact sequence



Splicing the two exact sequences, we get

$$0 \to \ker(f) \to \ker(g \circ f) \to \ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(g \circ f) \to \operatorname{coker}(g) \to 0,$$

which proves the claim. □

We now have the following useful fact.

Proposition 16.2: (*Tor via Flat Resolution*)

Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact additive functor between two Abelian categories, and suppose $\mathcal{A}$ has enough projectives. Then the Tor functors can be computed by flat resolutions, i.e, given $M, N \in \mathcal{A}$ and a *flat* resolution $F_\bullet \to M$, we have $\operatorname{Tor}_n^R(M, N) \cong H_n(F_\bullet \otimes N)$, and similarly in the second variable.

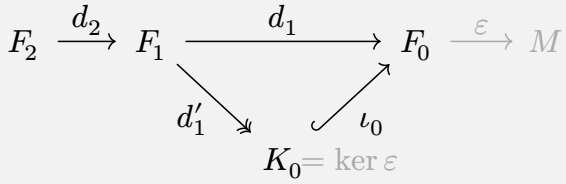
Proof : Let $M, N \in \mathcal{A}$ be fixed. Suppose $F_\bullet \xrightarrow{\varepsilon} M$ is a flat resolution. Since tensor is right exact, we have an exact sequence

$$F_1 \otimes N \xrightarrow{d_1 \otimes \mathrm{Id}_N} F_0 \otimes N \xrightarrow{\varepsilon \otimes \mathrm{Id}_N} M \otimes N \to 0.$$

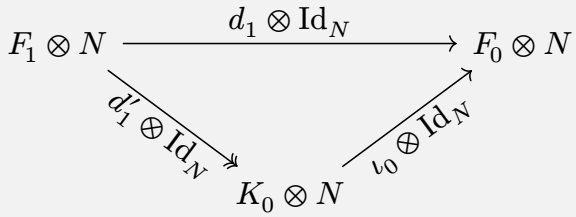
It follows that

$$H_0(F_\bullet \otimes N) = \frac{F_0 \otimes N}{\operatorname{im}(d_1 \otimes \operatorname{Id}_N)} \cong M \otimes N \cong \operatorname{Tor}_0^R(M, N).$$

For $n = 1$, we have the diagram



Tensoring by N, we have the commutative diagram



The right exactness of tensor shows that $d_1' \otimes \operatorname{Id}_N$ is an epimorphism, and hence, $\operatorname{im}(d_1 \otimes \operatorname{Id}_N) = \operatorname{im}(\iota_0 \otimes \operatorname{Id}_N)$ follows. Let

$$f \coloneqq \widetilde{d_1' \times \operatorname{Id}_N} : \frac{F_1 \otimes N}{\operatorname{im}(d_2 \otimes \operatorname{Id}_N)} \to K_0 \otimes N, \qquad g = \iota_0 \otimes \operatorname{Id}_N : K_0 \otimes N \to F_0 \otimes N.$$

The exact sequence $F_2 \to F_1 \to K_0 \to 0$ gives the exact sequence

$$F_2 \otimes N \xrightarrow{d_2 \otimes \operatorname{Id}_N} F_1 \otimes N \xrightarrow{d_1' \otimes \operatorname{Id}_N} K_0 \otimes N \to 0,$$

which implies

$$\ker(f) = \frac{\ker(d_1' \otimes \operatorname{Id}_N)}{\operatorname{im}(d_2 \otimes \operatorname{Id}_N)} = 0.$$

Also, f is an epimorphism, as it is induced by the epimorphism $d_1' \times \operatorname{Id}_N$. Now, by Lemma 16.1, we have the exact sequence

$$\underbrace{\ker(f)}_{0} \to \ker(g \circ f) \to \ker(g) \to \underbrace{\operatorname{coker}(f)}_{0}.$$

Hence, we have an isomorphism $\ker(g \circ f) \cong \ker(g)$. But $\ker(g \circ f) = \frac{\ker(d_1 \otimes \operatorname{Id}_N)}{\operatorname{im}(d_2 \times \operatorname{Id}_N)} = H_1(F_\bullet \otimes N)$. Hence, $H_1(F_\bullet \otimes N) \cong \ker(g) = \ker(\iota_0 \times \operatorname{Id}_N)$. Now, applying Theorem 15.12 to the exact sequence $0 \to K_0 \to F_0 \to M \to 0$ we have (part of) the exact sequence for Tor

$$\operatorname{Tor}_1^R(F_0, N) \to \operatorname{Tor}_1^R(M, N) \to K_0 \otimes N \xrightarrow{\iota_0 \otimes \operatorname{Id}_N} F_0 \otimes N.$$

Since F_0 is flat, we have $\operatorname{Tor}_1^R(F_0, N) = 0$, which implies

$$\operatorname{Tor}_1^R(M, N) = \ker(\iota_0 \otimes \operatorname{Id}_N) \cong H_1(F_\bullet \otimes N).$$

Let us now inductively assume that $\operatorname{Tor}_n^R(M, N) \cong H_n(F_\bullet \otimes N)$ for any flat resolution of any object in $\mathcal{A}$. We have part of the exact sequence

$$\mathrm{Tor}_{n+1}^R(F_0, N) \to \mathrm{Tor}_{n+1}^R(M, N) \to \mathrm{Tor}_n^R(K_0, N) \to \mathrm{Tor}_n(F_0, N).$$

As F_0 is flat, we have $\mathrm{Tor}_{n+1}^R(M, N) \cong \mathrm{Tor}_n(K_0, N)$. Now, $\cdots \to F_2 \to F_1 \to K_0 \to 0$ is a flat resolution of K_0, say, $F'_\bullet \to K_0$. Hence, by induction

$$\mathrm{Tor}_{n+1}^R(M, N) \cong \mathrm{Tor}_n^R(K_0) \cong H_n(F'_\bullet \otimes N) = H_{n+1}(F_\bullet \otimes N).$$

This proves the claim. □

16.2 Right Derived Functors and Ext

Simlar to the left derived functors of a right exact functor, we can compute the right derived functors of a left exact functor. In particular, suppose $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Suppose $\mathcal{A}$ has enough injectives. Given $A \in \mathcal{A}$, fix some injective resolution $A \xrightarrow{\varepsilon} I^\bullet$. Then, the i^{th} *right derived functor* of F is defined as

$$\mathfrak{R}^i F(A) := H^i(F(I_\bullet)) = H^i(0 \to F(I^0) \to F(I^1) \to \cdots).$$

Just like the left derived functors, it follows that $\mathfrak{R}^i F : \mathcal{A} \to \mathcal{B}$ is a well-defined functor (up to isomorphism), and $\mathfrak{R}^0 F = F$ naturally. Moreover, we have the following.

Theorem 16.3: (*Long Exact Sequence of Right Derived Functors*)

Left $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor between Abelian categories, where $\mathcal{A}$ has enough injectives. Then, given a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, there exists a natural long exact sequence

$$0 \to \underbrace{\mathfrak{R}^0 F(A)}_{F(A)} \to \underbrace{\mathfrak{R}^0 F(B)}_{F(B)} \to \underbrace{\mathfrak{R}^0 F(C)}_{F(C)} \to \mathfrak{R}^1 F(A) \to \mathfrak{R}^1 F(B) \to \mathfrak{R}^1 F(C) \to \cdots.$$

The (internal) $\hom$ functor in the category of modules, both the covariant and contravariant ones, are well-known left exact functors.

Definition 16.4: (*Ext Functor*)

For a fixed R-module, the right derived functors of the (covariant) $\hom$-functor $\hom_R(M, \_) : R\text{-Mod} \to R\text{-Mod}$ are called the *Ext functors*, and is denoted as $\mathrm{Ext}_R^n(M, \_) : R\text{-Mod} \to R\text{-Mod}$.

Similarly, for fixed R-module N, the right derived functors of the *contravariant* $\hom$-functor $\hom_R(\_, N) : R\text{-Mod} \to R\text{-Mod}$ are also called Ext functors, denoted as $\mathrm{Ext}_R^n(\_, N) : R\text{-Mod} \to R\text{-Mod}$. We can do this using a left projective resolution as well. Indeed, given a projective resolution $P_\bullet \to M$, it follows that $\mathrm{Ext}_R^i(M, N) = H_i(\hom(P_\bullet, N))$. As a consequence, if R is a PID, it follows that $\mathrm{Ext}_i^R(M, N) = 0$ for $i \geq 2$, and for $R = \mathbb{Z}$ we simply denote

$$\mathrm{Ext}(M, N) := \mathrm{Ext}_{\mathbb{Z}}^1(M, N), \qquad M, N \in \mathbb{Z}\text{-Mod} = \mathrm{Ab}.$$

In general, we have a bi-functor

$$\mathrm{Ext}_R^n(\_, \_) : R\text{-Mod} \to R\text{-Mod},$$

which is contravariant on the first vairable, and covariant in the second. Of couse, this requires *balancing of Ext*, i.e, we need to show that $\mathfrak{R}^i(\hom_R(M, \_))(N) \cong \mathrm{Ext}_R^i(M, N) \cong \mathfrak{R}^i(\hom_R(\_, N))(M)$.

16.3 Ext over a PID

Over a PID, injective modules are characterized as *divisible* ones.

Definition 16.5: (*Divisible Module*)

An R-module M is called *divisible* if given any regular element $0 \neq r \in R$ (i.e, $r \in R$ is a non-zero-divisor), and any $m \in M$, there is some $x \in M$ such that $rx = m$ holds. In other words, $rM = M$ for any regular element $r \in R$.

In general, when considering divisible modules, we assume that the ring is an *integral domain* (i.e, whenever $r_1 r_2 = 0$ for some $r_1, r_2 \in R$, we have $r_1 = 0$ or $r_2 = 0$). In this case, every nonzero element is a non-zero-divisor. Note that a PID is by definition an integral domain.

Theorem 16.6: (*Injective Modules over PID are Divisible*)

If R is an integral domain, then an injective R-module I is divisible. Moreover, if R is a PID, a divisible R-module D is injective.

Proof : Suppose R is an integral domain, and I is an injective R-module. Let $0 \neq r \in R$ and $m \in I$ be fixed. Consider the map $f : R \to R$ given by (left) multiplication by. Since $r \neq 0$ is not a zero-divisor, it follows that f is injective. Also consider the map $\varphi : R \to M$ given by $\varphi(1) = x$ (and extending linearly). As I is injective (Definition 14.25), we have an R-linear map $\Phi : R \to I$ solving the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & R & \xrightarrow{f} & R \\ & & \downarrow{\scriptstyle \varphi} & \swarrow{\scriptstyle \Phi} & \\ & & I & & \end{array}$$

Consider $x = \Phi(1)$. Then,

$$rx = r\Phi(1) = \Phi(r.1) = \Phi \circ f(1) = \varphi(1) = m.$$

As $0 \neq r \in R$ and $m \in I$ are arbitrary, we have I is divisible.

Now, suppose R is a PID and D is a divisible R-module. Let us consider a monomorphism $f : A \to B$ and a map $\varphi : A \to D$. To show the existence of $\Phi : B \to D$ such that $\Phi \circ f = \varphi$, we need to apply Zorn's lemma. As f is monic, without loss of generality, by identifying A with $f(A)$, we can assume that A is a submodule of B. Consider the family

$$\mathcal{F} = \{(C, \psi) \mid A \subset C \subset B \text{ as submodules}, \psi : C \to D, \psi|_A = \varphi\}.$$

The family is nonempty since $(A, \varphi) \in \mathcal{F}$. We have a partial order $\preceq$ on $\mathcal{F}$: for $(C_1, \psi_1), (C_2, \psi_2) \in \mathcal{F}$, we have $(C_1, \psi_1) \leq (C_2, \psi_2)$ if and only if $C_1 \subset C_2$ and $\psi_2|_{C_1} = \psi_1$. Suppose $\mathcal{J} = \{(C_i, \psi_i)\}$ is a chain in $\mathcal{F}$ with respect to $\preceq$. Let $C = \bigcup C_i$, and define $\psi : C \to D$ by $\psi|_{C_i} = \psi_i$. It is easy to see that C is a submodule of B with $A \subset C \subset B$, and ψ is a well-defined R-module map, such that $\psi|_A = \varphi$. Thus $(C, \psi) \in \mathcal{F}$ is an upper bound of the chain $\mathcal{J}$. Then by Zorn's lemma, there exists a maximal element $(C_0, \psi_0) \in \mathcal{F}$. We claim that $C_0 = B$. If not, then there is an element $x \in B \setminus C_0$. Let $C \subset B$ be the submodule generated by $C_0 \cup \{x\}$. Clearly, $A \subset C_0 \subsetneq C \subset B$. Let $I = \{r \in R \mid rx \in C_0\}$. Clearly, I is an ideal in R, and hence, $I = \langle a \rangle$ for some $a \in R$ (as R is a PID).

- If $a \neq 0$, divisibility of D implies that there is some $y \in D$ such that $ay = \psi_0(ax)$.
- If $a = 0$, choose some arbitrary $y \in D$.

In any case, we can now extend ψ_0 to $\psi : C \to D$ by setting $\psi(x) = y$. That is, define $\psi(c_0 + rx) = \psi_0(c_0) + ry$ for $c_0 + rx \in C$. Let us verify that this is well-defined. Indeed, if $rx \in C_0$ for some $r \in R$, we have $r \in I = \langle a \rangle \Rightarrow r = sa$ for some $s \in R$. Then, $\psi(rx) = ry = say = s\psi_0(ax) = \psi_0(sax) = \psi_0(rx)$. Thus, $\psi : C \to D$ is well-defined. This contradicts the maximality of (C_0, ψ_0). Hence, $C_0 = B$, and clearly $\Phi = \psi_0$ is an extension of φ as required. This proves that D is an injective module. □

Example 16.7: (*Injective Abelian Groups*)

It is easy to verify that $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{R}$ are divisible Abelian groups. Since $\mathbb{Z}$ is a PID, injective $\mathbb{Z}$-modules are precisely the divisible groups (Theorem 16.6). Hence, we have $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{R}$ as examples of injective $\mathbb{Z}$-modules.

Recall the fact that every R-module embeds in an injective module, i.e, R-Mod has enough injectives (Definition 15.1). Let us give a proof of this for $\mathbb{Z}$. The general statement is proved in Corollary 16.17.

Proposition 16.8: (*Enough Injective Abelian Groups*)

Every Abelian group can be embedded in a direct product of $\mathbb{Q}/\mathbb{Z}$, which is seen to be a divisible (and hence an injective) Abelian group.

Proof : Let A be an Abelian group. Fix $0 \neq A$, and consider the subgroup $\langle a \rangle \hookrightarrow A$.

- If a has infinite order in A, choose arbitrary $0 \neq x_a \in \mathbb{Q}/\mathbb{Z}$.
- If a has finite order, say, $n \geq 2$ in A, choose $0 \neq x_a \in \mathbb{Q}/\mathbb{Z}$ such that $\mathrm{ord}(x_a) \mid n$.

In any case, we have a map $\varphi_a : \langle a \rangle \to \mathbb{Q}/\mathbb{Z}$ defined by setting $\varphi(a) = x_a$. Since $\mathbb{Q}/\mathbb{Z}$ is divisible and hence injective, extend the map φ_a to a map $\Phi_a : A \to \mathbb{Q}/\mathbb{Z}$. Then, using the universal property of product, define a map $\Phi : A \to \prod_{0 \neq a \in A} \mathbb{Q}/\mathbb{Z}$ where each component is Φ_a. Clealry direct product of divisible groups is again divisible, and hence $\prod_{0 \neq a \in A} \mathbb{Q}/\mathbb{Z}$ is injective. Moreover, Φ is clearly injective. Indeed, if for some $0 \neq a \in A$ we have $\Phi(a) = 0$, then in particular, $0 = \varphi_a(a) = x_a$, a contradiction. Thus, we can embed A in to an injective Abelian group. □

We now note the following interesting result for a PID.

Proposition 16.9: (*Quotient of Injective*)

Let R be a PID. Then, any quotient of an injective R-module is again injective.

Proof : Since injective modules are precisely the divisible ones for a PID, let us consider D to be a divisible R-module, and let $\varphi : D \to E$ be an epimorphism. Let us show that E is divisible. Let $0 \neq r \in R$ and $e \in E$ be given. Choose some $d \in D$ such that $\varphi(d) = e$. Since D is divisible, there is some $y \in D$ such that $ry = d$. Let $x = \varphi(x) \in E$. Clearly, $rx = r\varphi(y) = \varphi(ry) = \varphi(d) = e$. Thus, E is divisible, and hence, an injective R-module. □

As a corollary, we have the following.

Corollary 16.10: ($\mathrm{Ext}^n_R = 0$ *for* $n \geq 2$ *over a PID* R)

Let R be a PID. Given any R-module N, there is an injective resolution $0 \to N \to I^0 \to I^1 \to 0$, and consequently, $\mathrm{Ext}^n_R(M, N) = 0$ for any $M \in R\text{-Mod}$ and $n \geq 2$.

Proof : Let N be an arbitrary R-module. We embed $0 \to N \overset{\eta}{\hookrightarrow} I^0$, where I^0 is injective. Let $I^1 = \mathrm{coker}(\eta)$, which is then again injective. Clearly, we have an exact sequence $0 \to N \to I^0 \to I^1 \to 0$, which is then an injective resolution.

As we have an injective resolution of length 2, it follows from the construction that $\mathrm{Ext}^n_R(M, N) = 0$ for any R-module M, and $n \geq 2$. □

Exercise 16.11: (*Computation of Ext*)

Compute the following.

1. $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, G)$ for any Abelian group G and $a > 1$.
2. $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z})$ for $a, b > 1$.
3. $\mathrm{Ext}^n_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$ for any Abelian group G.
4. $\mathrm{Ext}^n_k(V, W)$ for a field k and k-modules V, W.

Example 16.12: (*Group Cohomology*)

Let us give the dual example to Example 15.14. Let G be a group, R be a ring, and M be an $R[G]$-module. Define the *invariants* as

$$M^G := \{m \in M \mid g \cdot m = m, \forall g \in G\}.$$

This defines a functor $(\cdot)^G : R[G]\text{-Mod} \to R[G]\text{-Mod}$, which can be identified with $\mathrm{hom}_{R[G]}(R, \_)$. Thus, invariants being a hom functor is left exact. The right derived functors, denoted as $H^i(G, M)$ is called the *group cohomology* of G with coefficients in M. Clearly, $H^i(G, M) = \mathrm{Ext}^i_{R[G]}(R, M)$.

Remark 16.13: (*Hochschild (co)Homology*)

Another interesting class of derived functors appear in Algebra known as Hochschild (co)homology. In particular, given a possibly noncommutative algebra A over a ring R, and an A-bimodule, one defines the *Hochschild cohomology* as $HH^i(A, M) := \mathrm{Ext}^i_{A^{\mathrm{op}} \otimes_k A}(A, M)$, and the *Hochschild homology* as $HH_i(A, M) := \mathrm{Tor}_i^{A^{\mathrm{op}} \otimes_k A}(A, M)$. These are important tools in deformation theory of algebras.

Exercise 16.14: (*Ext and Projective/Injective*)

Let A be an R-module. Show that

1. A is projective if and only if $\mathrm{Ext}^1_R(M, A) = 0$ for all R-module M, and
2. A is injective if and only if $\mathrm{Ext}^1_R(A, N) = 0$ for all R-moduls N.

Remark 16.15: (*Interpreting Ext^n_R As Extensions*)

Given R-modules A, C, we call $B \in R\text{-Mod}$ to be an *extension* of C by A if we have a short exact sequence $0 \to A \to B \to C \to 0$. Call two extensions B_1, B_2 *equivalent* if there is an isomorphism $\varphi : B_1 \to B_2$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \nearrow & B_1 & \searrow & C & \longrightarrow & 0 \\ & & & \searrow & \downarrow \varphi & \nearrow & & & \\ & & & & B_2 & & & & \end{array}$$

One can verify that the equivalence classes of extensions is in bijection with the set $\mathrm{Ext}^1_R(C, A)$, where 0 corresponds to split extensions. In particular, if $\mathrm{Ext}^1_R(C, A) = 0$, then every extension of C by A splits.

Higher Ext modules are related to n-extensions, namely, an exact sequence $0 \to A \to B_1 \to \cdots \to B_n \to C$. Consider the equivalence relation on all n-extensions *generated* by the relation that there is a chain map

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & A & \longrightarrow & B_1 & \longrightarrow & \cdots & \longrightarrow & B_n & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & B_1 & \longrightarrow & \cdots & \longrightarrow & B_n & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Then, $\mathrm{Ext}^n_R(C, A)$ is in bijection with the equivalence classes of all n-extensions of C by A.

16.4 A Digression : Enough Injective R-Modules

Let us give a category theoretic proof of the fact that for any ring R, the category of R-modules has enough injectives! First, we have the following useful result.

Proposition 16.16: (*Adjoint and Injectives*)

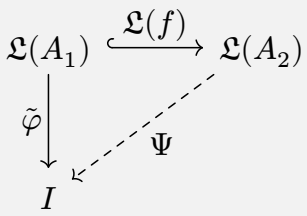
Suppose we have an adjunction $\mathfrak{L} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathfrak{R}$ of additive functors $\mathfrak{L}, \mathfrak{R}$ between two Abelian categories $\mathcal{A}, \mathcal{B}$. Assume that

- $\mathfrak{L}$ is *exact*, i.e, takes short exact sequence in $\mathcal{A}$ to short exact sequence in $\mathcal{B}$ (in particular, assume $\mathfrak{L}$ is left exact as any left adjoint is alsways right exact), and
- $\mathfrak{L}$ is *faithful*, i.e, given any $A_1, A_2 \in \mathcal{A}$ the induced map $\hom_{\mathcal{A}}(A_1, A_2) \to \hom_{\mathcal{B}}(\mathfrak{L}(A_1), \mathfrak{L}(A_2))$ is injective.

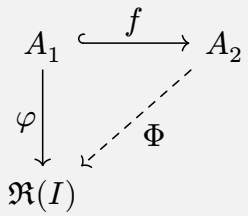
Then, $\mathcal{A}$ has enough injectives, provided $\mathcal{B}$ has enough injectives.

Proof : Suppose $\mathcal{B}$ has enough injectives. Let $A \in \mathcal{A}$. Then, there exists an injective object $I \in \mathcal{B}$ and a monomorphism $\iota : \mathfrak{L}(A) \hookrightarrow I$. By the adjuction $\hom_{\mathcal{B}}(\mathfrak{L}(A), I) \cong \hom_{\mathcal{A}}(A, \mathfrak{R}(I))$, we have a morphism $\tilde{\iota} : A \to \mathfrak{R}(I)$. We claim the following.

- **$\mathfrak{R}(I)$ is injective in $\mathcal{A}$:** We show that the right adjoint of an exact functor preserves injectives. Indeed, consider a monomorphism $f : A_1 \hookrightarrow A_2$ and a map $\varphi : A_1 \to \mathfrak{R}(I)$ in $\mathcal{A}$. Applying the adjunction, and the fact $\mathfrak{L}$ is exact (and in particular, preseves monomorphisms), we have the diagram



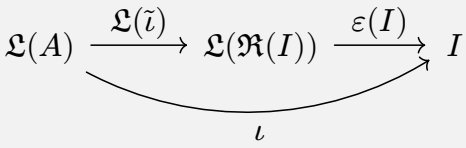
The extension Ψ exists since I is injective. But then again by the adjunction, we have a map $\Phi = \tilde{\Psi} : A_2 \to I$ solving



- **$\tilde{\iota}$ is a monomorphism:** We have an exact sequence $0 \to \ker(\tilde{\iota}) \to A \xrightarrow{\tilde{\iota}} \mathfrak{R}(I)$. Since $\mathfrak{L}$ is exact, we have an exact sequence

$$0 \to \mathfrak{L}(\ker(\tilde{\iota})) \to \mathfrak{L}(A) \xrightarrow{\mathfrak{L}(\tilde{\iota})} \mathfrak{L}(\mathfrak{R}(I)).$$

Now, using the unit $\varepsilon : \mathfrak{L} \circ \mathfrak{R} \Rightarrow \mathrm{Id}_{\mathcal{B}}$ of the adjunction, it folows that the composition is



Since the composition is monic, it follows that $\mathfrak{L}(\tilde{\iota})$ is also monic. But then $\mathfrak{L}(\ker(\tilde{\iota})) \to \mathfrak{L}(A)$ is the 0 map. Since $\mathfrak{L}$ is assumed to be faithful, it follows that $\ker(\tilde{\iota}) \to A$ is also the 0 map. In other words, ι is monic.

Thus, we have an embedding $\iota : A \hookrightarrow \mathfrak{R}(I)$, where $\mathfrak{R}(I)$ is injective in $\mathcal{A}$. Since $A \in \mathcal{A}$ was arbitrary, it follows that $\mathcal{A}$ has enough injectives. □

Then, as corollary we have that any module category has enough injectives.

Corollary 16.17: (R-Mod *has Enough Injectives*)

Given any ring R, the category R-Mod has enough injectives.

Proof : Consider the *forgetful functor* $U : R\text{-Mod} \to \mathbb{Z}\text{-Mod} = \mathrm{Ab}$.

- U is evidently additive.
- U has a right adjoint $\mathfrak{R} : \mathrm{Ab} \to R$-Mod given by *coextension of scalars*. Explicitly, we have $\mathfrak{R}(G) = R \otimes_{\mathbb{Z}} G$, where we treat the ring R as a $\mathbb{Z}$-module; then $\mathfrak{R}(G)$ is an R-module via the multiplication in R. Clearly $\mathfrak{R}$ is additvie as well.
- U being a left adjoint, is right exact. Moreover, any monomorphism of R-module is clearly an injective Abelian group map. Hence, U is exact.
- U is clearly faithful since any R-module map being 0 as a group map implies the map itself is 0.

Now, by Proposition 16.8, we have Ab has enough injectives. But then by Proposition 16.16, we have R-Mod has enough injectives. □

Unlike Proposition 16.9, over a general ring R it does not follow that a quotient of an injective module is again injective. Thus, for an aribitrary ring R, higher Ext modules may not vanish.

16.5 Universal Coefficient Theorems

The goal of this section is to compute the (co)homology of a a free chain complex after tensoring (or taking $\hom$) by an R-module. This leads to the definition of (co)homology with coefficients. We shall always assume that R is a PID; there are analogous theorems for a general ring R, which are usually stated via spectral sequences. If there is no confusion, we shall simply denote $\mathrm{Tor} = \mathrm{Tor}_1^R$ and $\mathrm{Ext} = \mathrm{Ext}_R^1$.

Theorem 16.18: (*UCT for Homology with Coefficients*)

Let R be a PID, $C_\bullet$ be a chain complex of free R-modules, and M be an arbitrary R-module. Then, there exists a short exact sequence

$$0 \to H_n(C_\bullet) \otimes M \to H_n(C_\bullet \otimes M) \to \mathrm{Tor}(H_{n-1}(C_\bullet), M) \to 0,$$

which is natural in both $C_\bullet$ and the coefficient module M. Moreover, the short exact sequence splits, the splitting is natural in the coefficient module, but may not be natural with respect to the chain complex.

Proof : Let us denote the module of cycles, boundaries, and the homology as

$$Z_n := \ker\left(C_n \xrightarrow{\partial} C_{n-1}\right), \qquad B_n := \mathrm{im}\left(C_{n+1} \xrightarrow{\partial} C_n\right), \qquad H_n := H_n(C_\bullet) = B_n/Z_n.$$

Since R is a PID, both Z_n and B_n are free, being submodules of the free R-module C_n (Proposition 14.7). In particular, we have a short exact sequence

$$0 \to Z_n \to C_n \to B_{n-1} \to 0,$$

which is split as B_{n-1} is a free (and hence projective) R-module. Hence, tensoring by M, we again get a split exact sequence

$$0 \to Z_n \otimes M \to C_n \otimes M \to B_{n-1} \otimes M \to 0.$$

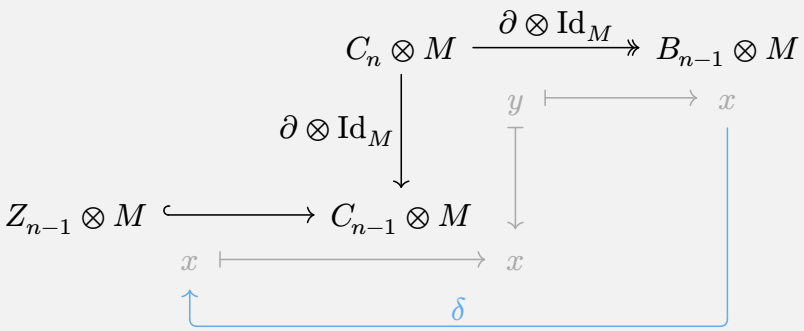
Treating $Z_\bullet \otimes M$ and $B_\bullet \otimes M$ as a chain complex with 0-differentials, and $C_\bullet \otimes M$ with the differential $\partial \otimes \mathrm{Id}_M$, we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes M \to C_\bullet \otimes M \to B_\bullet \otimes M[-1] \to 0,$$

where $B_\bullet \otimes M$ is shifted by degree -1. Applying Theorem 7.1, we then have a long exact seuqence in homology

$$\cdots \to B_n \otimes M \xrightarrow{\delta} Z_n \otimes M \to H_n(C_\bullet \otimes M) \to B_{n-1} \otimes M \xrightarrow{\delta} Z_{n-1} \otimes M \to \cdots$$

Here δ is the boundary map of the long exact sequence. Recall from Remark 7.2, we have the diagram



Hence, δ is nothing but the map $\iota \otimes \mathrm{Id}_M$, where $\iota : B_n \hookrightarrow Z_n$ is the inclusion.

We also have a short exact sequence

$$0 \to B_n \to Z_n \to H_n \to 0.$$

Since B_n, Z_n are free, we can treat this as a free (hence projective) resolution of H_n. In particular, it follows that

$$\mathrm{Tor}(H_n, M) \cong \ker(\iota \otimes \mathrm{Id}_M : B_n \otimes M \to Z_n \otimes M) = \ker(\delta).$$

Also, exactness of $B_n \otimes M \to Z_n \otimes M \to H_n \otimes M \to 0$ implies that

$$\mathrm{coker}(\delta) = \mathrm{coker}(\iota \otimes \mathrm{Id}_M) \cong H_n \otimes M.$$

Then, splicing the homology long exact sequence we get

$$0 \to H_n \otimes M \to H_n(C_\bullet \otimes M) \to \mathrm{Tor}\ (H_n, M) \to 0,$$

which is the required short exact sequence.

The naturality of the short exact sequence follows from the functoriality of Tor and the above construction. As for the splitting, *choose* a splitting $s_n : B_{n-1} \to C_n$ of the exact sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$, which gives a splitting $\sigma_n := s_n \otimes \mathrm{Id}_M$ of the exact sequence after tensoring. In particular, $\sigma_\bullet$ is a splitting for the short exact sequence of the chain complexes, which induces a splitting in the homology exact sequence. Since s_n was a choice independent of the module M, it is natural with respect to the coefficients, but may not be natural with respect to the chain complex. □

Let us explain the meaning of the splitting being *not* natural.

Example 16.19: (*Unnaturality of the Splitting in Homology UCT*)

Let us consider the map of chain complexes

$$\begin{array}{ccccccccccccc}
C_\bullet & : & & \overset{(4)}{0} & \longrightarrow & \overset{(3)}{0} & \longrightarrow & \overset{(2)}{\mathbb{Z}} & \xrightarrow{\times 2} & \overset{(1)}{\mathbb{Z}} & \xrightarrow{0} & \overset{(0)}{\mathbb{Z}} & \longrightarrow & 0 \\
{\scriptstyle f_\bullet}\downarrow & & & \downarrow & & \downarrow & & \| & & \downarrow & & \| & & \\
D_\bullet & : & & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow[\times 2]{} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Note that the homology is given as

$$H_n(C_\bullet) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}/2\mathbb{Z}, & n = 1 \\ 0, & \text{otherwise.} \end{cases} \qquad H_n(D_\bullet) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}/2\mathbb{Z}, & n = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the map $f_\bullet$ induces 0 everywhere except Id on $H_0(C_\bullet) \to H_0(D_\bullet)$. Next, tensoring by $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, we have the diagram

$$\begin{array}{rccccccccccc}
C_\bullet \otimes \mathbb{Z}_2\text{:} & \overset{(4)}{0} & \longrightarrow & \overset{(3)}{0} & \longrightarrow & \overset{(2)}{\mathbb{Z}_2} & \xrightarrow{0} & \overset{(1)}{\mathbb{Z}_2} & \xrightarrow{0} & \overset{(0)}{\mathbb{Z}_2} & \longrightarrow & 0 \\
{\scriptstyle f_\bullet \times \mathrm{Id}_{\mathbb{Z}_2}}\downarrow & \downarrow & & \downarrow & & \| & & \downarrow & & \| & & \\
D_\bullet \otimes \mathbb{Z}_2\text{:} & 0 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow[0]{} & \mathbb{Z}_2 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0
\end{array}$$

We have the homology with coefficients

$$H_n(C_\bullet \otimes \mathbb{Z}_2) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & n = 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases} \qquad H_n(D_\bullet \otimes \mathbb{Z}_2) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & n = 0, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

Passing to homology, we now have an extra identity map $\mathrm{Id} : H_2(C_\bullet \otimes \mathbb{Z}_2) \to H_2(D_\bullet \otimes \mathbb{Z}_2)$ induced by $f_\bullet \otimes \mathrm{Id}_{\mathbb{Z}_2}$.

Looking at the short exact sequence from Theorem 16.18, we have the diagram

$$\begin{array}{ccccccccc}
 & & & \overset{0}{} & & \overset{\mathbb{Z}_2}{} & & \overset{\mathbb{Z}_2}{} & \\
0 & \longrightarrow & H_2(C_\bullet) \otimes \mathbb{Z}_2 & \longrightarrow & H_2(C_\bullet \otimes \mathbb{Z}_2) & \longrightarrow & \mathrm{Tor}(H_1(C_\bullet), \mathbb{Z}_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_2(D_\bullet) \otimes \mathbb{Z}_2 & \longrightarrow & H_2(D_\bullet \otimes \mathbb{Z}_2) & \longrightarrow & \mathrm{Tor}(H_1(D_\bullet), \mathbb{Z}_2) & \longrightarrow & 0 \\
 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & 0 & &
\end{array}$$

If the splitting were natural, then we must have a commutative diagram

$$\begin{array}{ccc}
H_2(C_\bullet \otimes \mathbb{Z}_2) & \xrightarrow{\cong} & H_2(C_\bullet) \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}\,(H_1(C_\bullet), \mathbb{Z}_2) \\
\mathrm{Id}\downarrow & & \downarrow 0 \oplus 0 \\
H_2(C_\bullet \otimes \mathbb{Z}_2) & \xrightarrow[\cong]{} & H_2(C_\bullet) \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}\,(H_1(C_\bullet), \mathbb{Z}_2)
\end{array}$$

where the isomorphisms are induced by the splitting. But clearly the diagram does not commute, which shows that the splitting in UCT is not natural in the chain complexes.