

Algebraic Topology II (KSM4E02)

Instructor: Aritra Bhowmick

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cellular chain – cellular homology – cellular homology of $\mathbb{CP}^\infty$

12.1 Cellular Chain Complex and Homology

Let $H_\star$ be an additive ordinary homology theory with $H_0(\star) = \mathbb{Z}$. As example, we can assume $H_\bullet$ to be the singular homology (with $\mathbb{Z}$ -coefficient). Suppose X is obtained from $A \subset X$ by attaching n -cells via the map

$$(\Phi, \varphi) : \left(\bigsqcup_{\alpha \in \Lambda} D_\alpha^n, \bigsqcup_{\alpha \in \Lambda} S_\alpha^{n-1} \right) \rightarrow (X, A).$$

Proposition 12.1: (Homology Isomorphism and Attaching Cells)

The induced map

$$\sum H_\star(\Phi_\alpha) : \bigoplus_{\alpha \in \Lambda} H_\star(D_\alpha^n, S_\alpha^{n-1}) \rightarrow H_\star(X, A)$$

is an isomorphism.

Proof : Recall that Φ is a *relative homeomorphism*, i.e, a homeomorphism after taking quotient on both sides. Thus, we have the diagram

$$\begin{array}{ccccc} \bigoplus H_\star(D_\alpha^n, S_\alpha^{n-1}) & \xrightarrow{\cong} & H_\star(\bigsqcup D_\alpha^n, \bigsqcup S_\alpha^{n-1}) & \xrightarrow{H_\star(\Phi)} & H_\star(X, A) \\ & & \downarrow \cong & & \downarrow \cong \\ & & \tilde{H}_\star(\bigvee D_\alpha^n / S_\alpha^{n-1}) & \xrightarrow[\tilde{H}_\star(\tilde{\Phi})]{\cong} & \tilde{H}_\star(X/A) \end{array}$$

By [Theorem 10.14](#), the vertical quotient maps induce isomorphism in homology. The commutativity shows that $H_\star(\Phi)$ is an isomorphism. The proof then follows since $H_\star$ is assumed to be an additive homology, which gives the isomorphism on the left. $\square$

Let us describe an inverse map as well. For $\alpha \in \Lambda$, we have an inclusion $p_\alpha : (X, A) \hookrightarrow (X, X \setminus E_\alpha^n)$, where $E_\alpha^n = \Phi(\mathring{D}_\alpha^n)$, and the relative homeomorphism $\Phi_\alpha : (D_\alpha^n, S_\alpha^{n-1}) \rightarrow (X, X \setminus E_\alpha^n)$. Thus, we have the maps

$$H_k(X, A) \xrightarrow{H_k(p_\alpha)} H_k(X, X \setminus E_\alpha^n) \xleftarrow[\cong]{H_k(\Phi_\alpha)} H_k(D_\alpha^n, S_\alpha^{n-1}).$$

The image under this map (after inverting the isomorphism), gives a map $z \mapsto (z_\alpha)_{\alpha \in \Lambda}$. It follows that this is precisely the inverse of the isomorphism in [Proposition 12.1](#).

Now, suppose X is a CW complex. From the triple (X^{n+1}, X^n, X^{n-1}) , we have the boundary map

$$\partial : H_{k+1}(X^{n+1}, X^n) \rightarrow H_k(X^n, X^{n-1}).$$

It is easy to see that $\partial^2 = 0$. Indeed, the boundary maps of the triples are obtained from the diagram

$$\begin{array}{ccccc} & & H_k(X^n) & & \\ & \nearrow & & \searrow & \\ H_{k+1}(X^{n+1}, X^n) & \xrightarrow{\partial} & H_k(X^n, X^{n-1}) & \xrightarrow{\partial} & H_{k-1}(X^{n-1}, X^{n-2}) \\ & & \searrow & & \nearrow \\ & & H_{k-1}(X^{n-1}) & & \end{array}$$

Clearly, the consecutive red and blue maps compose to 0 being part of the long exact sequence of (X^n, X^{n-1}) . Thus, $\partial^2 = 0$.

Definition 12.2: (Cellular Chain Complex and Homology)

Given a CW complex X , the **cellular chain complex** is defined as $C_n^{\text{cell}}(X) := H_n(X^n, X^{n-1})$, with the boundary map $\partial_n : H_n(X^n, X^{n-1}) \rightarrow H_{n-1}(X^{n-1}, X^{n-2})$ obtained from the long exact sequence of the triple (X^{n+1}, X^n, X^{n-1}) . The homology of the chain complex is called the **cellular homology** of X , and is denoted as $H_\bullet^{\text{cell}}(X)$.

Exercise 12.3: (Cellular Chain as Free Groups on Cells)

Verify that the the cellular chain group $C_n^{\text{cell}}(X) = H_n(X^n, X^{n-1})$ is canonically identified with the free Abelian group generated by the n -cells in X . That is, verify that $C_n^{\text{cell}}(X) = \bigoplus_{\{\alpha \in \Lambda\}} \mathbb{Z}\langle e_\alpha^n \rangle$, where $\{e_\alpha^n \mid \alpha \in \Lambda\}$ is the collection of n -cells in the CW decomposition of X .

Proposition 12.4: (Cellular Homology is Functorial and Homotopy Invariant)

A cellular map $f : X \rightarrow Y$ induces a chain map $C_\bullet^{\text{cell}}(X) \rightarrow C_\bullet^{\text{cell}}(Y)$. Homotopic chain maps induce a chain homotopy.

Proof : Let $f : X \rightarrow Y$ be a chain map, i.e, $f(X^n) \subset Y^n$. Then, we have induced maps $H_n(f) : H_n(X^{n+1}, X^n) \rightarrow H_n(Y^{n+1}, Y^n)$ which commutes with boundary by the naturality of the long exact sequence of triple ([Theorem 3.8](#)).

Next, suppose $h : X \times I \rightarrow Y$ is a homotopy between two cellular maps $f, g : X \rightarrow Y$. Since $X \times I$ is a CW complex, by [Theorem 11.2](#), we can assume that $\varphi : X \times I \rightarrow Y$ is a cellular map as well. Note that $(X \times I)^n = X^n \times \partial I \cup X^{n-1} \times I$. Consider the map

$$s_n : H_n(X^{n+1}, X^n) \xrightarrow{\sigma} H_{n+1}(X^{n+1} \times I, X^{n+1} \times \partial I \cup X^n \times I) \xrightarrow{H_{n+1}(\varphi)} H_{n+1}(Y^{n+1}, Y^n),$$

where the first σ is the isomorphism from [Lemma 13.11](#) for the pair (X^{n+1}, X^n) . Working through the diagrams involving the triple (X^{n+1}, X^n, X^{n-1}) , one can show that $\partial s_n = H_n(g) - H_n(f) - s_{n-1}\partial$. That is, $s_\bullet$ is a chain homotopy. $\square$

Example 12.5: (Cellular Homology of $\mathbb{CP}^n$)

Recall the complex projective space $\mathbb{CP}^n$ is defined as the space of complex lines (i.e, 1-dimensional complex vector subspaces) of $\mathbb{C}^{n+1}$. In other words, $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus 0)/\sim$, where

$$(z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n), \quad 0 \neq \lambda \in \mathbb{C}.$$

We denote the equivalence class as $[z_0 : \dots : z_n]$, which gives rise to the *homogeneous coordinates* on $\mathbb{CP}^n$, making it into an n -dimensional complex manifold. Clearly, $\mathbb{CP}^0 = \{\star\}$ is a singleton.

Since every line is determined by the unit vector (up to sign) on the line, restricting to norm one vectors, we can equivalently get $\mathbb{CP}^n = S^{2n+1}/S^1$. Note that the unit vectors in $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is the $2n+1$ -sphere, and $S^1 = \{e^{i\theta} \mid 0 \leq \theta < 2\pi\}$ acts freely as an (Abelian) group. This is generalization of the antipode action which gives $\mathbb{RP}^n = S^n/\mathbb{Z}_2 = S^n/S^0$.

It is easy to see that the inclusion $\mathbb{C}^{n+1} \hookrightarrow \mathbb{C}^{n+2}$ as the hyperplane $\{z_{n+2} = 0\}$ gives rise to an inclusion

$$\begin{aligned} \mathbb{CP}^n &\hookrightarrow \mathbb{CP}^{n+1} \\ [z_0 : \dots : z_n] &\longmapsto [z_0 : \dots : z_n : 0]. \end{aligned}$$

Thus, we have

$$\{\star\} = \mathbb{CP}^0 \hookrightarrow \mathbb{CP}^1 \hookrightarrow \dots \hookrightarrow \mathbb{CP}^\infty,$$

where $\mathbb{CP}^\infty$ is given the weak topology from the inclusions.

Let us now give a CW decomposition of $\mathbb{CP}^n$. We claim $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a $2n$ -cell. Note that the inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ gives the inclusion $S^{2n-1} \hookrightarrow S^{2n+1}$, which induces the same map $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$. Then, any point $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$ is *uniquely* written as $[z_0 : \dots : z_{n-1} : \sqrt{1 - \|\mathbf{z}\|^2}]$, where $\mathbf{z} = (z_0, \dots, z_{n-1}) \in \mathbb{C}^n$ satisfies $\|\mathbf{z}\|^2 := \sum_{j=0}^n z_j \bar{z}_j \leq 1$. Treating $\mathbb{D}^{2n} \subset \mathbb{C}^n$ as the closed unit ball, we have a continuous map

$$\begin{aligned} \Phi : \mathbb{D}^{2n} &\longrightarrow \mathbb{CP}^n \\ \mathbf{z} &\longmapsto [z_0 : \dots : z_n : \sqrt{1 - \|\mathbf{z}\|^2}]. \end{aligned}$$

The boundary map $\varphi := \Phi|_{S^{2n-1}}$ maps to $\mathbb{CP}^{n-1}$ since the last coordinate is 0. In the interior, Φ is a homeomorphism, since we can define the inverse map

$$\begin{aligned} \mathbb{CP}^n \setminus \mathbb{CP}^{n-1} &\longrightarrow \mathring{\mathbb{D}}^{2n} \\ [z_0 : \dots : z_n] &\longmapsto \left(\frac{z_0}{z_n}, \dots, \frac{z_{n-1}}{z_n} \right). \end{aligned}$$

In other words, $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a $2n$ -cell via the attaching map φ . In particular, note that $\mathbb{CP}^1 = \{\star\} \cup_\varphi D^2 = S^2$ is the 2-sphere.

Now, we can immediately write the cellular chain complex $C_n = C_n^{\text{cell}}(\mathbb{CP}^n)$. It follows that

$$\begin{array}{ccccccc} 0 & \rightarrow & C_{2n} & \rightarrow & C_{2n-1} & \rightarrow & \dots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow 0 \\ & & \mathbb{Z} & & 0 & & \mathbb{Z} & 0 & \mathbb{Z} \end{array}$$

The boundary maps are forced to be 0. Hence, it follows that

$$H_k^{\text{cell}}(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}, & 0 \leq k \leq 2n, k \text{ even}, \\ 0, & \text{otherwise.} \end{cases}$$

This pattern continues for $\mathbb{CP}^\infty$ as well, and we have,

$$H_k^{\text{cell}}(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z}, & k \text{ even,} \\ 0, & k \text{ odd.} \end{cases}$$

Exercise 12.6: (*Cellular Homology of $S^n \times S^n$*)

Compute the cellular homology $X = S^n \times S^n$.

Hint : There is a CW decomposition of X with exactly 4 cells!