

Algebraic Topology II (KSM4E02)

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Hurewicz isomorphism – reduced singular homology – cofibration – singular homology of cofiber – cellular decomposition

10.1 Hurewicz Isomorphism

Let us now show that the Hurewicz map is an isomorphism for path connected spaces.

Theorem 10.1: (Hurewicz Isomorphism)

If X is a path connected space with a basepoint x_0 , then the Hurewicz map $\eta : \pi_1(X, x_0)^{\text{ab}} \rightarrow H_1(X)$ is an isomorphism.

Proof : For each $x \in X$, fix a path $u_x : x_0 \rightarrow x$, with $\bar{u}_x$ as the reversed pat $x \rightarrow x_0$. Assume u_{x_0} to be the constant path c_{x_0} . Next, for any $\sigma : \Delta^1 \rightarrow X$, denote $\sigma_0 = \sigma d_0 = \sigma(0, 1)$ and $\sigma_1 = \sigma d_1 = \sigma(1, 0)$. Then, $(u_{\sigma_1} \star \sigma) \star \bar{u}_{\sigma_0}$ is a loop at x_0 . Let us define the linear map

$$\begin{aligned} \xi' : S_1(X) &\rightarrow \pi_1(X, x_0)^{\text{ab}} \\ \sigma &\mapsto [(u_{\sigma_1} \star \sigma) \star \bar{u}_{\sigma_0}]. \end{aligned}$$

This is possible since $\pi_1(X, x_0)^{\text{ab}}$ is Abelian. Now, for any 2-simplex $\omega : \Delta^2 \rightarrow X$, denote the faces $\omega_j = \omega d_j : \Delta^1 \rightarrow X$, and then observe that $\omega_2 \star \omega_0 \simeq \omega_1$. Also, it is easy to see that

$$\begin{aligned} \xi'(\omega_1) &= [(u_{\omega_{11}} \star \omega_1) \star \bar{u}_{\omega_{10}}] = [(u_{\omega_{11}} \star (\omega_2 \star \omega_0)) \star \bar{u}_{\omega_{10}}] \\ &= [((u_{\omega_{21}} \star \omega_2) \star \bar{u}_{\omega_{20}}) \star ((u_{\omega_{01}} \star \omega_0) \star \bar{u}_{\omega_{00}})], \text{ since } \bar{u}_{\omega_{20}} \star u_{\omega_{01}} \simeq c_{x_0} \\ &= [(u_{\omega_{21}} \star \omega_2) \star \bar{u}_{\omega_{20}}] + [(u_{\omega_{01}} \star \omega_0) \star \bar{u}_{\omega_{00}}] \\ &= \xi'(\omega_2) + \xi'(\omega_0) \end{aligned}$$

Hence, $\xi'(\partial\omega) = 0$ which implies that ξ' factors as a map $\xi : \frac{S_1(X)}{B_1(X)} \rightarrow \pi_1(X)^{\text{ab}}$. Clearly, $\xi \circ \eta = \text{Id}$, and hence, η is injective. To show that η is surjective, let $\alpha \in H_1(X)$ be represented by some 1-cycle $a := \sum n_\sigma \sigma$ with $\partial a = 0$. By linearity, we have a map $u : S_0(X) \rightarrow S_1(X)$. Then, for any $\sigma : \Delta^1 \rightarrow X$, we have $u(\partial\sigma) = u(\sigma d_0 - \sigma d_1) = u_{\sigma_0} - u_{\sigma_1} = u_{\sigma_0} + \bar{u}_{\sigma_1}$. In particular, we have

$$\eta\left(\sum n_\sigma \xi(\sigma)\right) = \sum n_\sigma ([u_{\sigma_1}] + [\sigma] + [\bar{u}_{\sigma_1}]) = \sum n_\sigma ([\sigma] + [u\partial\sigma]) = \alpha + u([\partial a]) = \alpha,$$

as $\partial a = 0$. This implies η is surjective. □

Exercise 10.2:

Compute the homology groups of $S_g := \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_g$, the torus with g -holes.

Hint : Recall, $\pi_1(S_g) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod [a_i b_i] = 1 \rangle$ by Van Kampen's theorem.

Remark 10.3: (The Hurewicz Isomorphism)

The more general Hurewicz isomorphism states the following. Suppose X is a $(n-1)$ -connected, i.e., $\pi_i(X) = 0$ for $i < n$. Then, the Hurewicz map $h : \pi_n(X) \rightarrow H_n(X)$ is an isomorphism. As the Hurewicz map commutes with the suspension map, it follows moreover that $h : \pi_{n+1}(X) \rightarrow H_{n+1}(X)$ is an epimorphism. The isomorphism lets us inductively show that

$$\pi_i(S^n) = \begin{cases} 0, & i < n \\ \mathbb{Z}, & i = n. \end{cases}$$

In general, computing the higher homotopy groups of the sphere is a nontrivial problem.

10.2 Reduced Singular Homology

Suppose X is a space, and $c : X \rightarrow \star$ is the constant map. Recall, the reduced homology is defined as the kernel $\tilde{H}_n(X) := \ker \left(H_n(X) \xrightarrow{H_n(c)} H_n(\star) \right)$. Let us interpret it in a different way for singular homology. Consider the chain map $\varepsilon_\bullet : S_\bullet(X) \rightarrow S_\bullet(\star)$ given by $\varepsilon_n = 0$ for $n \neq 0$, and

$$\varepsilon_0 \left(\sum n_\sigma \sigma \right) = \left(\sum n_\sigma \right) \sigma_0,$$

where $\sigma_0 : \Delta^0 \rightarrow \star$ is the unique 0-simplex.

Exercise 10.4:

Using [Proposition 6.10](#), verify that $H_\bullet(c) = H_\bullet(\varepsilon) : B_\bullet(X) \rightarrow H_\bullet(\star)$.

Let us now define a chain homotopy $s_n : S_n(X) \rightarrow S_{n+1}(\star)$ by setting $s_n(\sigma) = (-1)^n \sigma_{n+1}$ on generators and extending linearly. Here, $\sigma_{n+1} : \Delta^n \rightarrow \star$ is the unique $(n+1)$ -simplex. Note that

$$\begin{aligned} s\partial(\sigma) + \partial s(\sigma) &= s \left(\sum_0^n (-1)^i \sigma d_i \right) + (-1)^n \partial \sigma_{n+1} \\ &= (-1)^{n-1} \sum_0^n (-1)^i \sigma_n + (-1)^n \sum_0^{n+1} (-1)^i \sigma_n \\ &= (-1)^n \left[- \sum_0^n (-1)^i \sigma_n + \sum_0^{n+1} (-1)^i \sigma_n \right] \\ &= (-1)^n (-1)^{n+1} \sigma_n = -\sigma_n = \varepsilon_n(\sigma) - S_n(c)(\sigma). \end{aligned}$$

Then, $H_\bullet(\varepsilon) = H_\bullet(c)$ follows.

Definition 10.5: (Augmented Singular Chain Complex)

Given a space X , the **augmented singular chain complex** is defined as the chain complex $S_2(X) \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0$, where $\varepsilon(\sum n_\sigma \sigma) = \sum n_\sigma$. This map ε is called the **augmentation map**.

Observe that considering $\mathbb{Z}$ as a chain complex concentrated at degree 0, we can look at the augmentation map as a chain map by setting it zero every nonzero degree.

Exercise 10.6:

Verify that the reduced singular homology is the homology of the augmented singular chain complex.

10.3 Cofibration and Homology

Let us recall the notion of cofibration.

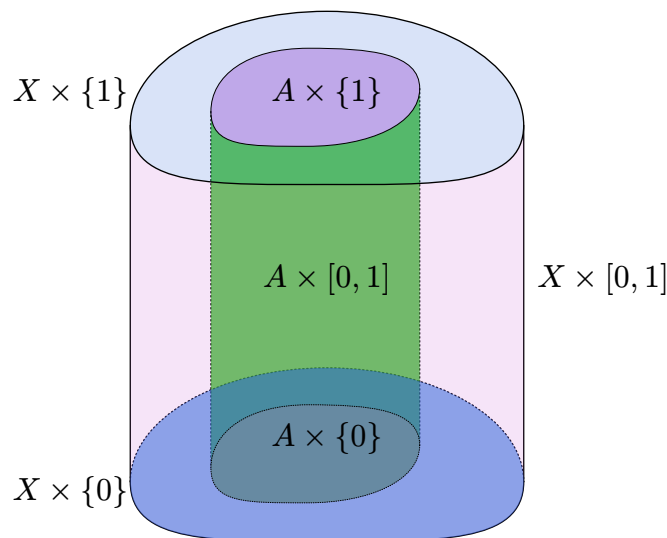
Definition 10.7: (Cofibration)

A map $f : A \rightarrow X$ is called a **cofibration** if it has the *homotopy extension property* against any space, i.e., given a homotopy $h : A \times [0, 1] \rightarrow Y$ and a map $F : X \rightarrow Y$ with $F \circ f = h|_{A \times 0}$, there exists a homotopy $H : X \times [0, 1] \rightarrow Y$ such that $H|_{X \times \{t\}} \circ f = h|_{X \times \{t\}}$. In other words, the commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & Y^{[0,1]} \\ f \downarrow & \nearrow H & \downarrow \text{ev}_0 \\ X & \xrightarrow{F} & Y \end{array}$$

admits a solution H which commutes with other maps.

The homotopy extension is easier to visualize if we only consider an inclusion map $\iota : A \hookrightarrow X$.



Homotopy extension for $\iota : A \hookrightarrow X$

Most of the times, we will only concern ourselves about an inclusion being a cofibration. In fact, given any function $f : A \rightarrow X$, consider the *mapping space*

$$M_f := X \cup_f (A \times [0, 1]),$$

where we identify $(a, 0) \sim f(a)$. It is easy to see that M_f (strongly) deformation retracts onto X by collapsing $A \times [0, 1]$. On the other hand, we have the inclusion $\iota : A \hookrightarrow M_f$ given by $a \mapsto (a, 1)$. We then have a *homotopy* commutative diagram

$$\begin{array}{ccc} & & M_f \\ & \nearrow \iota & \uparrow \text{s.d.r.} \\ A & \xrightarrow{f} & X \end{array}$$

Exercise 10.8: (*Mapping Cylinder is a Cofibration*)

Given any map $f : A \rightarrow X$, show that $\iota : A \rightarrow M_f$ is a cofibration.

Let us now observe a few easy topological consequences about cofibrations.

Lemma 10.9: (*Composition of Cofibration*)

Composition of cofibrations is again a cofibration.

Proof : Suppose $f : X \rightarrow Y, g : Y \rightarrow Z$ are cofibrations. Let us consider the homotopy extension diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & W^I \\ \downarrow f & & \downarrow \text{ev}_0 \\ Y & & W \\ \downarrow g & & \downarrow \\ Z & \xrightarrow{F} & W \end{array}$$

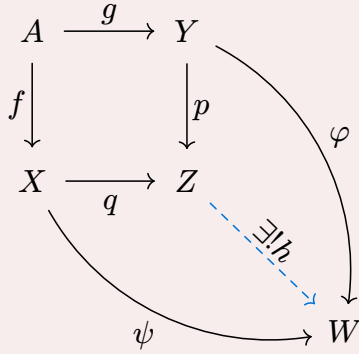
$g \circ f$ (curved arrow from X to Z)

We first get a lift $H_1 : Y \rightarrow W^I$ such that $\text{ev}_0 \circ H_1 = F \circ g$ and $H_1 \circ f = h$. Then, we get another lift $H_2 : Z \rightarrow W^I$ such that $H_2 \circ g = H_1$ and $\text{ev}_0 \circ H_2 = F$. Then, we have $H_2 \circ (g \circ f) = H_1 \circ f = h$, and $\text{ev}_0 \circ H_2 = F$. Thus, H_2 is the required extension, proving that $g \circ f$ is a cofibration. $\square$

Next, we consider pushout of a cofibration. Let us now recall the definition first.

Definition 10.10: (Pushout of Spaces)

Given continuous maps $X \xleftarrow{f} A \xrightarrow{g} Y$, the **pushout** is defined via the following universal property.



Given any map $\varphi : Y \rightarrow W, \psi : X \rightarrow W$ with $\varphi \circ g = \psi \circ f$, there exists a unique map $h : Z \rightarrow W$ such that $h \circ p = \varphi, h \circ q = \psi$.

Explicitly, one can construct the pushout as the identification space $Z = (X \sqcup Y)/\sim$, where $\sim$ is the smallest equivalence relation such that $f(a) \sim g(a)$ for $a \in A$.

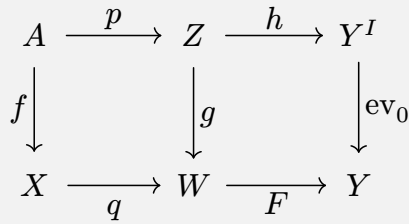
Example 10.11:

Given a map $f : A \rightarrow X$, the mapping cone $C(f)$ is the pushout of the diagram $CA \hookleftarrow A \xrightarrow{f} X$. The pushout of the sphere as the boundary $D^n \hookleftarrow S^{n-1} \hookrightarrow D^n$ is the sphere S^n .

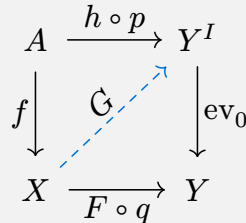
Lemma 10.12: (Pushout of a Cofibration)

Suppose $f : A \rightarrow X$ is a cofibration. Then, any pushout of f is again a cofibration.

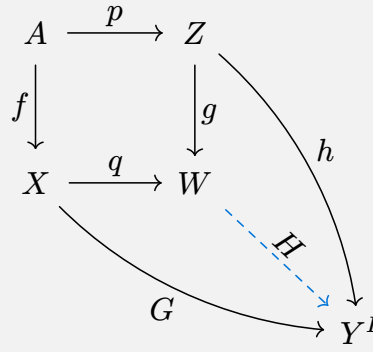
Proof : Consider a pushout diagram and a homotopy extension problem for the middle arrow



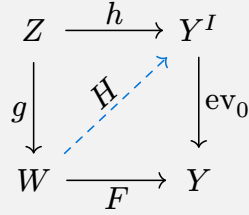
As f is a cofibration, we have a lift



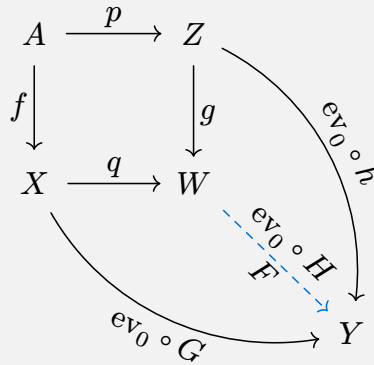
Next, we use the pushout to get a unique map



We claim that H is the solution in



We already have $H \circ g = h$ from the pushout diagram. Next, we again look at the pushout



It follows that both $\text{ev}_0 \circ H$ and F solves the pushout diagram. Hence, by the uniqueness, we have $\text{ev}_0 \circ H = F$. This proves that H is a homotopy extension, as required. Thus, the pushout $g : Z \rightarrow W$ is a cofibration. $\square$

Lemma 10.13: (*Contractible Cofibered Subspace*)

Suppose $A \subset X$ is contractible, and the inclusion $\iota : A \hookrightarrow X$ is a cofibration. Then, the quotient $q : X \rightarrow X/A$ is a homotopy equivalence.

Proof : Since A is contractible, there exists a homotopy $h : A \times [0, 1] \rightarrow A$ such that $h|_{A \times \{0\}} = \text{Id}_A$ and $h|_{A \times \{1\}} = \{a_0\}$ for some $a_0 \in A$. As $\iota : A \hookrightarrow X$ is a cofibration, we have a homotopy extension, $H : X \times [0, 1] \rightarrow X$ with $H_0 := H|_{X \times \{0\}} = \text{Id}_X$. Observe that $H_1 := H|_{X \times \{1\}} : X \rightarrow X$ maps A to a point $\{a_0\}$, and thus, we have a map $r : X/A \rightarrow X$ such that $r \circ q = H_1$. By construction, $H : r \circ q \simeq \text{Id}_X$ is a homotopy. On the other hand, the homotopy $H_t : X \rightarrow X$ maps $H_t(A) \subset A$, and hence, passing to quotient we have the homotopy $\tilde{H}_t : X/A \rightarrow X/A$. Clearly $\tilde{H}_0 = \text{Id}_{X/A}$. Also, $qr([x]) = qrq(x) = qH_1(x) = \tilde{H}_1q(x) = \tilde{H}_1([x])$, i.e., $\tilde{H}_1 = q \circ r$. Thus, $\tilde{H} : q \circ r \simeq \text{Id}_{X/A}$ is a homotopy. This proves that $q : X \rightarrow X/A$ is a homotopy equivalence. $\square$

Theorem 10.14: (Homology of Quotient by a Cofibered Subspace)

Suppose $A \subset X$ is a subspace such that the inclusion $\iota : A \hookrightarrow X$ is a cofibration. Then, the quotient map $q : X \rightarrow X/A$ induces an isomorphism $H_n(X, A) \rightarrow H_n(X/A, \star) \cong \tilde{H}_n(X/A)$.

Proof : Consider the mapping $C(\iota)$ which is given as the pushout

$$\begin{array}{ccc} A & \longrightarrow & CA \\ \downarrow \iota & & \downarrow j \\ X & \longrightarrow & C(\iota) \end{array}$$

As ι is a cofibration, by Lemma 10.12, we have j is a cofibration. But CA is contractible. Hence, by Lemma 10.13, we have $q : C(\iota) \rightarrow C(\iota)/CA = X/A$ is a homotopy equivalence. Now, the inclusion map $(X, A) \rightarrow (C(\iota), CA)$ induces an isomorphism in homology since we can first excise out the coning point, and then apply a deformation retract. Also, $(C(\iota), CA) \rightarrow (X/A, \star)$ induces an isomorphism as the quotient map is a homotopy equivalence. Hence, composing, it follows that the quotient $(X, A) \rightarrow (X/A, \star)$ induces an isomorphism in homology.

The isomorphism $H_n(X/A, \star) \cong \tilde{H}_n(X/A)$ follows from applying the quotient map $(Y, \emptyset) \rightarrow (Y/\emptyset, \emptyset/\emptyset) = (Y, \star)$ for any space Y . $\square$

Example 10.15: (CW pair)

Let X be a CW-complex, and $A \subset X$ a subcomplex. Then the inclusion $A \hookrightarrow X$ is a cofibration. In fact, if X is a locally finite CW complex (i.e, only finitely many cells of each dimension), then we can drop the assumption that A is a sub-complex and only require that $A \subset X$ is closed and a CW complex on its own.

10.4 Cellular Decomposition

Let us recall the definition of a (relative) CW complex. Given a pair (X, A) , we say X is obtained from A by *attaching an n -cell* if there is a pushout diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\varphi} & A \\ \downarrow & & \downarrow \iota \\ D^n & \xrightarrow{\Phi} & X \end{array}$$

Now, the inclusion $S^{n-1} \hookrightarrow D^n$ is trivially a cofibration. Indeed, we have a retract of $D^n \times [0, 1]$ to $D^n \times \{0\} \cup S^{n-1} \times [0, 1]$ obtained by “pushing everything radially from the center of $D^n \times \{1\}$ ”. Composing with this retract, we can always get the desired homotopy extension from D^n , while keeping any homotopy on S^{n-1} fixed. Then, it follows from Lemma 10.12 that $\iota : A \hookrightarrow X$ is also a cofibration. The map $\varphi : S^{n-1} \rightarrow A$ is called the *attaching map* for the n -cell, and Φ is called the *characteristic map*. Note that A is necessarily closed in X , and Φ is a homeomorphism of the interior $\mathring{D}^n$ onto $X \setminus A$. Indeed, X is homeomorphic to the mapping cone $C(\varphi) = A \cup_{\varphi} \underbrace{C(S^{n-1})}_{D^n}$. In other words, $\Phi : (D^n, S^{n-1}) \rightarrow (X, A)$ is a *relative homeomorphism*, i.e,

the induced map

$$\tilde{\Phi} : D^n / S^{n-1} \rightarrow X/A$$

is a homeomorphism.

Example 10.16:

The n -sphere is obtained from the point $\{\star\}$ by attaching an n -cell, where the attaching map is the constant map. On the other hand, we can also realize the n -sphere from the n -disc by attaching a single n -cell, where the attaching map is the identity map on the boundary. The real projective space $\mathbb{RP}^n$ is obtained from $\mathbb{RP}^{n-1}$ by attaching an n -cell. The complex projective space $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a $2n$ -cell.

Remark 10.17:

Suppose X is a T_2 -space with a closed subspace $A \subset X$. Let $\Phi : D^n \rightarrow X$ be continuous map, which restricts to a homeomorphism of $\mathring{D}^n$ onto $X \setminus A$. Then, X is obtained from A by attaching an n -cell, via the attaching map $\varphi := \Phi|_{S^{n-1} = \partial D^n}$.

More generally, we can attach multiple n -cells at once. Given a pair (X, A) , we say X is obtained from A by *attaching n -cells* if there exists a pushout diagram

$$\begin{array}{ccc} \bigsqcup_{\alpha \in J} S_{\alpha}^{n-1} & \xrightarrow{\varphi} & A \\ \downarrow & & \downarrow \iota \\ \bigsqcup_{\alpha \in J} D_{\alpha}^n & \xrightarrow{\Phi} & X \end{array}$$

Here, J is an arbitrary (possibly empty) indexing set. The restriction $\varphi_{\alpha} := \varphi|_{S_{\alpha}^{n-1}}$ is called the *attaching map* for the n -cell $E_{\alpha}^n := \mathring{D}_{\alpha}^n$. It follows that A is closed in X , and Φ induces a homeomorphism of $\bigsqcup E_{\alpha}^n$ with $X \setminus A$, in particular $X \setminus A$ is a union of components, each being an n -cell. In case $J = \emptyset$, we have $X = A$.

Exercise 10.18:

Suppose X is obtained from A by attaching n -cells, with attaching maps $\{\varphi_{\alpha} : S_{\alpha}^{n-1} \rightarrow A\}_{\alpha \in J}$. Show that a map $f : A \rightarrow Y$ extends to a map $F : X \rightarrow Y$ if and only if each $f \circ \varphi_{\alpha}$ is null-homotopic (whence the extension is determined by the choice of null-homotopies).

Definition 10.19: (CW Complex)

Let $A \subset X$ be given. A **CW decomposition** of the pair (X, A) is a sequence of subspaces

$$A = X^{-1} \subset X^0 \subset X^1 \subset \dots \subset X,$$

such that the following holds.

1. $X = \bigcup_{i \geq 0} X^i$, with the colimit topology (i.e, a map $f : X \rightarrow Y$ is continuous if and only if $f|_{X^n} : X^n \rightarrow Y$ is continuous for all n).
2. For each $n \geq 0$, the space X^n is obtained from X^{n-1} by attaching n -cells.

The pair (X, A) equipped with a CW decomposition is called a **relative CW complex**. When $A = \emptyset$, we say X is a **CW complex**. The space X^n is called the **n -skeleton** of the CW complex. We say (X, A) is a **finite CW complex** (resp. **countable CW complex**) if there are only finitely (resp. countably) many cells attached. We say (X, A) is a **locally finite CW complex** if for each n , only finitely many n -cells are attached.

Exercise 10.20:

Suppose (X, A) is a relative CW complex. Show that the inclusion $A \hookrightarrow X$ is a cofibration. More generally, given a CW complex X , and a subcomplex $Y \subset X$ (i.e, Y is closed and consists of cells of X), the inclusion $Y \hookrightarrow X$ is a cofibration.

Hint : Use [Lemma 10.12](#) to the cofibration $\sqcup S^{n-1} \hookrightarrow \sqcup D^n$, and observe that the composition of cofibrations is again a cofibration ([Lemma 10.9](#)). Inductively extend the homotopy $X^n \times [0, 1]$ to $X^{n+1} \times [0, 1]$. The final homotopy is continuous as the topology on X is a colimit topology.

In the above definition, there is no restriction on the topology on A . If A is a T_2 -space, then X is a T_2 -space, moreover the topology on X is given by the colimit topology with respect to the family consisting of A and the closure of cells. This leads to the following definition.

Definition 10.21: (Whitehead Complex)

A **Whitehead complex** is a space X , along with a **cell decomposition** $\{e_\lambda \mid \lambda \in \Lambda\}$, where each e_λ is homeomorphic to $\mathring{D}^n$ for some n , such that the following holds.

1. X is a T_2 -space.
2. For each n -cell, there is a map $\Phi_\lambda : D^n \rightarrow X$ such that $\Phi|_{\mathring{D}^n} : \mathring{D}^n \rightarrow e_\lambda$ is a homeomorphism, and Φ maps the boundary ∂D^n into X^{n-1} , which is the union of cells of dimension $\leq n-1$.
3. (Closure Finiteness) The closure $\bar{e}_\lambda$ of each cell intersects only a finitely many cells.
4. (Weak Topology) The topology on X is induced by the colimit topology of the family $\{\bar{e}_\lambda \mid \lambda \in \Lambda\}$, i.e, $U \subset X$ is open if and only if $U \cap \bar{e}_\lambda$ is open for all $\lambda \in \Lambda$.

It follows that any Whitehead complex carries a CW decomposition, and conversely, a CW complex is a Whitehead complex.

Remark 10.22:

Note that the phrase “ X is a CW complex” technically means that “ X is a space equipped with a CW decomposition, and the corresponding colimit topology”. There are spaces which does not admit any CW decomposition, e.g., the Hawaiian earring. To see this, one first proves that a space admitting a CW decomposition must be locally contractible, but Hawaiian earring is not.

Finally, let us understand cell decomposition on a product of CW complexes. Suppose X, Y are given are CW complexes, with $\{e_\alpha^n, \varphi_\alpha\}$ and $\{f_\beta^n, \psi_\beta\}$ as the respective collection of cells and attaching maps. Then, on $X \times Y$ we have a cell decomposition given by $\{e_\alpha^n \times f_\beta^n, \varphi_\alpha \times \psi_\beta\}$. In particular, we have the n -skeleton $(X \times Y)_n = \cup_{i+j=n} X_i \times Y_j$. Note that the weak topology on $X \times Y$ generated by the skeletons maybe strictly finer than the product topology! The two topologies match if any of the following holds true.

- Either of X or Y is compact (or more generally, locally compact). Note, X, Y being CW complexes, is already T_2 .
- If both X and Y have countably many cells, i.e, they are countable CW complexes.

In general, the weak topology on $X \times Y$ matches with the *compactly generated topology* on $X \times Y$.

Example 10.23: (CW decomposition of $S^p \times S^q$)

The sphere S^p can be constructed using one 0-cell e^0 and one p -cell e^p . Similarly, S^q can be constructed using one 0-cell f^0 and one q -cell f^q . Then, on $S^p \times S^q$ we have a cell decomposition:

0-Cells

$$e^0 \times f^0$$

 p -Cells

$$e^p \times f^0$$

 q -Cells

$$e^0 \times f^q$$

 $p + q$ -Cells

$$e^p \times f^q$$

The attaching map $\omega : S^{p+q-1} \rightarrow S^p \vee S^q$ for the top cell represents the *universeal Whitehead bracket* : given maps $[f] \in \pi_p(X), [g] \in \pi_q(X)$, this is an element $[f, g] \in \pi_{p+q-1}(X)$ represented by the composition

$$S^{p+q-1} \xrightarrow{\omega} S^p \vee S^q \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X$$