Quiz 1

Course: Algebraic Topology I (KSM3E02) **Instructor:** Aritra Bhowmick

Time: 12:00PM – 1:00PM, 20th August, 2026 **Total marks:** 20

Q1. Suppose $X = \mathbb{R}^3$ and A is one of the subspaces as below. Describe $X \setminus A$ as the wedge of circles and spheres up to homotopy equivalence. No need to provide explanations. $[5 \times 2 = 10]$

a. A is an infinite line.

Solution : The answer is S^1.

To see this, assume that A simply the z-axis. Then, the projection to the xy-plane is clearly a deformation retract. Thus, $\mathbb{R}^3 \setminus A \simeq \mathbb{R}^2 \setminus \{0\} \simeq S^1$.

b. A is the union of n-many parallel infinite lines.

Solution : The answer is $\bigvee_{i=1}^n S^1$.

We can assume that the lines are parallel to the z-axis. Again, the projection onto the xy-plane gives a deformation retract. Hence,

$$\mathbb{R}^3 \setminus A \simeq \mathbb{R}^2 \setminus \{n\text{-many points}\} \simeq \bigvee_{i=1}^n S^1.$$

c. A is the union of n-many infinite lines all passing through the origin.

Solution : The answer is $\bigvee_{i=1}^{2n-1} S^1$.

To see this, we can radially collapse the outside of the unit sphere, which gives strongly deformation retract of $\mathbb{R}^3 \setminus A$ onto the solid closed ball D^3 with n-many lines passing through origin removed. Note that each line cuts the boundary sphere in 2-points. We can also radially move everything away from the origin, and get a strong deformation retract onto S^2 with $2n$-many points removed! But this is homotopy equivalent to $(2n-1)$-fold wedge of S^1.

d. A is the unit circle S^1 centered at origin in the xy-plane.

Solution : The answer is $S^1 \vee S^2$.

You can thicken up S^1 to a solid torus. Then, removing S^1 is same as removing the interior of the solid torus. Indeed, from each point on S^1, you can radially push in the directions perpendicular to S^1 to get a strong deformation retract. Assume that the solid torus is exactly bounded by the sphere of radius r, collapse everything outside of this sphere. Thus, we have a strong deformation retract to the solid ball D^3 with the interior of the solid torus (that touches the ball!) removed. Finally, project down along z-axis to get another strong deformation retract to the torus with a disc attached in the middle hole. Collapse this disc and you have the pinched sphere, which is $S^1 \vee S^2$ up to homotopy equivalence.

Another way to think about this might be as follows. Add an extra point $\{\infty\}$ to $\mathbb{R}^3$, so that we have S^3. If you remove a solid torus from S^3, you actually get another solid torus back! The infinity point is actually in the interior of this solid torus. So if you remove it, you get homotopy equivalence to a torus with a disc attached in the “tube-like hole”. Collapse the disc, and you have the pinched sphere, which is $S^1 \vee S^2$ up to homotopy equivalence.

e. $A = \{(x, y, 0) \mid (x-1)^2 + y^2 = 1\} \cup \{(x, y, 0) \mid (x+1)^2 + y^2 = 1\}$.

Solution : The answer is $S^1 \vee S^1 \vee S^2$.

The space A is basically a figure-8 shape, in the xy-plane. Again, thicken it up to get a solid 2-torus. You can perform the deformation retractions as in the previous problem and get a 2-torus with two disks attached in the two holes. If you collapse both of them, you get the sphere S^2, pinched twice! This you can see is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.

Another way would be to again add an $\{\infty\}$ point, to get the S^3. You can imagine that the complement of the interior of a 2-torus in S^3 is another solid 2-torus, whose interior contains the $\{\infty\}$ point. Removing the infinity gives you the $S^2 \vee S^1 \vee S^1$ back.

Q2. Given a map $f : X \to Y$, show that the following are equivalent. $[5 + 5 = 10]$

a. $f : X \to Y$ is null-homotopic.

b. There exists a retraction $C_f \to Y$ from the mapping cone of f to the base X.

Solution : Suppose $f : X \to Y$ is null-homotopic. So, we have a homotopy $h : X \times [0,1] \to Y$ such that $h_0 = f$ and $h_1 = c_{y_0}$, where $c_{y_0} : X \to Y$ is the constant map that sends everything to the point $y_0 \in Y$. In particular, $h(X \times \{1\}) = \{y_0\}$, and hence, h induces a map in the quotient space $CX = (X \times [0,1])/(X \times \{0\})$, i.e, we have a map $\tilde{h} : C(X) \to Y$. Note that $\tilde{h}|_{X\times\{1\}} = f$, and hence, we have a well defined map $r : C_f \to Y$ in the quotient $C(X) \sqcup Y \to C_f$. This map r is clearly a retraction.

Conversely, suppose we have a retraction $r : C_f \to Y$. Consider the composition

$$F : X \times [0,1] \to C(X) \hookrightarrow C(X) \sqcup Y \to C_f \xrightarrow{r} Y,$$

where we have the two quotient maps. Since $X \times \{0\}$ is mapped to the cone point, it is clear that $F|_{t=0}$ is constant. On the other hand,

$$F(x,1) = r([f(x)]) = f(x),$$

since the retraction fixes the point $f(x) \in Y$. This shows that F is homotopy between f and a constant map, i.e, f is null-homotopic.

Q3. Given a space X, show that the following are equivalent. $[5+5=10]$

a. X is contractible.

b. Given any space Y, any map $Y \to X$ is null-homotopic.

Solution : Suppose X is contractible. Then, there exists a point $x_0 \in X$ and a homotopy $h : X \times [0,1] \to X$ such that $h_0 = \mathrm{Id}_X$ and $h_1 = c_{x_0}$, the constant map. Then, for any map $f : Y \to X$, we have a homotopy

$$H : Y \times [0,1] \xrightarrow{f \times \mathrm{Id}} X \times [0,1] \xrightarrow{h} X,$$

which is evidently a homotopy of f to a constant $Y \to X$.

Conversely, suppose for any space Y, any map $f : Y \to X$ is null-homotopic. In particular, $\mathrm{Id}_X : X \to X$ is null-homotopic. But this is equivalent to X being contractible!