Course notes for

Algebraic Topology I (KSM3E02)

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Day 1 : 4th August, 2026

`quotient topology — homotopy — homotopy equivalence`

Unless otherwise mentioned, throughout this course, a space means a topological space and a map $f : X \to Y$ between spaces is always assumed to be continuous.

1.1 A Primer on Quotient Topology

Let us recall few important concepts from point-set topology that crops up in algebraic topology all the time!

Definition 1.1: (*Quotient Topology*)

Given a space X, a set Y, and set map $q : X \to Y$, the *quotient topology* on Y induced by q is the *finest* topology (i.e, the *largest* topology) on Y such that q becomes continuous. We say q is a quotient map.

Explicitly, the quotient topology on Y induced by $q : X \to Y$ is given as

$$\{U \subset Y \mid q^{-1}(U) \subset X \text{ is open}\}.$$

It is easy to verify that the quotient topology as defined above is indeed a topology. In order to understand the definition, note that if we give Y the indiscrete topology then any map $f : X \to Y$ is continuous; on the other hand, if Y has a finer topology, a map into it may fail to be continuous. So, we aim for the finest possible topology on Y. A map $f : X \to Y$ is called a quotient map if the following holds:

$$U \subset Y \text{ is open if and only if } f^{-1}(U) \subset X \text{ is open.}$$

An important property of the quotient topology is the following.

Proposition 1.2: (*Universal Property of the Quotient Topology*)

Suppose $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ are given topological spaces, and $f : X \to Y$ is a set function. Then, the following are equivalent.

1. $\mathcal{T}_Y$ is the quotient topology induced by f.
2. $\mathcal{T}_Y$ is the unique topology having the following property: given any space Z and a set map $g : Y \to Z$, we have g is continuous if and only if $g \circ f$ is continuous.

Quotient topology crops up from equivalence relations. Given an equivalence relation $\sim$ on a space X, denote $X/_\sim$ as the set of equivalence classes. Then, we have a set map $q : X \to X/_\sim$ which maps each element to its equivalence class. We equip $X/_\sim$ with the quotient topology induced by this map. As a consequence, if we have a continuous map $f : X \to Z$ such that $a \sim b \Rightarrow f(a) = f(b)$ (i.e, f respects the equivalence relation), then there is an induced map $\tilde{f} : X/_\sim \to Z$, which is moreover continuous (Prove it!).

Example 1.3: (*Projective Spaces*)

On $X := \mathbb{R}^{n+1} \setminus \{0\}$ consider the equivalence relation: $\mathrm{u} \sim \mathrm{v}$ if and only if $\mathrm{u} = \lambda \mathrm{v}$ for some nonzero scalar $\lambda \in \mathbb{R}$. The induced quotient space, denoted as $\mathbb{RP}^n$, is known as the n^{th} real projective space. Can you see that $\mathbb{RP}^1$ is nothing but the circle S^1?

Let us now recall some constructions that one can perform on topological spaces, all of which are given quotient topology.

1.2 Why Algebraic Topology?!

In point-set topology, we encounter many properties of a space, and then verify which of them are invariant under homeomorphism. Then, this property can be used to distinguish between spaces. As an example, consider the intervals $[0, 1]$ and $[0, 1)$: they can be homeomorphic since one of them is compact, the other is not, and we know that homeomorphism preserves compactness. Now, if we consider the intervalse $(0, 1)$ and $[0, 1)$, then compactness is not good enough as none of them are compact. To distinguish them we can use the following trick.

> If possible, suppose $f : [0, 1) \to (0, 1)$ is a homeomorhpism. Denote $z = f(0)$. Observe that we have an induced map
>
> $$\tilde{f} : (0, 1) \to (0, 1) \setminus \{z\},$$
>
> which is also a homeomorphism. But this is impossible since deleting a point makes $(0, 1)$ disconnected, and we know that connectedness is preserved under homeomorphism.

This trick can be used to distinguish (up to homeomorphism) between $\mathbb{R}$ and $\mathbb{R}^2$ as well. But the trick fails when you try to distinguish between $\mathbb{R}^2$ and $\mathbb{R}^3$.

The point of algebraic topology is to weak the notion of homeomorphism, and yet be able to distinguish between a spaces. Indeed, we shall define the notion of *simply connectedness* and see that $\mathbb{R}^2$ with a point removed is simply connected, but $\mathbb{R}^3$ with a point removed is not.

More generally, we shall try to meaningfully attach algebraic objects (like groups, rings, chain complexes, etc.) to a space, which are (sometimes) computable. Then, using them we shall try to distinguish between spaces. The first step of course is to define the weakened notion of homeomorphism!

1.3 Homotopy and Homotopy Equivalence

Definition 1.4: (*Homotopy*)

Given maps $f, g : X \to Y$, a *homotopy* between them is a map $h : X \times [0, 1] \to Y$ such that

$$h(x, 0) = f(x), \qquad h(x, 1) = g(x), \qquad\qquad x \in X.$$

We denote this as $h : f \simeq g$, and say f, g are *homotopic* maps.

Intuitively, a homotopy is a continuous way to deform one map to another.

Proposition 1.5: (*Homtopy is an Equivalence Relation*)

Being homotopic is an equivalence relation on the set $C(X, Y)$ of continuous maps $X \to Y$.

<u>**Proof**</u> **:** We explicitly check the properties.

- **Reflexivity:** Given any $f : X \to Y$, we have a homotopy $h : f \simeq f$ given by $h(x, t) = f(x)$ for $0 \leq t \leq 1$.
- **Symmetry:** Suppose $h : f \simeq g$ is as homotopy. Define the *reversed homotopy* as

$$\begin{aligned} k : X \times [0, 1] &\to Y \\ (x, t) &\mapsto h(x, 1 - t). \end{aligned}$$

  It is easy to see that $k : g \simeq f$.
- **Transitivity:** Suppose $\varphi : f \simeq g, \psi : g \simeq h$ are homotopies. Then, define the map

$$\begin{aligned} \zeta : X \times [0, 1] &\to Y \\ (x, t) &\mapsto \begin{cases} \varphi(x, 2t), & 0 \leq t \leq \frac{1}{2} \\ \psi(x, 2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases} \end{aligned}$$

  Via the *pasting lemma*, it follows that ζ is continuous, and clearly, $\zeta : f \simeq h$ is a homotopy.

Thus, we have $\simeq$ is an equivalence relation on $C(X, Y)$. $\square$

Definition 1.6: (*Homotopy Equivalence*)

A map $f : X \to Y$ is called a *homotopy equivalence* if there is a map $g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. We say that X and Y are *homotopy equivalent* as spaces, and denote it as $X \simeq Y$.

Intuitively, homotopy equivalence allows you to deform a space: think of a space as a rubber sheet; you are allowed to squish it, stretch it, but you cannot tear it! Explicitly writing down homotopy equivalence is not always feasible as we shall soon see.

Exercise 1.7: (*Homotopy Equivalence is an Equivalence Relation*)

Show that being homotopy equivalence is an equivalence relation between the class of all topological spaces.

<u>**Hint**</u> **:** Use Exercise 2.1.

Day 2 : 5th August, 2026

`homotopy extension — homotopy lifting`

2.1 Composition and Homtopies

There are two ways to compose homotopies. We can compose them *vertically*

$$\begin{tikzcd}
X \arrow[r, bend left=70, "f"] \arrow[r, "g"] \arrow[r, bend right=70, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow H"] \arrow[r, phantom, shift right=3, "\Downarrow K"] \arrow[r, phantom, shift left=1, "\Downarrow L"] & Y
\end{tikzcd}$$

This is nothing but the transitivity. We can also compose them *horizontally*

$$\begin{tikzcd}
X \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "H\Downarrow"] \arrow[rr, bend left=80, "g_1 \circ f_1"] \arrow[rr, bend right=80, "g_2 \circ f_2"'] & Y \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "K\Downarrow"] \arrow[phantom, "\Downarrow L"] & Z
\end{tikzcd}$$

> **Exercise 2.1:**
>
> Construct the horizontal composition of two homotopies.

2.2 Retraction

> **Definition 2.2:** (*Retraction*)
>
> Given a subspace $A \subset X$, a *retraction* is a map $r : X \to A$ such that $r \circ \iota = \mathrm{Id}_A$, where $\iota : A \hookrightarrow X$ is the inclusion. We say A is a *retract* of X. In other words, we have a commuting diagram
>
> $$\begin{tikzcd}
A \arrow[r, hook, "\iota"'] & X \arrow[l, bend right, "r"']
\end{tikzcd}$$

As an example, any point in a space is a retract. We shall see that the boundary circle is *not* a retract of the closed disc (and more generally, boundary sphere is not a retract of the closed ball). Try to justify that $A = \{0, 1\} \subset [0, 1]$ is *not* a retract of $[0, 1]$!

Exercise 2.3:

Suppose X is a T_2-space (i.e, Hausdorff). If $A \subset X$ is a retract, verify that A must be a closed subset.

Hint : For a T_2-space X, if we have two maps $f, g : X \to Y$, then the diagonal $\Delta = \{x \in X \mid f(x) = g(x)\}$ is a closed subset.

Let us now specialize homotopy equivalence to the case of retraction, which is easier to visualize.

Definition 2.4: (*Deformation Retract*)

A subspace $A \subset X$ is a *deformation retract* if there is a retraction $r : X \to A$ satisfying $r \circ \iota = \mathrm{Id}_A$, and there is a homotopy $\iota \circ r \simeq \mathrm{Id}_X$. Explicitly, we need a homotopy $H : X \times [0, 1] \to X$ such that

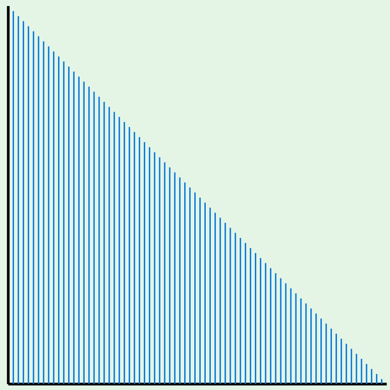
$$H(x,1) = x, \quad x \in X, \qquad H(x,0) \in A, \quad x \in X, \qquad H(a,0) = a, \quad a \in A.$$

We say $A \subset X$ is a *strong deformation retract* if moreover the homotopy restricts to identity on A, i.e, if $H(a,t) = a$ for all $a \in A$ and $0 \le t \le 1$.

It is clear from the definition that a deformation retract is always a homotopy equivalence. There are (pathological) examples which are deformation retracts, but *not* strongly.

Example 2.5:

Consider the following comb-like space.



Explicitly, the space X can be identified as the following subspace of $\mathbb{R}^2$

$$X = ([0,1] \times \{0\}) \cup \bigcup_{r \in [0,1] \cap \mathbb{Q}} \{r\} \times [0, 1-r]$$

That is, keep the horizontal interval $[0,1] \times \{0\}$, and then for each rational point r on it add the vertical line of length $1-r$.

We claim that X strongly deformation retrats onto any point on the horizontal line. Indeed, it is easy to construct a linear homtopy which pushes everything down to the horizontal line, and then collapses the line to a designated point; this designated point is never moved throughout.

On the other hand, for any point *not* on the horizontal line, there is always a deformation retract: just perform the previous deformation to the point, say, $(0,0)$ and then slide this point to any other point, speeding up as necessary. But for such a point, there cannot be a strong deformation retract. To see this intuitively, observe that if there is a strong deformation retract to a point, then in any neighborhood of that point must contain a smaller neighborhood which also collapses to that same point. But this is clearly not the case since you can always find a neighborhood of such a point which cannot be path connected (and hence cannot collapse to a point).

Let us give a name to spaces which are homotopy equivalent to a point.

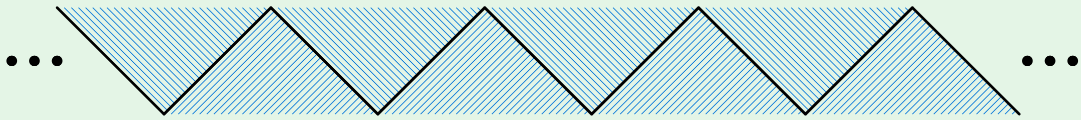
Definition 2.6: (*Contractible Space*)

A space X is said to be *contracitble* if it is homotopy equivalent to a point. We denote it as $X \simeq \star$.

We have seen that any space admits a retraction to any point in it. It is easy to see that if a space deformation retracts to a point, then it is contractible, and vice versa. This does not hold true for *strong* deformation retract.

Example 2.7:

This example builds upon Example 2.5. Consider the following subspace X of $\mathbb{R}^2$



This is obtained from the comb-like space in Example 2.5 by arranging them in the infinite zigzag pattern!

We claim that this space X is *contractible*, but does not *strongly* deformation retract to any point. Intuitively, a strong deformation retract to a point x_0 will imply that there for any neighborhood of x_0, there is a smaller neighborhood which contracts to x_0: this is clearly not the case here (do a case-by-case analysis of different kinds of points).

In order to see why this space is contractible, we do the following. Firstly, for each point x in the space, we have a unit-speed path $r_x : [0, \infty) \to X$ which follows the *bristle* of the comb where x belongs till it hits the bold zigzag line, and then moves along the zigzag to the right. Clearly if x is already on the zigzag, it just starts following it. Consider the following map

$$\begin{aligned} F : X \times [0,1] &\to X \\ (x,t) &\mapsto r_x(t). \end{aligned}$$

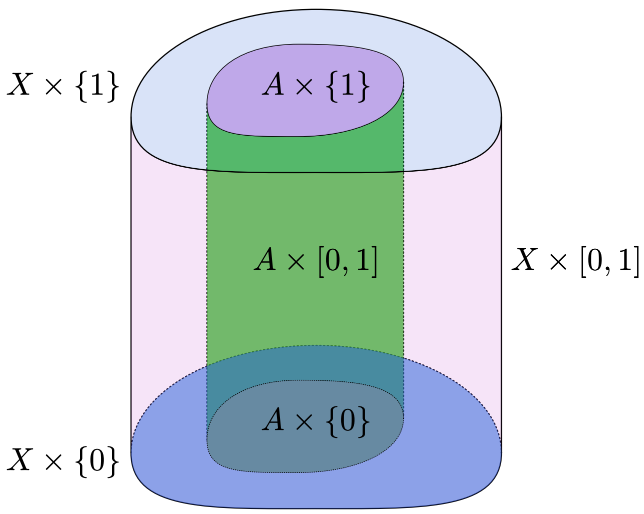
One can check that F is a continuous map, and hence a homotopy between the Id_X to a map which ends in the zigzag. Finally, the zigzag is just a copy of $\mathbb{R}$, and hence contractible. Thus, composing we can get a contracting homotopy. In other words, $X \simeq \star$.

Caution 2.8:

According to Hatcher, a "deformation retract" is precisely what we call a *strong* deformation retract. Hatcher also defines a notion of "deformation retract in the weak sense": this asks for a deformation retraction $H : X \times [0,1] \to X$ such that $H(a,t) \in A$ for all $a \in A$ and for all time $0 \leq t \leq 1$.

2.3 Homotopy Extension Property

Suppose $A \subset X$ is a subspace. Assume that we are given a map $f : X \to Y$, and a homotopy $h : A \times [0,1] \to Y$ from $f|_A : A \to Y$.



We say, the pair (X, A) has *homotopy extension property* (or *HEP*) with respect to the space Y if there exists a homotopy $H : X \times [0,1] \to Y$ from f, which extends the homotopy h. That is,

$$H(x,0) = f(x) \quad x \in X, \qquad H(a,t) = h(a,t) \quad a \in A, 0 \leq t \leq 1.$$

The schematic picture explains what is going on. We are given a continuous map on the subspace

$$A \times [0,1] \cup X \times \{0\} \subset X \times [0,1],$$

and we wish to extend it to the wholse space $X \times [0,1]$.

Definition 2.9: (*Cofibration*)

The inclusion map $\iota : A \hookrightarrow X$ is called a *(Hurewicz) cofibration* if it has homotopy extension property with respect to every space Y.

Remark 2.10:

More generally, we can define HEP for any map $f : A \to X$, not just inclusion maps. Indeed, this can be formulated via the following diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & Y^{[0,1]} \arrow[d, "e"] \\ X \arrow[r, "g"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

Here, $Y^{[0,1]}$ denotes the space of continuous maps $[0,1] \to Y$, and $e : Y^{[0,1]} \to Y$ is evaluation at 0. A homotopy $h : A \times [0,1] \to Y$ can be interpreted as a map $h : A \to Y^{[0,1]}$. The map $f : A \to X$ is said to have HEP against the space Y if for any given map $g : X \to Y$ and a homotopy $h : A \times [0,1] \to Y$, such that $h|_{A \times \{0\}} = g \circ f$, there exists a homotopy $H : X \times [0,1] \to Y$ which makes the diagram commutative. One then defines $f : A \to X$ to be a *(Hurewicz) cofibration* if it has HEP against all space Y.

Turns out any cofibration $f : A \to X$ is actually injective, and moreover, f is a homeomorphism onto its image. Thus, we are not losing any generality with our definition.

Let us now see a prototypical example of a cofibration!

Example 2.11: ($S^{n-1} \hookrightarrow D^n$ *is a Cofibration*)

Consider the inclusion $\iota : S^{n-1} \hookrightarrow D^n$ of the unit $(n-1)$-sphere S^{n-1} as the boundary of the closed unit disc D^n in $\mathbb{R}^n$. We claim that this has HEP against any space, and hence, is a cofibration. Suppose $f : D^n \to Y$ is a map, and $h : S^{n-1} \times [0,1] \to Y$ is a homotopy of $f|_{S^{n-1}}$. Consider the subspace

$$X := S^{n-1} \times [0,1] \cup D^n \times \{0\} \subset D^n \times [0,1].$$

Clearly, we are given a map $F : X \to Y$, which we intend to extend to $D^n \times [0,1]$. One can imagine X as a hollowed out cylinder (with the bottom intact, basically like a bucket). One can define a *retraction* $r : D^n \times [0,1] \to X$ as follows. Consider the point $z = \mathbf{0} \times \{2\}$, the center of the disc $D^n \times \{2\}$. Given any point p in the solid cylinder $D^n \times [0,1]$, draw a straight line from z which passes through p, and then record the (unique) point q where this line hits the bucket X. Define $r : D^n \times [0,1] \to X$ which maps $p \mapsto q$. It is easy to see that r is continuous, and moreover, r is identity when restricted to the subspace X. Define the homotopy $D^n \times [0,1] \to Y$ as the composition $D^n \times [0,1] \xrightarrow{r} X \xrightarrow{F} Y$. Since r is identity on X, this composition is indeed an extension of F. Thus, $\iota : S^{n-1} \hookrightarrow D^n$ is a cofibration.

Exercise 2.12:

The construction of retraction map in Example 2.11 is also known as the *orthographic projection.* Figure out an explicit formula for the map!

Day 3 : 6th August, 2026

`contractible spaces — disjoint union — attaching spaces — mapping cylinder — mapping cone`

3.1 A Detour into Contractible Spaces

We have already defined contractible spaces (Definition 2.6). In Example 2.7 we saw that contractible spaces need to *strongly* deformation retract onto any point.

Exercise 3.1:

Given a space X, verify that the following are equivalent.

1. X is contractible.
2. X admits a deformation retract to some $x_0 \in X$.

Next, let us see some nontrivial but important examples of contractible spaces.

Example 3.2: (*Path Space is contractible*)

Given a space X, fix a point $x_0 \in X$. Consider the space of all paths in X starting at x_0

$$\mathcal{P} = \mathcal{P}(X, x_0) = \{\gamma : [0,1] \to X \mid \gamma(0) = x_0\}.$$

Give it the compact-open topology. Recall: the compact-open topology on the space $C(X, Y)$ of continuous maps is generated by the sets

$$\{f : X \to Y \mid f(C) \subset U\}, \quad C \underset{\text{compact}}{\subset} X, \quad U \underset{\text{open}}{\subset} Y.$$

Since $[0,1]$ is locally compact, it follows that the evaluation map

$$\begin{aligned} e : \mathcal{P} &\to X \\ \gamma &\mapsto \gamma(1) \end{aligned}$$

is continuous. We claim that the path space $\mathcal{P}$ is contractible.

Intuitively, the contracting homotopy is given by contracting every path back to the origin x_0, like sucking a bunch of spaghetti! Explicitly, we can consider the map

$$\begin{aligned} H : \mathcal{P} \times [0,1] &\to \mathcal{P} \\ (\gamma, t) &\mapsto (s \mapsto \gamma((1-t)s)). \end{aligned}$$

Clearly, we have $H|_{t=0} = \mathrm{Id}_{\mathcal{P}}$, and $H|_{t=1}$ maps every path γ to the constant path $\gamma_0 : s \mapsto x_0$. Thus, H is a strong deformation retract of $\mathcal{P}$ to the *point* γ_0. The fact that H is continuous is a consequence of the compact-open topology.

Example 3.3: (S^∞ *is contractible*)

Recall that the 0^{th} sphere S^0 is nothing but two points, clearly it is not even connected. We know S^1 is path connected, and later see that it is not *simply connected,* in particular S^1 is not contractible. Similarly, using other machinery (namely, either homology or higher homotopy groups) one can show that the n -sphere S^n is not contractible either.

Let us define the infinite-dimensional sphere S^∞. There are many ways to do so. Recall we have the following inclusions

$$S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow S^3 \hookrightarrow \cdots,$$

where the inclusions are the middle sphere. One defines S^∞ as the colimit of these spaces. Without going into this abstract construction, let us define it in another way.

Consider the space X to be the collection of sequences $\mathrm{x} = (x_0, x_1, x_2, ...)$ of real numbers, such that all but finitely many of them are 0 (i.e, for some large enough K, we have $x_n = 0$ for $n \geq K$). This space X is a normed linear space via the ℓ^2-norm

$$\|\mathrm{x}\| = \sqrt{\sum x_i^2}.$$

Note that the sum is a finite sum. One then defines

$$S^\infty = \{\mathrm{x} \in X \mid \|\mathrm{x}\| = 1\}$$

as the unit norm sphere in X.

We claim that S^∞ is contractible. In order to see this, consider the shift map $T : X \to X$ defined as

$$T(x_0, x_1, ...) = (0, x_0, x_1, ...).$$

One can check that T is continuous and injective. Note that for any $\mathrm{x} \in S^\infty$, we have $T(\mathrm{x})$ is *not* a multiple of x. In particular, the line $t\mathrm{x} + (1-t)T(\mathrm{x})$ is never 0. Consequently, we have a continuous map

$$\begin{aligned} F : S^\infty \times [0,1] &\to S^\infty \\ (\mathrm{x}, t) &\mapsto \frac{t\mathrm{x} + (1-t)T(\mathrm{x})}{\|t\mathrm{x} + (1-t)T(\mathrm{x})\|}. \end{aligned}$$

This is a homotopy between Id and the normalized shift map $f : \mathrm{x} \mapsto \frac{T(\mathrm{x})}{\|T(\mathrm{x})\|}$. Next, note that the point $x_0 = (1, 0, 0, ...) \in S^\infty$ is *not* in the image of this normalized shift map. Consequently, the line joining x_0 with any point in the image of f does not pass through the origin, and we have another homotopy

$$\begin{aligned} G : S^\infty \times [0,1] &\to S^\infty \\ (x, t) &\mapsto \frac{tx_0 + (1-t)f(\mathrm{x})}{\|tx_0 + (1-t)f(\mathrm{x})\|}. \end{aligned}$$

This is a homotopy from f to the constant map at x_0. Composing the two homotopies, we get a deformation retract of S^∞ to the point x_0. In particular, S^∞ is contractible.

3.2 Disjoint Union

Let us now define a few important constructions in algebraic topology, which are all special instances of quotient spaces.

Example 3.4: (*Disjoint Union*)

Given two spaces X, Y, consider the *disjoint union* $X \sqcup Y$ as a set. Explicitly, you can consider

$$X \sqcup Y := \{(x, X) \mid x \in X\} \cup \{(y, Y) \mid y \in Y\},$$

and then you can identify X, Y as *subsets* of $X \sqcup Y$ in the obvious way. Next, give it the following topology:

$U \subset X \sqcup Y$ is open if and only if both $U \cap X$ and $U \cap Y$ are open.

Verify that this topology is induced by the following equivalence relation:

$(u, A) \sim (v, B)$ if and only if $(u, A) = (v, B)$.

$X \sqcup Y$ is also known as the *topological sum* of the spaces.

The above description of the disjoint union topology is very cumbersome. The point is that given two subsets $A, B \subset C$, you can construct the union $A \cup B \subset C$, if there are elements repeating in both A and B they are not considered as distinct elements. On the other hand, for disjoint union, you want them to be distinct! That is why, any element $a \in A$ is *tagged* by a unique ID, say, (a, A) and similarly, $b \in B$ is tagged as (b, B). As an explicit example, consider $A = \{1, 2\}, B = \{2, 3\}$, then $A \cup B$ has three elements, namely $\{1, 2, 3\}$, whereas $A \sqcup B$ has 4 elements: $\{(1, A), (2, A), (2, B), (3, B)\}$.

Exercise 3.5:

Consider the spaces $A = [0, \infty)$ and $B = (-\infty, 0]$. Clearly $A \cup B = \mathbb{R}$. Justify that $A \sqcup B$ is *not* homeomorphic to $\mathbb{R}$.

Hint : What are the connected components of $A \sqcup B$?

Exercise 3.6:

Consider $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ as subspaces of $\mathbb{R}$. Is $A \sqcup B$ homeomorphic to $\mathbb{R}$?

The concept of disjoint union easily generalizes to arbitrary collection of spaces. Indeed, given spaces $\{X_\alpha\}_{\alpha \in \Lambda}$, we have the disjoint union $\bigsqcup_{\alpha \in \Lambda} X_\alpha$, where the topology is given as $U \subset \bigsqcup_{\alpha \in \Lambda} X_\alpha$ is open if and only if $U \cap X_\alpha$ is open for each $\alpha \in \Lambda$.

Exercise 3.7: (*Universal Property of Disjoint Union*)

Verify that given any space Z and any continuous maps $f : X \to Z, g : Y \to Z$, there exists a *unique* continuous map $h : X \sqcup Y \to Z$ such that $h|_X = f$ and $h|_Y = g$. Moreover, this property can be used to define the disjoint union:

if there exists a space W such that for any $f : X \to Z, g : Y \to Z$ there is a unique map $h : W \to Z$ satisfying $h|_X = f, h|_Y = g$, then W is (uniquely) homeomorphic to $X \sqcup Y$.

One denotes, $f \sqcup g : X \sqcup Y \to Z$. Do this exercise for an arbitrary collection of spaces.

3.3 Attaching Spaces

Suppose $A \subset X$ is a subspace, and $f : A \to Y$ is a map. We would like to *attach* (or glue) X to Y along this map.

Definition 3.8: (*Attaching Space*)

Given a subspace $A \subset X$ and a map $f : A \to Y$, the *attaching space* $X \cup_f Y$ is defined as the quotient space induced by the following equivalence relation on the disjoint union $X \sqcup Y$: $u \sim v$ if and only if one of the following holds

i. $u = v$.

ii. $u, v \in A$ and $f(u) = f(v)$.

iii. $u \in A$ and $v = f(u)$.

iv. $v \in A$ and $u = f(v)$

The map f is called the *attaching map*.

At first glance, this equivalence relation might seem hard to parse. Instead, we look at the universal property for the attaching spaces. Firstly, note that we have two canonical maps $p : X \to X \cup_f Y, q : Y \to X \cup_f Y$, both of which are continuous. Given an arbitrary space Z, suppose we have maps $\varphi : X \to Z, \psi : Y \to Z$. Then, as discussed earlier, we have a unique map $\varphi \sqcup \psi : X \sqcup Y \to Z$. Now, given a subspace $A \subset X$, and a map $f : A \to Z$, suppose moreover that $\varphi|_A = \psi \circ f$. Then, there exists a *unique* map $h : X \cup_f Y \to Z$ such that $h \circ p = \varphi, h \circ q = \psi$. That is, we have the diagram

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d, "p"] \arrow[ddr, bend left, "\varphi"] & \\
Y \arrow[r, "q"'] \arrow[drr, bend right, "\psi"'] & X \cup_f Y \arrow[dr, dashed, "\exists! h"] & \\
& & Z
\end{tikzcd}$$

Here is an example, we shall see more soon.

Example 3.9:

- X, Y be the closed unit disc in $\mathbb{R}^2$, A be the unit circle, and $f : A \to Y$ be the inclusion map. Then, $X \cup_f Y$ is homeomorphic to the 2-sphere S^2.
- X be the closed unit disc in $\mathbb{R}^2$, Y be the space with a single point, A be the unit circle and $f : A \to Y$ be the (obvious) constant map. Then, $X \cup_f Y$ is again homeomorphic to S^2.

The last example is also known as an *identification space*.

Definition 3.10: (*Identification Space*)

Given a subspace $A \subset X$, the *identification space* X/A is defined as the attaching space $X \cup_f \{\star\}$, where $f : A \to \{\star\}$ is the constant map. Alternatively, X/A is defined as the quotient space induced from the following equivalence relation on X:

$$u \sim v \text{ if and only if either } u, v \in A, \text{ or } u \notin A, v \notin A, u = v.$$

Intuitively, the identification space collapses the subspace $A \subset X$, and identifies it to a single point.

Exercise 3.11:

Given X as the closed unit ball in $\mathbb{R}^n$ and $A \subset X$ as the closed unit sphere S^{n-1}, verify that X/A is homeomorphic to the n-sphere S^n.

Exercise 3.12:

Suppose $A, B \subset X$ are subspaces such that $X = \mathring{A} \cup \mathring{B}$, i.e, the interiors of A and B covers X. Show that the attaching space $A \cup B$ along the intersection $A \cap B$ is homeomorphic to X.

3.4 Mapping Cylinder

The concept of attaching spaces gives rise to many important constructions in algebraic topology.

Definition 3.13: (*Mapping Cylinder*)

Given $f : X \to Y$ a map, the *mapping cylinder* is constructed as the attaching space $M_f := (X \times [0, 1]) \cup_f Y$, where we have identified f as the attaching map $f : X \times \{0\} \to Y$. We have an inclusion map

$$\begin{aligned} \iota : X &\hookrightarrow M_f \\ x &\mapsto [(x, 1)], \end{aligned}$$

and a retraction map

$$\begin{aligned} r : M_f &\to Y \\ [(x, t)] &\mapsto f(x) \\ [y] &\mapsto y. \end{aligned}$$

Proposition 3.14: (*Mapping Cylinder and Cofibration*)

Given any $f : X \to Y$, the inclusion $\iota : X \hookrightarrow M_f$ in the mapping cylinder is a cofibration, and $r : M_f \to Y$ is a homotopy equivalence. In fact, r is a strong deformation retract.

This construction gives us the following commutative diagram

$$\begin{tikzcd} & M_f \arrow[dr, bend left, two heads, "r", "s.d.r"'] & \\ X \arrow[ur, bend left, hook, "\iota"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

One says that any map $f : X \to Y$ can be replaced (up to homotopy) by a cofibration $\iota : X \to M_f$ into the mapping cylinder.

3.5 Homotopy Lifting Property

The dual concept of homotopy *extension* is homotopy *lifitng*. Given a map $p : E \to B$, we say p has *homotopy lifting property* (or *HLP*) against a spaces X if given any map $f : X \to E$ and any homotopy $h : X \times [0,1] \to B$ satisfying $h|_{t=0} = p \circ f$, there exists a lift $H : X \times [0,1] \to E$ such that

$$H(x,0) = f(x) \quad x \in X, \qquad p \circ H(x,t) = h(x,t) \quad x \in X, 0 \le t \le 1.$$

In other words, given a commutative diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

there exists a lift H making everything commute.

Definition 3.15: (*Fibration*)

A map $p : E \to B$ is called a *(Hurewicz) fibration* if it has homotopy lifting property against any space X.

Let us now see few examples of fibrations.

Example 3.16:

Given any space X, we have the constant map $p : X \to \{\star\}$, and it is easy to see a fibration: the lift is given by constant homotopy. More generally, given a product $B \times F$, consider the projection $\pi : B \times F \to B$. Then, π is a fibration. Indeed, given any diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d, hook] & B \times F \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"'] & B \end{tikzcd}$$

define $H(x,t) = f(x)$. It is easy to see that H is a homotopy lifting here.

Remark 3.17:

A more general class of fibrations is given by Serre fibrations: a map $p : E \to B$ is a *Serre fibration* if it has HLP against all CW complexed. Equivalently, one can define Serre fibrations as those maps $p : E \to B$ which has HLP against all the closed n-discs D^n.

One can show that a fiber bundle (which is a *twisted* product space) is a Serre fibration, and it is a Hurewicz fibration if B is a paracompact space. We shall later define *covering spaces,* which are examples of fiber bundles. In particular, they are Serre fibrations.

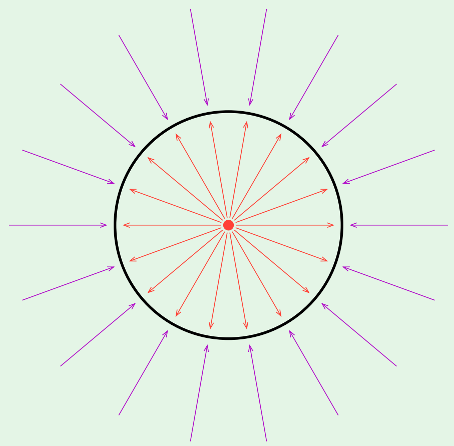
Day 4 : 11th August, 2026

`examples of homotopy equivalence — wedge sum — topologist's sine curve — Hawaiian earring — local contractibility`

4.1 Examples of Homotopy Equivalence

Example 4.1: (*Puncturing the Euclidean Plane*)

Consider the space $X = \mathbb{R}^2$. If you remove a single point, you can show that the space is homotopy equivalent to the circle S^1. In fact, you can get a strong deformation retract to a circle of radius 1 centered at the puncture point.



Example 4.2: (*Puncturing the Euclidean Plane Again*)

Let's say you remove 2-points, say, p, q from $\mathbb{R}^2$. This time, you can again get a strong deformation retract as follows. First, determine the mid point, say, x of the line joining p and q. Say, the distance between p and q is $2d$. Draw the two circles of radii d, centered at p and q respectively; they touches exactly at the point x. Then, push everything inside the two discs away from the punctures to the boundary circles. You get a strong deformation retract onto the plane with two open discs removed. Next, draw the circle of radius $2d$ about the point x and push everything from outside to the boundary circle. You are left with a closed disc, with two open discs removed. Finally, collapse everything from top and bottom to the original circles of radius d. You are left with a figure-8 shape.

Before puncturing more holes, let us define a the notion of wedge sum.

Definition 4.3: (*Wedge Sum*)

Let $\{(X_\alpha, x_\alpha) \mid \alpha \in \Lambda\}$ be a collection of pointed spaces, i.e, a space with a point in it. The *wedge sum*, denoted $\bigvee_{\alpha\in\Lambda} X_\alpha$ is defined as the identification space $\frac{\bigsqcup_{\alpha\in\Lambda} X_\alpha}{\bigsqcup_{\alpha\in\Lambda}\{x_\alpha\}}$. That is, we consider the disjoint union $\bigsqcup_{\alpha\in\Lambda} X_\alpha$, and then identify all the basepoints $\bigsqcup_{\alpha\in\Lambda}\{x_\alpha\}$ to a single point.

The figure-8 shape in Example 4.2 is simply the wedge sum $S^1 \vee S^1$. To be precise, we should fix a basepoint in S^1, but it is easy to see that the wedge sum of path connected spaces are independent of such choice.

Example 4.4: (*Puncturing the Euclidean Plane Again and Again and…*)

Let us now remove n-many points from $\mathbb{R}^2$. Again, by suitably choosing a radius, you can strongly deforme this to $\mathbb{R}^2$ with n-many disjoint open discs removed. The trick is to collapse the outside of the boundary circles to a single point, and thus, we get n-fold wedge of circles. This is no longer a strong deformation retract.

Here is a different way to see this. Firstly, we claim that there is a *homeomorphism* of $\mathbb{R}^2$ to itself which takes any prescribed n-many points to another set of prescribed n-many points. Using this homeomorphism, we can assume that the punctures are arranged symmetrically along the unit circle. Then, draw n-many circles about the punctures which touches the neighboring two circles exactly at one point. Radially perform deformation retract so that you are left with the touching circles and the inside of the disc (which looks like a cookie after taking n-many symmetric bites). Finally, note that the inside is a contractible subspace, and hence you can collapse it. This gives the same space as the n-fold wedge sum of the circles.

Exercise 4.5: (*Neckless of Circles*)

Arrange n-many circles like a necklace: any circle touches exactly two neighboring circle, each at exactly one point. Can you identify this space as a wedge sum?

Exercise 4.6:

Let X be the space obtained from the sphere S^2 by removing $(n+1)$-many points. Can you describe X homotopy equivalent to certain wedge sum?

Before giving more examples of homotopy equivalent spaces, let us state a very useful result that we have already used a few times.

Proposition 4.7: (*Collapsing Contractible Subspaces*)

Let $A \subset X$ be given. Suppose A is contractible and the inclusion $\iota : A \hookrightarrow X$ is a cofibration. Then, the quotient map $q : X \to X/A$ is a homotopy equivalence.

Proof : Since A is contractible, we have a contracting homotopy $h : A \times [0,1] \to A$ which homotopes Id_A to a constant map. Since $\iota : A \hookrightarrow X$ is a cofibration, by HEP we can extend this homotopy to $H : X \times [0,1] \to X$, with $H|_{X\times\{0\}} = \mathrm{Id}_X$. Note that $H_1 := H|_{X\times\{1\}} : X \to X$ maps A to a single point, and hence, induces a map $r : X/A \to X$ such that $r \circ q = H_1$. In particular, H is a homotopy of Id_X with $r \circ q$.

On the other hand, the maps $H_t := H|_{X\times\{t\}}$ maps A into A, since H extends h. Thus, again passing to the quotient, we have a map $\tilde{H} : X/A \times [0,1] \to X/A$. One can show that this map is continuous, and hence, a homotopy of $\mathrm{Id}_{X/A}$. Note that

$$\tilde{H}_1([x]) = \tilde{H}_1(q(x)) = qH_1(x) = qrq(x) = qr([x]).$$

Thus, $\tilde{H}$ is a homotopy of $\mathrm{Id}_{X/A}$ with $q \circ r$.

This proves that $q : X \to X/A$ is a homotopy equivalence, with homotopy inverse $r : X/A \to X$. $\square$

Remark 4.8:

The following is a result due to J. H. C. Whitehead : Suppose $q : X \to Y$ is a quotient map and Z is locally compact. Then, the product $p := q \times \mathrm{Id}_Z : X \times Z \to Y \times Z$ is a quotient map.

Since $[0, 1]$ is locally compact, in particular, we have $\tilde{q} : X \times [0, 1] \to X/A \times [0, 1]$ a quotient map. Now, in Proposition 4.7, the composition $q \circ H : X \times [0, 1] \to X/A$ takes $A \times \{t\}$ to a point for all t, and hence induces a map $\tilde{H} : X/A[0, 1] \to X/A$. By the property of quotient topology, $\tilde{H}$ is a continuous map.

Example 4.9: (*Pinching a Sphere*)

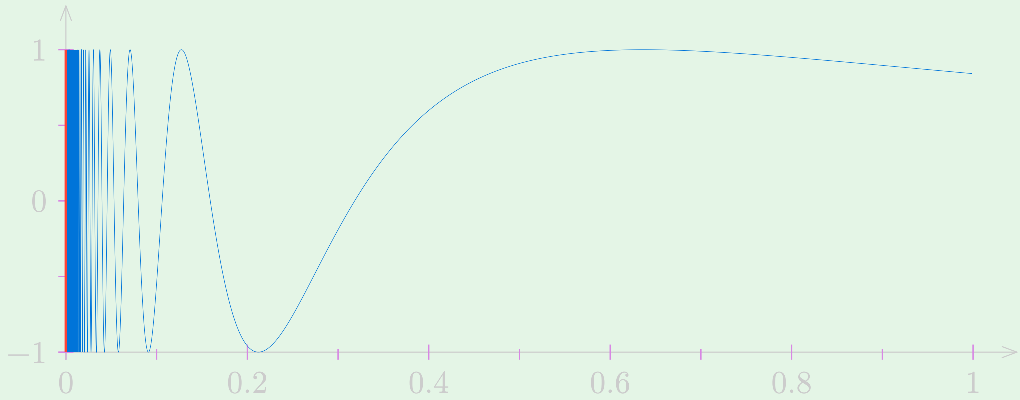
Consider the sphere S^2, along with the line $[-1, 1]$ along the z-axis which touches the sphere at the north and south poles. Since the line segment is contractible, we can collapse it. This spaces is known as the *pinched sphre,* which is obtained by identifying two distinct points on the sphere. We claim that this is also homotopy equivalent to a wedge sum, namely, $S^2 \vee S^1$. To see this slide one end of the line segment to the other end and thus getting a circle. To justify this in another way, draw an arc of a great circle passing through the two points. This arc is contractible, and hence, collapsing it will give a homotopy equivalence. In particular, we have that the pinched sphere is homotopy equivalent to $S^2 \vee S^1$.

Caution 4.10:

We have been using Proposition 4.7 without checking whether the inclusion of the subspace is a cofibration. They are in fact cofibrations, since there are CW complex structures adapted to these inclusions. There are examples where this is not the case!

Example 4.11: (*Topologist's Sine Curve*)

Consider the following topologist's sine curve

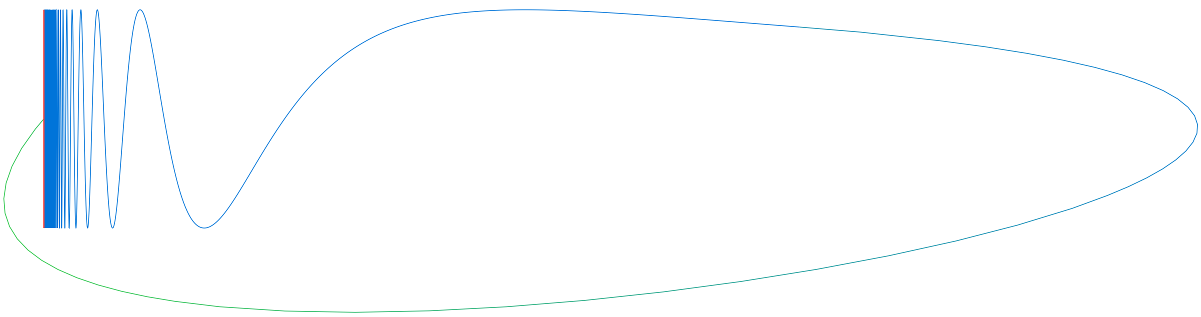


$$X = \{0\} \times [0,1] \cup \left\{ \left(x, \sin\frac{1}{x} \right) \mid 0 < x \leq 1 \right\}.$$

This space connected, but not path connected! Indeed, one can see that X is the closure of the graph of $\sin\frac{1}{x}$ over the segment $(0,1]$, the points on the vertical segment $A := \{0\} \times [0,1]$ being the limit points. Since the graph is homeomorphic to $(0,1]$ (why?!), it is clearly connected, and hence the closure X is also connected. On the other hand, one can show that X has two path components: one is the graph of $\sin\frac{1}{x}$ and the other path component is the vertical segment. Thus, X is not contractible.

Now, the subspace A is contractible being an interval. We claim that X/A is homeomorphic to $[0,1]$, hence contractible, and consequently, cannot be homotopy equivalent to X. Consider the projection map $\pi : X \to [0,1]$. Observe that π maps A to the single point $\{0\}$, and thus passes to the quotient $\tilde{\pi} : X/A \to [0,1]$. We have $\tilde{\pi}$ is bijective. Also, X/A is compact (being the quotient of the compact space X) and $[0,1]$ is Hausdorff. Hence, $\tilde{\pi}$ is a homeomorphism.

Continuing with the previous example, let us construct the *Warsaw circle*.



This is obtained from the topologist's sine curve, by adding a curve starting at $(1, \sin(1))$ and ending at the origin. The Warsaw circle is path connected, but not locally path connected at any of the points of vertical interval.

Proposition 4.12:

Let W denote the Warsaw circle as above. Suppose K is a sequentially compact, locally path connected space. Then, any map $K \to W$ is null-homotopic, i.e, homotopic to a constant map.

<u>Proof</u> : Let us breakdown the Warsaw circle as follows.

- Denote the closed arc attached to the topologist's sine curve by C. Clearly, C is homeomorphic to $[0,1]$.
- Denote the vertical segment $V := \{0\} \times [0,1]$
- Denote the segment $S = \left\{(x, \sin\frac{1}{x}) \mid 0 < x \le 1\right\}$, which is homeomorphic to $(0,1]$.

We have, $W = V \cup S \cup C$. Now, for each $0 < \varepsilon \le 1$, denote the subspaces

$$S_\varepsilon = \left\{\left(x, \sin\frac{1}{x}\right) \middle| \varepsilon \le x \le 1\right\}, \quad W_\varepsilon = V \cup S_\varepsilon \cup C.$$

Observe that W_ε is clearly contractible, and hence, any map $f : K \to W_\varepsilon$ is null-homotopic. We claim that any map $f : K \to W$ is actually contained in some W_ε. In fact, we show that $f(K) \cap S \subset S_\varepsilon$ for some $\varepsilon > 0$. Suppose not. Let us denote $u(x) = \pi_1(f(x))$, projection of f onto the first component. Then, for each $n \ge 1$, there exists some $x_n \in K$ such that $0 < u(x_n) < \frac{1}{n}$. Since K is sequentially compact, passing to a subsequence, we have $x_n \to x_0$. Since u is continuous, it follows that $u(x_0) = 0$. Thus, $f(x_0)$ is on the vertical segment V. Choose small ball U around $f(x_0)$, such that $U \cap W \not\supset C$. As f is continuous, there is an open neighborhood $x_0 \in V \subset X$ such that $f(V) \subset U \cap W$. Since X is locally path connected, we can assume that V is path connected. Now, for some $n \gg 1$, we have $x_n \in V$. Get a path $\gamma : [0,1] \to V$ joining x_n to x_0. It follows that $f \circ \gamma$ is a path joining $f(x_0) \in V$ to $f(x_n) \in S$. But any such path must traverse the entire curve C. This is a contracdiction, since the path $f \circ \gamma$ is contained in U, and $U \cap W \not\supset C$. This proves the claim. $\square$

Since each sphere S^n is sequentially compact and locally path connected, it follows from Proposition 4.12 that any map $S^n \to W$ is null-homotopic. (This leads to the fact that the Warsaw circle is *weakly contractible,* i.e, every homotopy group of the Warsaw circle is zero with respect to any basepoint).

Exercise 4.13:

If X is contractible, show that any map $f : Y \to X$ is null-homotopic.

Proposition 4.14:

The Warsaw circle is *not* contractible.

Proof : Firstly, observe that the identification space W/V, obtained by collapsing the vertical segment, is homeomorphic to S^1. Thus, we have a map $f : W \to S^1$. We show that this map is *not* null-homotopic. If possible, suppose $h : f \simeq c$ is a homotopy to a constant map $c : W \to S^1$. Consider the map $p : \mathbb{R} \to S^1$, given by $p(\theta) = (\cos\theta, \sin\theta)$. It is well-known (we shall see a prove later) that p is a fibration, i.e, p has homotopy lifting property against all spaces. Since c is a constant map, we can always get a lift $\tilde{c} : W \to \mathbb{R}$ such that $p \circ \tilde{c} = c$. Then, by HLP, we have a lift of h to $\tilde{h} : W \times [0,1] \to \mathbb{R}$. In particular, we have a lift $\tilde{f} : W \to \mathbb{R}$ such that $p \circ \tilde{f} = f$. Now, f is injective everywhere but V, and hence, $\tilde{f}$ is also injective, except possible on V, i.e, $\tilde{f}|_{W \setminus V}$ is injective. Now, $f(V)$ is a point, and $\tilde{f}(V) \subset p^{-1}(f(V)) = \mathbb{Z}$. As V is path connected, the only possibility is that $\tilde{f}(V)$ is also a singleton. Thus, we can pass to the quotient, and get a map $\hat{f} : W/V \to \mathbb{R}$. It follows that $\hat{f}$ is injective. But $W/V \cong S^1$, and (by the intermediate value theorem) there is no injective continuous map $S^1 \to \mathbb{R}$. This is a contradiction. Hence, $f : W \to S^1$ cannot be null-homotopic. This implies that W is not contractible. $\square$

Thus, the Warsaw circle is an example of a weakly contractible space, which is not contractible!

Recall that we can define wedge sum of arbitrary many (pointed) spaces.

Exercise 4.15: (*Countable Infinite Wedge of Circles*)

Consider the space

$$X = \bigcup_{n \geq 1} \{(x, y) \mid (x-n)^2 + y^2 = n^2\} \subset \mathbb{R}^2.$$

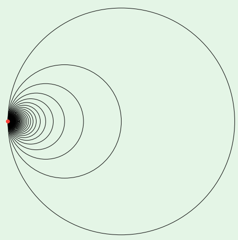
Show that X is homeomorphic to the infinite wedge $\bigvee_{n \geq 1} S^1$. Can you directly show that $\bigvee_{n \geq 1} S^1$ is noncompact? Show that given a compact subspace $K \subset X$, there exists some large N such that

$$K \subset \bigcup_{n=1}^{N} \{(x, y) \mid (x-n)^2 + y^2 = n^2\} \subset X.$$

The next (in)famous example is similar to the infinite wedge of circle, but is vastly different from it!

Example 4.16: (*Hawaiian Earring*)

Consider the space



$$H := \bigcup_{n \geq 1} \left\{ (x, y) \,\middle|\, \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2} \right\} \subset \mathbb{R}^2.$$

This space is known as the Hawaiian earring. Clearly, H is compact (being closed and bounded), path connected, locally path connected space. One can see that the space X/A considered in Caution 4.10 is actually homeomorphic to the Hawaiian earring! We claim that H is not even homotopy equivalent to the countable infinite wedge of circles.

Intuitively, it is clear that H is not contractible, we shall justify this using fundamental group. Let us consider the point $x_0 = (0, 0) \in H$. Any open neighborhood of x_0 will contain infinitely many copies of circles (in fact, a shrunk down copy of the Hawaiian earring itself). Thus, any open neighborhood of x_0 fails to be contractible. But this is not the case for $\bigvee_{n=1}^{\infty} S^1$. This implies that H is not homotopy equivalent to the countable infinite wedge of circles.

Let us give a name to the property discussed above.

Definition 4.17: (*Locally Contractible*)

A space X is called *locally contractible* if given any point $x \in X$ and a neighborhood $x \in U \subset X$, there exists an open set $x \in V \subset U$ such that V is contractible.

We have seen that the Hawaiian earring is *not* locally contractible, but the infinite wedge $\bigvee_{n\geq 1} S^1$ evidently is. Thus, they cannot be homeomorphic. In fact, every CW complex is locally contractible. Hence, Hawaiian earring cannot be given the structure of a CW complex.

Caution 4.18:

In general, local contractibility is not preserved under homotopy equivalence. Indeed, consider the usual comb space

$$X = ([0,1]\times\{0\}) \cup \bigcup_{n\geq 1}\left\{\left(x, \frac{x}{n}\right) \,\middle|\, 0 \leq x \leq 1\right\}.$$

X fails to be locally contractible at the point $(1,0)$. But X contractible, i.e, homotopy equivalent to a point, which is tautologically locally contractible!

Remark 4.19:

It is a fact that the Hawaiian earring is not even homotopy equivalent to any CW complex. But to prove this we need more sophisticated argument. More generally, it does not even have the weak homotopy type of a CW complex.

Day 5 : 13th August, 2026

`relative homotopy – pointed space – basepoint preserving homotopy – attaching cells`

5.1 Relative Homotopy

Let us now generalize the notion of homotopy in the following way.

Definition 5.1: (*Relative Homotopy*)

Let X, Y be spaces, and $A \subset X$ is a subspace. A homotopy $h : X \times [0,1] \to Y$ is said to be *relative* to A if h fixes the points of A, i.e, if $h(a,t) = h(a,0)$ for all $a \in A$ and all $0 \leq t \leq 1$. If $f = h|_{t=0}$ and $g = h|_{t=1}$, we denote a relative homotopy as $h : f \simeq g \quad (\mathrm{rel}\ A)$.

Example 5.2:

Suppose $A \subset X$ is a retract, $r : X \to A$ be the retraction map. Then, r is *strong* deformation retract precisely when there is a relative homotopy $\iota \circ r \simeq \mathrm{Id}_X \quad (\mathrm{rel}\ A)$.

A special case of relative homotopy is given by basepoint preserving homotopies.

Definition 5.3: (*Pointed Spaces and Maps*)

A *pointed space* (X, x_0) is a space X along with a fixed *basepoint* $x_0 \in X$. A *pointed map* (or *based map*) $f : (X, x_0) \to (Y, y_0)$ between pointed spaces is a continuous map $f : X \to Y$ such that $f(x_0) = y_0$. A *basepoint preserving homotopy* is a homotopy $h : X \times [0,1]$ such that $h(x_0, t) = y_0$ for all $0 \leq t \leq 1$. A *basepoint preserving homotopy equivalence* is a pointed map $f : (X, x_0) \to (Y, y_0)$ such that there is another map $g : (Y, y_0) \to (X, x_0)$, and basepoint preserving homotopies $g \circ f \simeq \mathrm{Id}_X$, $f \circ g \simeq \mathrm{Id}_Y$.

It is easy to see that a basepoint preserving homotopy between two pointed maps $f, g : (X, x_0) \to (Y, y_0)$ is precisely a relative homotopy $f \simeq g \quad (\mathrm{rel}\ \{x_0\})$.

So far, all the examples that we have seen, we were not very careful about basepoints! Turns out in many nice situations, we don't need to worry about basepoints. The next few results justify this claim. First, we need a definition.

Definition 5.4: (*Well-pointed Space*)

A pointed space (X, x_0) is called *well-pointed* if $\{x_0\}$ is closed in X, and the inclusion map $\{x_0\} \hookrightarrow X$ is a cofibration, i.e, has homotopy extension property against all spaces.

Lemma 5.5:

Let $(X, x_0), (Y, y_0)$ be pointed spaces. Suppose (X, x_0) is well-pointed, and Y is path connected. Then, given any map $f : X \to Y$, there exists a based map $f' : (X, x_0) \to (Y, y_0)$ such that $f \simeq f'$.

Proof : Since Y is path connected, we can get a path $\gamma : [0, 1] \to Y$ such that

$$\gamma(0) = f(x_0), \quad \gamma(1) = y_0.$$

Consider the subspace $Z = (X \times \{0\}) \cup (\{x_0\} \times [0, 1]) \subset X \times [0, 1]$. Since $\{x_0\}$ is closed in X, by pasting lemma, it follows that we have a map $F : Z \to Y$ defined as

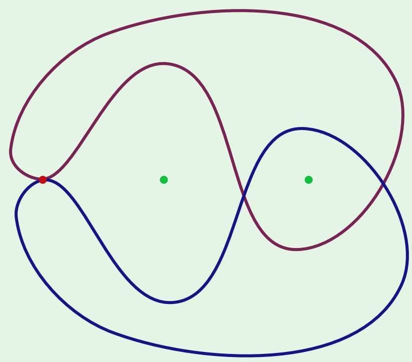
$$F(x, 0) = f(x), \quad F(x_0, t) = \gamma(t).$$

Since $\{x_0\} \hookrightarrow X$ has homotopy extension property against Y, it follows that F can be extended to a map $H : X \times [0, 1] \to Y$. Clearly F is a homotopy from f to a map $f' := F|_{t=1}$. By construction, $f'(x_0) = F(x_0, 1) = \gamma(1) = y_0$. Thus, f' is a based map, and $f \simeq f'$. $\square$

The above result is an example of a *strictification*. One says that an arbitrary map can be *strictified* to a based map, under the appropriate hypothesis. Since a homotopy is also a map, you may try to strictify it in a similar way, but that does not always work!

Example 5.6:

Consider the double-punctured plane $X = \mathbb{R}^2 \setminus \{p = (1, 0), q = (2, 0)\}$, and fix the origin $x_0 = (0, 0)$ as the base point in X.



Then, as in the above picture, we can have two map $\gamma_1, \gamma_2 : S^1 \to X$. Let us fix a basepoint on S^1, say, z_0, and make sure that $\gamma_1(z_0) = x_0 = \gamma_2(z_0)$. It is easy to see that $\gamma_1 \simeq \gamma_2$ if we don't care about basepoints: just slide the two curves avoiding the puncture at p. But γ_1 and γ_2 are *not* based homotopic (as they represent distinct elements in the fundamental group $\pi_1(X, x_0)$).

A homotopy which does not respect basepoints is also known as *free* homotopy. The next result says that under some nice conditions (which are met more often than not!), one can strictify a homotopy equivalent.

Theorem 5.7:

Let $(X, x_0), (Y, y_0)$ be well-pointed, path connected spaces. Suppose $f : X \to Y$ is a homotopy equivalence, with homotopy inverse $g : (Y, y_0) \to (X, x_0)$, and homotopies $\varphi : g \circ f \simeq \mathrm{Id}_X, \psi : f \circ g \simeq \mathrm{Id}_Y$. Then, there exists a basepoint-preserving homotopy equivalence $f' : (X, x_0) \to (Y, y_0)$, with homotopy inverse $g' : (Y, y_0) \to (X, x_0)$, and basepoint preserving homotopies $\varphi' : g' \circ f' \simeq \mathrm{Id}_X, \psi' : f' \circ g' \simeq \mathrm{Id}_Y$, such that $f \simeq f', g \simeq g', \varphi \simeq \varphi', \psi \simeq \psi'$.

In other words, a homotopy equivalence $X \to Y$ can be homotoped in to a basepoint-preserving homotopy equivalence $(X, x_0) \to (Y, y_0)$.

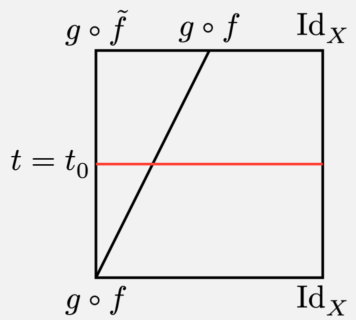
__Proof__ **:** Applying Lemma 5.5 to f, we can first get a based map $\tilde{f} : (X, x_0) \to (Y, y_0)$, and a homotopy $\zeta : \tilde{f} \simeq f$. We have a composed homotopy $\Phi : g \circ \tilde{f} \simeq g \circ f \simeq \mathrm{Id}_X$, given explicitly as

$$\Phi(x, t) = \begin{cases} g\zeta(x, 2t), & 0 \leq t \leq \frac{1}{2} \\ \varphi(x, 2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

We have a homotopy of homotopies, say, $\chi : \varphi \simeq \Phi$ given explicitly as

$$\chi(x, s, t) = \begin{cases} g\zeta(x, (1 - t) + 2st), & 0 \leq s \leq \frac{t}{2} \\ \varphi\left(x, \frac{2s - t}{2 - t}\right), & \frac{t}{2} \leq s \leq 1. \end{cases}$$

It is eay to see that χ is indeed continuous, and is the desired homotopy. In order to understand χ, consider the schematic diagram



At some time $t = t_0$, the homotopy χ travels the tail-end of the homotopy $g\zeta$ in double speed, and then follows the homotopy φ in a suitable faster speed; length of the tail grows from 0 to all of it as t moves from 0 to 1. The above formula of χ can be deduced from this diagram.

Next, we homotope g to fix basepoint, but in a controlled way. Consider the path

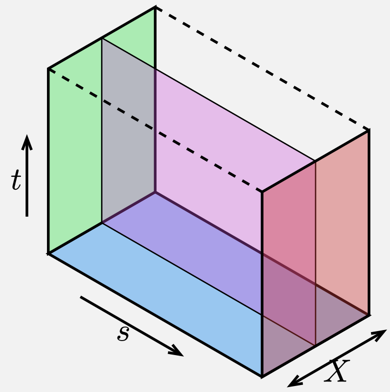
$$\gamma : t \mapsto \chi(x_0, t, 1) = \Phi(x_0, t)$$

in X, which joins $g\tilde{f}(x_0) = g(y_0)$ to x_0. Then, we have map

$$g \cup \gamma : (Y \times \{0\}) \cup (\{y_0\} \times [0, 1]) \to X.$$

Since (Y, y_0) is well-pointed, we have a based map $g' : (Y, y_0) \to (X, x_0)$, and a homotopy, $\xi : g' \simeq g$. Just like above, we have the composed homotopy $\tilde{\Phi} : g' \circ \tilde{f} \simeq g \circ \tilde{f} \simeq \mathrm{Id}_X$. Moreover, the same trick as above, gives a homotopy of homotopies $\Phi \simeq \tilde{\Phi}$; composing we get homotopy $\varphi \simeq \tilde{\Phi}$.

Now, we need to homotope the homotopy $\tilde{\Phi}$ to a basepoint-preserving homotopy. We should do this relative to the boundary $\{0, 1\}$ for the time parameter. Consider the subspace



$$B = (\{x_0\} \times [0,1] \cup X \times \{0,1\}) \times [0,1] \subset X \times [0,1] \times [0,1]$$

Since (X, x_0) is well-pointed, it follows that $B \hookrightarrow X \times [0,1]^2$ is again a cofibration. Now, we have a map defined on B.

- On $X \times [0,1] \times \{0\}$ it is the homotopy $\tilde{\Phi}$.
- On $X \times \{0\} \times [0,1]$ it is the map $g' \circ \tilde{f}$ constantly.
- On $X \times \{1\} \times [0,1]$ it is the map Id_X constantly.
- Finally, we need to define a map on $\{x_0\} \times [0,1] \times [0,1]$. Observe that all the four edges of this square is already prescribed. One the bottom edge $\{x_0\} \times [0,1] \times \{0\}$ it is the curve

$$\begin{aligned} \eta : t \mapsto \tilde{\Phi}(x_0, t) &= \begin{cases} \xi\Big(\tilde{f}(x_0), 2t\Big), & 0 \le t \le \frac{1}{2} \\ \Phi(x_0, 2t-1), & \frac{1}{2} \le t \le 1 \end{cases} \\ &= \begin{cases} \xi(y_0, 2t), & 0 \le t \le \frac{1}{2} \\ \gamma(2t-1), & \frac{1}{2} \le t \le 1 \end{cases} \\ &= \begin{cases} \gamma(1-2t), & 0 \le t \le \frac{1}{2} \\ \gamma(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}. \end{aligned}$$

  Thus, the curve γ is traced backward from x_0 to $g(y_0)$, and then back to x_0. We can then fill the square by homotoping this path back to the constant path x_0

Then, using the HEP, we have a homotopy

$$\varphi' : g' \circ \tilde{f} \simeq \mathrm{Id}_X \quad (\mathrm{rel}\ x_0),$$

and moreover, a homotopy of homotopies $\varphi \simeq \tilde{\Phi} \simeq \varphi'$ obtained from composition.

Finally, we fix the homotopy ψ. We already have,

$$\tilde{f} \circ g' \underset{\tilde{f} \circ \xi}{\simeq} \tilde{f} \circ g \underset{\zeta \circ g}{\simeq} f \circ g \underset{\psi}{\simeq} \mathrm{Id}_Y.$$

We perform the previous steps for by switching the roles (f, g) by (g', f). Note that we do not need to modify g' as it is already basepoint preserving. We get a homotopy $f' \simeq \tilde{f}$ such that $f' : (X, x_0) \to (Y, y_0)$ is based, and a homotopy $\psi' : f' \circ g' \simeq \mathrm{Id}_Y \quad (\mathrm{rel}\ \{y_0\})$. Clearly, $\psi \simeq \psi'$ follows from the construction. Also,

$$f' = f' \circ \mathrm{Id}_X \underset{\mathrm{rel}\ \{x_0\}}{\simeq} f' \circ \Big(g' \circ \tilde{f}\Big) = (f' \circ g') \circ \tilde{f} \underset{\mathrm{rel}\ \{x_0\}}{\simeq} \mathrm{Id}_Y \circ \tilde{f} = \tilde{f}.$$

The last homotopy is relative to x_0 since $\tilde{f}(x_0) = y_0$, and the homotopy $f' \circ g' \simeq \mathrm{Id}_Y$ is relative to y_0. But then we have $f' \simeq \tilde{f} \quad \mathrm{rel}\ \{x_0\}$. Composing, we have $f' \circ g' \simeq \tilde{f} \circ g' \simeq \mathrm{Id}_X$, relative to x_0. Clearly, the homotopies themselves are homotopic as well. This completes the claim. □

5.2 Mapping Cone

Let us revisit the construction of mapping cylinder.

Definition 5.8: (*Mapping Cone*)

Given a map $f : X \to Y$, the *mapping cone* C_f (or $C(f)$) is defined as the identification space $M_f/X \times \{1\}$. Explicitly, we have

$$C_f = \frac{(X \times [0,1]) \cup_f Y}{X \times \{1\}},$$

where the attaching map $f : X \to Y$ is on $X \times \{0\}$. We have an inclusion $Y \hookrightarrow C_f$.

The role of 0 and 1 in the definition is just a choice. One could have easily attached the space Y along $X \times \{0\}$, and collapsed $X \times \{1\}$; this would give a naturally homeomorphic copy of the mapping cone.

There are two special examples of mapping cones, which are important on their own.

Definition 5.9: (*Cone and Suspension*)

Given a space, the *cone* on X is denoted by $C(X)$ and is defined as the mapping cone of the identity map Id_X. Explicitly, we have the identification space

$$C(X) := (X \times [0,1])/(X \times \{0\}).$$

The *suspension* of the space X, denoted as ΣX is defined as the mapping cone of the constant map $X \to \{\star\}$. Explicitly, we have

$$\Sigma X := C(X)/(X \times \{1\})$$

Exercise 5.10:

Verify that there are homeomorphisms

$$C(D^n) = D^{n+1}, C(S^n) = D^n, \Sigma(D^n) = D^{n+1}, \Sigma(S^n) = S^{n+1},$$

where D^n is the closed n-disc and S^n is the n-sphere.

Exercise 5.11:

Given any space, show that the cone $C(X)$ is contractible. In fact, prove that it strongly deformation retracts onto the coning point, which is the equivalence class of $[(x, 0)]$ for any $x \in X$.

Let us prove a very important property of the mapping cone, which is true for the mapping cylinder as well.

Proposition 5.12: (*Mapping Cone is Homotopy Invariant*)

Suppose $f \simeq g : X \to Y$ are homotopic maps. Then the mapping cones $C_f \simeq C_g$ are homotopy equivalent. Similarly, the mapping cylinders $M_f \simeq M_g$ are also homotopy equivalent.

Proof : Suppose $h : f \simeq g$ is a homotopy. We first construct the map

$$\begin{aligned} \tilde{\varphi} : (X \times [0,1]) \sqcup Y &\to C_g \\ (x,t) &\mapsto \begin{cases} [(x,2t)], & 0 \le t \le \frac{1}{2} \\ [h(x, 2-2t)], & \frac{1}{2} \le t \le 1 \end{cases} \\ y &\mapsto [y]. \end{aligned}$$

Clearly this is continuous with the disjoint union topology since at $t = \frac{1}{2}$ and $t = 1$ the function definition matches. Intuitively, $\tilde{\varphi}$ maps half of the cylinder $X \times \left[0, \frac{1}{2}\right]$ to the cone part of C_g, and then for the rest of the cylinder $X \times \left[\frac{1}{2}, 1\right]$ it follows the homotopy h back from g to f. Moreover, we have $\tilde{\varphi}(x,0) = [(x,0)] = \{\star\}$ and $\tilde{f}(x,1) = [h(x,0)] = [f(x)] = \tilde{\varphi}(f(x))$. Thus, passing to the quotient, we have a continuous map $\varphi : C_f \to C_g$. Similarly, we have a map $\psi : C_g \to C_f$ induced by the map

$$\begin{aligned} \tilde{\psi} : (X \times [0,1]) \sqcup Y &\to C_f \\ (x,t) &\mapsto \begin{cases} [(x,2t)], & 0 \le t \le \frac{1}{2} \\ [h(x, 2t-1)], & \frac{1}{2} \le t \le 1 \end{cases} \\ y &\mapsto [y]. \end{aligned}$$

We claim that $\varphi : C_f \to C_g$ is a homotopy equivalence, with homotopy inverse $\psi : C_g \to C_f$. Observe that

$$\begin{aligned} \psi \circ \varphi : C_f &\to C_f \\ [(x,t)] &\mapsto \begin{cases} [(x,4t)], & 0 \le t \le \frac{1}{4} \\ [h(x, 4t-1)], & \frac{1}{4} \le t \le \frac{1}{2} \\ [h(x, 2-2t)], & \frac{1}{2} \le t \le 1 \end{cases} \\ [y] &\mapsto [y] \end{aligned}$$

Thus, to get a homotopy $\psi \circ \varphi \simeq \mathrm{Id}_{C_f}$ we only need to scale the homotopies suitably. Indeed, consider the map

$$\begin{aligned} \alpha : C_f \times [0,1] &\to C_f \\ ([(x,s)],t) &\mapsto \begin{cases} \left[\left(x, \frac{4s}{3t+1}\right)\right], & 0 \le s \le \frac{3t+1}{4} \\ [h(x, 4s-(3t+1))], & \frac{3t+1}{4} \le s \le \frac{t+1}{2} \\ [h(x, 2-2s)], & \frac{t+1}{2} \le s \le 1 \end{cases} \\ [y] &\mapsto [y] \end{aligned}$$

The continuity follows from the pasting lemma by checking that the function is well-defined at each boundary values. Finally, we have $\alpha|_{t=0} = \psi \circ \varphi$ and $\alpha|_{t=1} = \mathrm{Id}_{C_f}$. Thus, $\alpha : \psi \circ \varphi \simeq \mathrm{Id}_{C_f}$ is a homotopy. Similarly, we can define a homotopy $\beta : \varphi \circ \psi \simeq \mathrm{Id}_{C_g}$. This proves the claim. $\square$

Exercise 5.13:

Suppose $f : X \to Y$ is null-homotopic. Show that the mapping cone C_f is homotopy equivalent to $\Sigma X \vee Y$

Exercise 5.14:

Suppose $A \subset X$ is a subspace and the inclusion $\iota : A \hookrightarrow X$ is a cofibration. Then, show that the mapping cone C_ι is homotopy equivalent to X/A. Give a counterexample if we drop the cofibration hypothesis.

5.3 Attaching Cells

Given a space X and a map $f : S^{n-1} \to X$, we say that the mapping cone C_f is *obtained from X by attaching an n-cell to X via the map f*. Explicitly, we have

$$C_f = D^n \cup_f X,$$

which justifies the terminology.

Proposition 5.15:

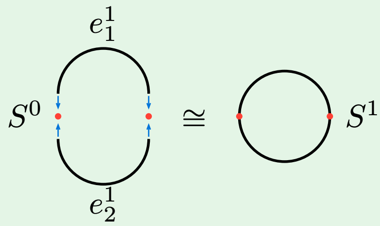
Given $f : S^{n-1} \to X$, there is a homeomorphic copy of the open n-disc in the mapping cone C_f, which is called an *n-cell*.

Proof : Recall that $D^n = C(S^{n-1})$. Indeed, we can *uniquely* write any element $x \in D^n \setminus \{0\}$ as $x = (z, r)$, where $z \in S^{n-1}$ and $r = \|x\|$. Consider the obvious map $g : \mathring{D}^n \to C_f$ given as the inclusion $\mathring{D}^n$ follows by the quotient map $q : D^n \sqcup X \to C_f$. We see that g is a continuous map, which is bijective onto the image. Next, consider an open set $U \subset \mathring{D}^n$. Then, $g(U) \subset C_f$ is a saturated set and $q^{-1}(U) = U$ is open in $D^n \sqcup X$. Hence, g is an open map as well. Consequently, g is a homeomorphism onto its image. □

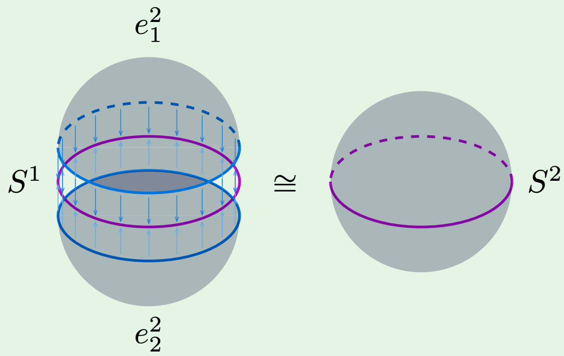
As a consequence of Proposition 5.12, the space obtained from attaching a cell is determined up to homotopy equivalence if we homotope the attaching map. Moreover, we have a canonical inclusion map $X \hookrightarrow C_f$ as the base of the mapping cone.

Example 5.16: (*Attaching cells to get the sphere*)

We can get the n-sphere S^n by attaching an n-cell to a point (via the obvious constant map), i.e, $S^n = D^n \cup_{S^{n-1} \to \star} \{\star\}$. On the other hand, we can get S^n in a different way as well. Let us start with S^0, which is a disjoint union of two points.



Attach **two** 1-cells, say, e_1^1 and e_2^1 as in the picture, and we get S^1. Next, let us begin with S^1.



Attach **two** 2-cells, say, e_1^2 and e_2^2 as the upper and lower hemispheres as in the picture. We then get S^2. Continuing this we can get S^n by attaching exactly *two* cells in each dimensions.

Day 6 : 18th August, 2026

`relative CW complex — weak topology`

6.1 Relative CW Complex

A (relative) CW complex is constructed via subsequently attaching (possibly infinitely many) cells of higher and higher dimension. The first step is to understand how to attach infinitely many n-cells to a space. Suppose we are given a family of maps $\{f_\alpha : S_\alpha^{n-1} \to X\}_{\alpha\in\Lambda}$. Then, we have a canonical map from the disjoint union

$$\bigsqcup_{\alpha\in\Lambda} : \bigsqcup_{\alpha\in\Lambda} S_\alpha^{n-1} \to X.$$

The disjoint union of spheres is naturally a subset of the disjoint union of closed n-discs, i.e, we have $\sqcup S_\alpha^{n-1} \hookrightarrow \sqcup D_\alpha^n$. Hence, we can perform the attaching space construction (Definition 3.8)! The space

$$\left(\bigsqcup_{\alpha\in\Lambda} D_\alpha^n\right) \cup_{\sqcup f_\alpha} X$$

is called *the space obtained from X by attaching n-cells, via the family $\{f_\alpha : S_\alpha^{n-1} \to X\}$*. The attaching map is also called the *characteristic map* for the corresponding n-cell.

Definition 6.1: (*Relative CW Complex*)

A *relative CW decomposition* is the following data

$$A = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \cdots,$$

where

- $X_1 = A$ is a (possibly empty) space, and
- X_n is obtained from X_{n-1} by attaching n-cells via a (possibly empty) collection of maps $\{f_\alpha : S_\alpha^{n-1} \to X_{n-1}\}_{\alpha\in\Lambda}$.

On the union $X = \cup_{n\geq -1} X_n$, we consider *weak topology*, and say (X, A) is a *relative CW complex*. If there is no confusion, we will not mention the underlying CW decomposition. For each n, the subspace X_n is called the *n-skeleton* of the CW complex. If $X_n = X_{n+1} = \cdots$, then we have $X = X_n$, and say $\dim X \leq n$. The map $f_\alpha : S_\alpha^{n-1} \to X_{n-1}$ is called the *characteristic map* for the n-cell. It extends to a continuous map $D_\alpha^n \to X_n$, which restricts to a homeomorphism of the interior $\mathring{D}_\alpha^n$ onto its image in X_n, which is called an *n-cell* in the CW complex.

Let us explain what we mean by the weak topology.

Definition 6.2: (*Weak Topology*)

Given an increasing collection of spaces

$$Y_0 \subset Y_1 \subset \cdots,$$

the *weak topology* on the union $Y = \cup_{i\geq 0} Y_i$ is defined as follows: $U \subset Y$ is open if and only if $U \cap Y_i$ is open for each i.

The **W** in CW complex stands for **w**eak topology!

Exercise 6.3:

Show that the weak topology on the union $\cup_{n\geq 1}(-n,n)$ is homeomorphic to $\mathbb{R}$. On the other hand, verify that the weak topology on $\bigcup_{n\geq 1}\{0,1,\frac{1}{2},\cdots,\frac{1}{n}\}$ is *not* homeomorphic to the subspace $K=\{0\}\cup\{\frac{1}{n}\mid n\geq 1\}\subset\mathbb{R}$.

It is easy to see that if we have $Y_0\subset Y_1\subset\cdots Y_n=Y_{n+1}=\cdots$, then $Y=\cup Y_i=Y_n$, and moreover, the weak topology on Y is the same as the topology on Y_n. In particular, for a finite dimensional CW complex, say, $\dim X=n$ the weak topology is the same as that on n^{th} skeleton on X.

Example 6.4: (*Spheres as CW Complexes*)

In Example 5.16 we have seen how to construct the spheres by attaching two cells at each step. As an example, the construction gives S^2 the structure of a 1-dimensional CW complex with two 0-cells, two 1-cells, and two 2-cells. We can continue this way and obtain an increasing chain of spaces

$$S^0\subset S^1\subset S^2\subset\cdots.$$

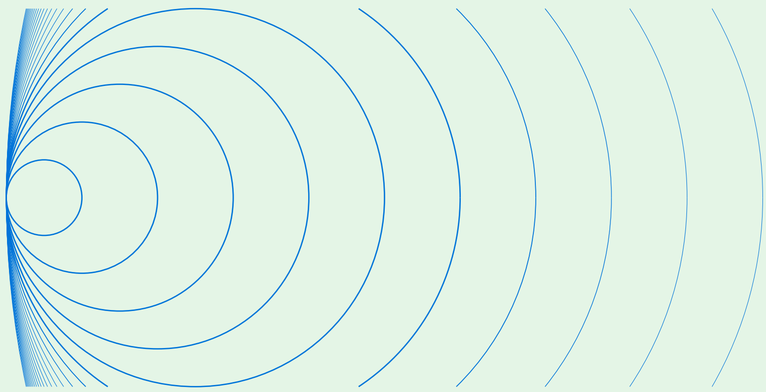
The union is called the infinite-dimensional sphere $S^\infty=\cup_{n\geq 0}S^n$, where we give the weak topology.

Example 6.5: (*Infinite Wedge of Circles*)

Consider $X_0=\{\star\}$, a singleton. Then, get a countable infinite collection of 1-discs, say, $\{D_i^1\}_{i=1}^\infty$. Since the target is a singleton, the attaching maps from the boundaries must be the constant map. Consequently, we get

$$X_1=\left(\bigsqcup_{i\geq 1}D_i^1\right)\cup_{\text{const}}\{\star\}.$$

We claim that this space is homeomorphic to the subspace



$$Z=\bigcup_{i\geq 1}\{(x,y)\mid (x-i)^2+y^2=i^2\}\subset\mathbb{R}^2.$$

Indeed, consider the map that wraps the i^{th} cell around the circle of radius i, and send the 0-cell to the origin $(0,0)$. This map is clearly bijective. The continuity and openness follow from the quotient topology. Thus, the CW complex obtained by infinitely many 1-cells to a point is the infinite wedge of circles. On the other hand, we have seen that this space is not the same as the Hawaiiaan earring.

Day 7 : 19th August, 2026

`CW complex is` T_2

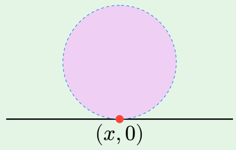
7.1 Hausdorffness of CW Complexes

A desirable property of a CW complex is that it is T_2 (i.e, Hausdorff). In general, attaching spaces need not be T_2, even if we start with T_2-spaces.

Example 7.1:

The attaching space obtained from some T_2-spaces need not always be T_2! Let us justify the hypothesis. Firstly, consider $A = (0, 1] \subset [0, 1]$, and attach A to a single point via the constant map. One can easily check that the space obtained is not T_2. Indeed, the space is homeomorphic to the two-point set $\{a, b\}$ with topology $\{\emptyset, \{a\}, \{a, b\}\}$, known as the Sierpinski space.

Next, consider the space X known as the Moore plane. As a set, $X = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. Any point (x, y) with $y > 0$, has usual open discs as a neighborhood basis. For a point $(x, 0) \in X$, a basic open set looks like an open disc touching the x-axis at $(x, 0)$, along with the point itself. This space known to be T_3 but not normal. We have the subspace $A = \mathbb{R} \times \{0\} \subset X$, which is closed and *discrete*.



Consider the discrete space $Y = \{a, b\}$, and the map $f : A \to Y$ which sends $f(\mathbb{Q}) = a$ and $f(\mathbb{R} \setminus \mathbb{Q}) = b$. Then, one can verify that in the attaching space $X \cup_f Y$, the two points a and b cannot be separated by open sets.

We have the following general result that is applicable while attaching cells to a T_2-space.

Proposition 7.2: (T_2 *Attaching Space*)

Suppose X is a T_3 (i.e, T_2 + regular) and Y is a T_2-space. Let $A \subset X$ be closed and is a neighborhood retract (i.e, there is an open neighborhood $A \subset U \subset X$ and a retraction $r : U \to A$). Then, for any $f : A \to Y$, the attaching space $X \cup_f Y$ is T_2.

<u>**Proof**</u> **:** Suppose $\alpha_1, \alpha_2 \in X \cup_f Y$ are distinct points. Consider the quotient map $q : X \sqcup Y \to X \cup_f Y$. Let us do case by case analysis.

- If $\alpha_1, \alpha_2 \in X \cup_f Y \setminus Y$, then $\alpha_i = [x_i]$ for some *unique* $x_i \in X \setminus A$. Since X is T_2, we have open sets $x_i \in V_i \subset X$ such that $V_1 \cap V_2 = \emptyset$. As $A \subset X$ is closed, we have $U_i = V_i \setminus A$ is open, and moreover, U_i is a saturated set. Thus, $q(U_i)$ are open sets, which setparate the points α_i.
- Suppose $\alpha_1 \in X \cup_f Y \setminus Y$ and $\alpha_2 \in Y$. Then, $\alpha_1 = [x]$ for some unique $x \in X \setminus A$ and $\alpha_2 = [y]$ for some $y \in Y$. Since X is regular, we have open sets $x \in V, A \subset W$ such that $V \cap W = \emptyset$. Consider the open set $O = W \sqcup Y$, which is saturated. Then, $q(V)$ and $q(O)$ are disjoint open sets in $X \cup_f Y$, which separates α_1 and α_2.
- Finally, suppose $\alpha_i \in Y$. Then, $\alpha_i = [y_i]$ for some $y_i \in Y$. As Y is T_2, we have open sets $y_i \in V_i \subset Y$ such that $V_1 \cap V_2 = \emptyset$. Consider $W_i = f^{-1}(V_i)$, which is an open set in A. Now, since A is a neighborhood retract, there is an open set $A \subset U \subset X$, and a retraction $r : U \to A$. Consider the sets $U_i = r^{-1}(W_i) = (f \circ r)^{-1}(V_i)$, which are open, and clearly $U_1 \cap U_2 = \emptyset$. We claim that W_i is

saturated. Indeed, for any $x \in U_i \setminus A$, we have the equivalence class of x is a singleton. As r is retraction, we have $U_i \cap A = W_i = f^{-1}(V_i)$, which is also saturated. Thus, $U_i \subset X$ is saturated. Consequently, we have open sets $O_i = q(U_i \sqcup V_i) \subset X \cup_f Y$. By construction, $O_1 \cap O_2 = \emptyset$, and $\alpha_i \in O_i$.

Thus, the attaching space $X \cup_f Y$ is T_2. □

In Proposition 7.2, we required that the subspace $A \subset X$, along which we are attaching, be closed and a *neighborhood retract.* This property is immediate for $S^{n-1} \subset D^n$. Indeed, we can consider open shell $S^{n-1} \times \left(\frac{1}{2}, 1\right]$ as a neighborhood, along with the radial projection map as a retraction. This same argument shows that

$$\bigsqcup_{\alpha \in \Lambda} S^{n-1}_\alpha \subset \bigsqcup_{\alpha \in \Lambda} D^n_\alpha$$

is a closed neighborhood retract as well. Clearly, the disjoint union $\sqcup D^n_\alpha$ is a T_3-space. Thus, we can use Proposition 7.2 to show that a (relative) CW complex is T_2.

Theorem 7.3: (*CW Complex is T_2*)

Let A be a T_2-space, and (X, A) be a relative CW complex. Then, X is a T_2-space.

Proof : Since for any $n \geq 1$ we have $\sqcup S^{n-1}_\alpha \subset \sqcup D^n_\alpha$ is a closed neighborhood retract, inductively using Proposition 7.2 it follows that each skeleton $X_n \subset X$ is a T_2-space. The induction base case is $X_{-1} = A$, which is assumed to be T_2. If we have $X = X_N$ for some N large, then we are done, as the weak topology is the same.

Let us consider the general case. Suppose $x, y \in X$ are distinct points. Then, there exists some N large such that $x, y \in X_{n_0}$. But as X_{n_0} is T_2, we have a some open sets $U_{n_0}, V_{n_0} \subset X_{n_0}$ such that $x \in U_{n_0}, y \in V_{n_0}$ and $U_{n_0} \cap V_{n_0} = \emptyset$. Now, the argument of Proposition 7.2 applies again, and we can inductively construct open sets $U_k, V_k \subset X_k$ for $k \geq n_0$ such that $U_k \cap V_k = \emptyset$, and $U_k \cap X_{k-1} = U_{k-1}, V_k \cap X_{k-1} = V_{k-1}$. For notation convenience, set $U_k = X_k \cap U, V_k = X_k \cap V$ for $k < n_0$. Clearly $U_k \subset X_k$ is open in the subspace topology for $k < n_0$. Now, consider the union $U = \cup_k U_k, V = \cup_k V_k$. By construction, $U \cap V = \emptyset$ and $x \in U, y \in V$. Also, in the weak topology on X, we have U, V are open as $U \cap X_k = U_k, V \cap X_k = V_k$ are open for each k. Thus, the CW complex X is T_2. □

Remark 7.4:

In fact one can show that a CW complex X is a T_6-space. That is, in a CW complex X, given two disjoint closed sets $A, B \subset X$, there exists a continuous map $f : X \to [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

As a consequence of Theorem 7.3, we have the following useful fact.

Corollary 7.5:

Given a CW complex X and an n-cell e^n_α, we have the closure $\overline{e^n_\alpha} = \Phi(D^n_\alpha)$, where $\Phi_\alpha : D^n_\alpha \to X_n$ is induced from the characteristic map.

<u>Proof</u> : Recall that $e^n_\alpha = \Phi_{\alpha\left(\mathring{D}^n_\alpha\right)}$ is a homeomorphic copy of the open n-disc. Now, D^n_α is compact, and hence, $\Phi_{\alpha(D^n_\alpha)} \subset X_n$ is also compact. As X is T_2, we have $\Phi_{\alpha(D^n_\alpha)}$ is closed. Thus, $\overline{e^n_\alpha} \subset \Phi_{\alpha(D^n_\alpha)}$. Conversely, let

$$x \in \Phi_\alpha\left(\partial D^n_\alpha \setminus \mathring{D}^n_\alpha\right) = \varphi_\alpha(S^{n-1}_\alpha).$$

For any open neighborhood $x \in U \subset X^n$, we have $\Phi^{-1}(U)$ is open, where $\Phi : \sqcup D^n_\alpha \to X^n$ is obtained from the attaching map. But that would imply $U \cap \Phi\left(\mathring{D}^n\right) \neq \emptyset$ $\square$

Day 8 : 20th August, 2026

`closure finiteness — cellular map — subcomplex`

8.1 Closure Finiteness of CW Complexes

Before defining the closure finiteness, let us recall the notion of cells. Given a CW complex X, we have the *characteristic map* $\varphi_\alpha : S^{n-1}_\alpha \to X_{n-1}$, which extends to a map $\Phi_\alpha : D^n \to X_n \subset X$. Since X is T_2 (Theorem 7.3), the image of the compact set D^n is a closed set in X. The map Φ restricts to a homeomorphism of the interior $\mathring{D}^n$ onto its image (Proposition 5.15). We say, $\Phi(D^n) \subset X$ is a *closed* n*-cell,* and $\Phi\left(\mathring{D}^n\right) \subset X$ is an *open* n*-cell.* In general, an open cell is *not* an open set in X, but an open n-cell is always open in the n-skeleton X_n. Moreover, we can justify that the closure of an n-cell is actually the image of the closed n-disc under the characteristic map.

Proposition 8.1: (*Closure of a Cell*)

Let $\varphi : S^{n-1} \to X^{n-1}$ be the characteristic map for the open cell $e^n = \Phi\left(\mathring{D}^n\right)$, where Φ is the induced map. Then, $\overline{e^n} = \Phi(D^n)$, i.e, the closure of a cell is the corresponding closed cell.

Proof : Since D^n is compact and X is T_2, we have $\Phi(D^n)$ is closed in X. Hence,

$$e^n \subset \Phi\left(\mathring{D}^n\right) \subset \Phi(D^n) \Rightarrow \overline{e^n} \subset \Phi(D^n).$$

For the converse, we have

$$\overline{\mathring{D}^n} = D^n \Rightarrow \Phi(D^n) \subset \overline{\Phi\left(\mathring{D}^n\right)} = \overline{e^n}.$$

Hence, the claim follows. □

The *closure finteness* property of a CW complex (which justifies the C in CW) says that the closure of an open cell intersects finitely many open cell.

Remark 8.2: (*Cell Decomposition*)

If X is a CW complex, then observe that the open cells are pairwise disjoint and forms a partition of the set underlying X. This is known as the *cell decomposition* of the CW complex. With this in mind, by a cell we shall mean an open cell, unless otherwise stated.

Let us now observe an useful general fact.

Proposition 8.3: (*Compacts in CW Complex*)

Let X be a CW complex, and $K \subset X$ is compact. Then, K intersects finitely many (open) cells. In other words, in the cell decomposition of X, any compact set is contained in the union of finitely many cells.

Proof : If possible, suppose there infinitely many points $x_i \in K$ such that each x_i is contained in a unique cell of X. Let $S \subset \{x_i\}$ be an arbitrary subset. We claim that S is closed in X. In the weak topology, we need to show that $S \cap X_n$ is closed in X_n for each n. Clearly, $S \cap X_0$ is closed in X_0, since X_0 has discrete topology (Verify!). Inductively, assume that $S \cap X_{n-1}$ is closed in X_{n-1}. For each characteristic map $\varphi_\alpha : S^{n-1}_\alpha \to X_{n-1}$, we have $\varphi^{-1}_\alpha(S \cap X_{n-1})$ is closed in S^{n-1}_α. Consider the induced map $\Phi_\alpha : D^n_\alpha \to X_n$. Then, by our assumption, $\Phi^{-1}_\alpha(S)$ consists of $\varphi^{-1}_\alpha(S \cap X_{n-1})$, and *at most* one more point. In particular, $\Phi^{-1}_\alpha(S \cap X_n)$ is closed in D^n_α. Now, for the quotient map $q : (\sqcup D^n_\alpha) \sqcup X_{n-1} \to X_n$, we have

$$q^{-1}(S \cap X_n) = \left(\sqcup \Phi^{-1}_\alpha(S \cap X_n)\right) \sqcup (S \cap X_{n-1}).$$

It follows that each components of this disjoint union is closed. But then we have $S \cap X_n$ is closed in X_n with the quotient topology. Hence, by induction, it follows that $S \cap X_n$ is closed in X_n for each n. Thus, in the weak topology, any subset $S \subset \{x_i\}$ is closed in X. In particular, it follows that $\{x_i\}$ is a *discrete* closed subspace of X. But this is a contradiction, since $\{x_i\}$ is an infinite discrete closed subspace of the compact set K. Thus, K intersects finitely many cells. □

As a consequence, we get that CW complexes are closure finite.

Corollary 8.4:

Any CW complex X is closure finite, i.e, closure of any cell intersects finitely many cells.

Proof : In Proposition 8.1, we have seen that the closure of a cell is the corresponding closed cell, which is compact. Then, by Proposition 8.3 it follows that the closure of a cell must be contained in finite union of cells. □

8.2 Cellular Maps and Sub-complexes

Definition 8.5:

Given CW complexes X, Y, a *cellular map* between them is a continuous map $f : X \to Y$ such that $f(X_n) \subset Y_n$ for each n, where X_n and Y_n are the respective n-skeleta.

Note that a cellular map need not send an n-cell in X to another n-cell in Y!

Example 8.6:

Consider the CW complex structure on the n-sphere which consists of one 0-cell and one n-cell. Then, if $n < m$, then the only cellular map $S^n \to S^m$ is the constant map! This is because the n-skeleton of S^n is S^n itself, whereas the n-skeleton of S^m consists of only the 0-cell.

We have a very powerful theorem.

Theorem 8.7: (*Cellular Approximation Theorem*)

Any continuous map $f : X \to Y$ between two CW complexes can be homotoped to a cellular map.

Remark 8.8:

An immediate consequence of Theorem 8.7 is that any map $S^n \to S^m$ with $n < m$ is null-homotopic, since the only cellular map is the constant one!

Next, let us define the notion of sub-complexes of a CW complex.

Definition 8.9:

Given a CW complex, a *subcomplex* is a subset $A \subset X$ such that A is the union of a subcollection of cells in X, and moreover, if $e \subset X$ is a cell in A, then $\overline{e} \subset A$. Equivalently, if $X = \cup_{n \geq 0} X_n$ is the union of the skeleta, then $A = \cup A_n$, where $A_n := A \cap X_n$ is obtained from A_{n-1} via attaching some n-cells from X.

Example 8.10:

Give S^n the CW complex structure consisting of one 0-cell and one n-cell. Then, there can only be two possible subcomplexes: one consists of the 0-cell, and the whole of S^n.

Exercise 8.11:

Construct S^n via attaching two cells in each dimension (Example 5.16). Verify that there are $(3n+4)$-many possible distinct subcomplexes.

Hint : Write the cell decomposition

$$S^n = (e^0_+ \sqcup e^0_-) \sqcup (e^1_+ \sqcup e^1_-) \sqcup \cdots \sqcup (e^n_+ \sqcup e^n_-).$$

Note that

$$\overline{e^k_\pm} = e^k_\pm \sqcup \bigsqcup_{i<k} (e^i_+ \sqcup e^i_-).$$

We have some easy observations involving subcomplexes.

Proposition 8.12:

Given a CW composition X, with skeletal decomposition $X = \cup_{n \geq 0} X_n$. We have the following.

1. Any subcomplex $A \subset X$ is closed.
2. Given a subcomplex $A \subset X$, the inclusion map $\iota : A \hookrightarrow X$ is cellular.
3. Each X_n is a subcomplex, and thus, $X_n \hookrightarrow X$ is cellular.

Proof : Let $A \subset X$ be a subcomplex. To show that $A \subset X$ is closed, we need to check that $A \subset X_n$ is closed for each $n \geq 0$. Since X_0 is a discrete set, we have $A \subset X_n$ is closed. Inductively, assume that

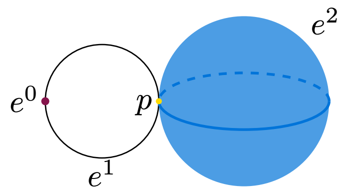
$A \subset X_{n-1}$ is closed. Now, for any n-cell $e \subset X$, either $\bar{e} \subset A$ or $e \cap A = \emptyset$. Then, for the characteristic map $\Phi : D^n \to X_n$ we have

$$\Phi^{-1}(A \cap X_n) = \begin{cases} D^n, & \bar{e} \subset A \\ \Phi^{-1}(A \cap X_{n-1}), & e \cap A = \emptyset. \end{cases}$$

Clearly, this is closed by the induction hypothesis. But then it follows that $A \cap X_n$ is closed. Inductively, we get that $A \subset X$ is closed in the weak topology.

The inclusion map of a subcomplex is cellular by definition. Finally, any n-skeleton is union of all the cells of dimension $\leq n$, and the closure of any k-cell is contained in X_k. Thus, $X_k \subset X$ is a subcomplex. □

In general, a closed cell need not be a subcomplex of a CW complex. Indeed, consider the CW complex X obtained from one 0-cell e^0, one 1-cell e^1, and one 2-cell attached to a point p on the 1-cell. In this CW structure, the closure of the 2-cell is the sphere S^2, but this S^2 is *not* a subcomplex of X.



A (non-regular) CW complex

Definition 8.13: (*Regular CW Complex*)

A CW complex is called *regular* if every closed n-cell e is a homeomorphic image of the closed disc D^n under the characteristic map.

Example 8.14:

The CW structure on S^n with two cells is not regular, whereas the CW structure with two cells in each dimension is a regular CW structure.

In a regular CW complex, we have the following.

Proposition 8.15:

If X is a regular CW complex, then the closure of any cell is a subcomplex of X.

Finally, we have the following result.

Theorem 8.16: (*Cofibration in CW Complexes*)

Given a CW complex X and a subcomplex $A \subset X$, the inclusion map $A \hookrightarrow X$ is a cofibration, i.e, it has homotopy extension property against all spaces.

In fact, one can more generally show that if X is a *locally finite* CW complex (i.e, each point has a neighborhood which intersects only finitely many cells), and $A \subset X$ is a closed subset which is also a CW complex (but not necessarily a subcomplex of X), then $A \hookrightarrow X$ is again a cofibration.