Course notes for

Algebraic Topology I (KSM3E02)

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`closure finiteness — cellular map — subcomplex`

8.1 Closure Finiteness of CW Complexes

Before defining the closure finiteness, let us recall the notion of cells. Given a CW complex X, we have the *characteristic map* $\varphi_\alpha : S_\alpha^{n-1} \to X_{n-1}$, which extends to a map $\Phi_\alpha : D^n \to X_n \subset X$. Since X is T_2 (Theorem 7.3), the image of the compact set D^n is a closed set in X. The map Φ restricts to a homeomorphism of the interior $\mathring{D}^n$ onto its image (Proposition 5.15). We say, $\Phi(D^n) \subset X$ is a *closed* n*-cell*, and $\Phi\left(\mathring{D}^n\right) \subset X$ is an *open* n*-cell.* In general, an open cell is *not* an open set in X, but an open n-cell is always open in the n-skeleton X_n. Moreover, we can justify that the closure of an n-cell is actually the image of the closed n-disc under the characteristic map.

Proposition 8.1: (*Closure of a Cell*)

Let $\varphi : S^{n-1} \to X^{n-1}$ be the characteristic map for the open cell $e^n = \Phi\left(\mathring{D}^n\right)$, where Φ is the induced map. Then, $\overline{e^n} = \Phi(D^n)$, i.e, the closure of a cell is the corresponding closed cell.

Proof : Since D^n is compact and X is T_2, we have $\Phi(D^n)$ is closed in X. Hence,

$$e^n \subset \Phi\left(\mathring{D}^n\right) \subset \Phi(D^n) \Rightarrow \overline{e^n} \subset \Phi(D^n).$$

For the converse, we have

$$\overline{\mathring{D}^n} = D^n \Rightarrow \Phi(D^n) \subset \overline{\Phi\left(\mathring{D}^n\right)} = \overline{e^n}.$$

Hence, the claim follows. □

The *closure finteness* property of a CW complex (which justifies the C in CW) says that the closure of an open cell intersects finitely many open cell.

Remark 8.2: (*Cell Decomposition*)

If X is a CW complex, then observe that the open cells are pairwise disjoint and forms a partition of the set underlying X. This is known as the *cell decomposition* of the CW complex. With this in mind, by a cell we shall mean an open cell, unless otherwise stated.

Let us now observe an useful general fact.

Proposition 8.3: (*Compacts in CW Complex*)

Let X be a CW complex, and $K \subset X$ is compact. Then, K intersects finitely many (open) cells. In other words, in the cell decomposition of X, any compact set is contained in the union of finitely many cells.

Proof : If possible, suppose there infinitely many points $x_i \in K$ such that each x_i is contained in a unique cell of X. Let $S \subset \{x_i\}$ be an arbitrary subset. We claim that S is closed in X. In the weak topology, we need to show that $S \cap X_n$ is closed in X_n for each n. Clearly, $S \cap X_0$ is closed in X_0, since X_0 has discrete topology (Verify!). Inductively, assume that $S \cap X_{n-1}$ is closed in X_{n-1}. For each characteristic map $\varphi_\alpha : S_\alpha^{n-1} \to X_{n-1}$, we have $\varphi_\alpha^{-1}(S \cap X_{n-1})$ is closed in S_α^{n-1}. Consider the induced map $\Phi_\alpha : D_\alpha^n \to X_n$. Then, by our assumption, $\Phi_\alpha^{-1}(S)$ consists of $\varphi_\alpha^{-1}(S \cap X_{n-1})$, and *at most* one more point. In particular, $\Phi_\alpha^{-1}(S \cap X_n)$ is closed in D_α^n. Now, for the quotient map $q : (\sqcup D_\alpha^n) \sqcup X_{n-1} \to X_n$, we have

$$q^{-1}(S \cap X_n) = \left(\sqcup \Phi_\alpha^{-1}(S \cap X_n)\right) \sqcup (S \cap X_{n-1}).$$

It follows that each components of this disjoint union is closed. But then we have $S \cap X_n$ is closed in X_n with the quotient topology. Hence, by induction, it follows that $S \cap X_n$ is closed in X_n for each n. Thus, in the weak topology, any subset $S \subset \{x_i\}$ is closed in X. In particular, it follows that $\{x_i\}$ is a *discrete* closed subspace of X. But this is a contradiction, since $\{x_i\}$ is an infinite discrete closed subspace of the compact set K. Thus, K intersects finitely many cells. □

As a consequence, we get that CW complexes are closure finite.

Corollary 8.4:

Any CW complex X is closure finite, i.e, closure of any cell intersects finitely many cells.

Proof : In Proposition 8.1, we have seen that the closure of a cell is the corresponding closed cell, which is compact. Then, by Proposition 8.3 it follows that the closure of a cell must be contained in finite union of cells. □

8.2 Cellular Maps and Sub-complexes

Definition 8.5:

Given CW complexes X, Y, a *cellular map* between them is a continuous map $f : X \to Y$ such that $f(X_n) \subset Y_n$ for each n, where X_n and Y_n are the respective n-skeleta.

Note that a cellular map need not send an n-cell in X to another n-cell in Y!

Example 8.6:

Consider the CW complex structure on the n-sphere which consists of one 0-cell and one n-cell. Then, if $n < m$, then the only cellular map $S^n \to S^m$ is the constant map! This is because the n-skeleton of S^n is S^n itself, whereas the n-skeleton of S^m consists of only the 0-cell.

We have a very powerful theorem.

Theorem 8.7: (*Cellular Approximation Theorem*)

Any continuous map $f : X \to Y$ between two CW complexes can be homotoped to a cellular map.

Remark 8.8:

An immediate consequence of Theorem 8.7 is that any map $S^n \to S^m$ with $n < m$ is null-homotopic, since the only cellular map is the constant one!

Next, let us define the notion of sub-complexes of a CW complex.

Definition 8.9:

Given a CW complex, a *subcomplex* is a subset $A \subset X$ such that A is the union of a subcollection of cells in X, and moreover, if $e \subset X$ is a cell in A, then $\overline{e} \subset A$. Equivalently, if $X = \cup_{n\geq 0} X_n$ is the union of the skeleta, then $A = \cup A_n$, where $A_n := A \cap X_n$ is obtained from A_{n-1} via attaching some n-cells from X.

Example 8.10:

Give S^n the CW complex structure consisting of one 0-cell and one n-cell. Then, there can only be two possible subcomplexes: one consists of the 0-cell, and the whole of S^n.

Exercise 8.11:

Construct S^n via attaching two cells in each dimension (Example 5.16). Verify that there are $(3n+4)$-many possible distinct subcomplexes.

Hint : Write the cell decomposition

$$S^n = (e^0_+ \sqcup e^0_-) \sqcup (e^1_+ \sqcup e^1_-) \sqcup \cdots \sqcup (e^n_+ \sqcup e^n_-).$$

Note that

$$\overline{e^k_\pm} = e^k_\pm \sqcup \bigsqcup_{i<k} (e^i_+ \sqcup e^i_-).$$

We have some easy observations involving subcomplexes.

Proposition 8.12:

Given a CW composition X, with skeletal decomposition $X = \cup_{n\geq 0} X_n$. We have the following.

1. Any subcomplex $A \subset X$ is closed.
2. Given a subcomplex $A \subset X$, the inclusion map $\iota : A \hookrightarrow X$ is cellular.
3. Each X_n is a subcomplex, and thus, $X_n \hookrightarrow X$ is cellular.

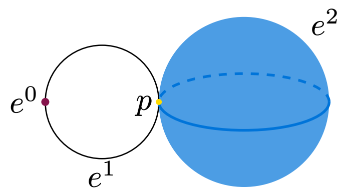
Proof : Let $A \subset X$ be a subcomplex. To show that $A \subset X$ is closed, we need to check that $A \subset X_n$ is closed for each $n \geq 0$. Since X_0 is a discrete set, we have $A \subset X_n$ is closed. Inductively, assume that $A \subset X_{n-1}$ is closed. Now, for any n-cell $e \subset X$, either $\overline{e} \subset A$ or $e \cap A = \emptyset$. Then, for the characteristic map $\Phi : D^n \to X_n$ we have

$$\Phi^{-1}(A \cap X_n) = \begin{cases} D^n, & \overline{e} \subset A \\ \Phi^{-1}(A \cap X_{n-1}), & e \cap A = \emptyset. \end{cases}$$

Clearly, this is closed by the induction hypothesis. But then it follows that $A \cap X_n$ is closed. Inductively, we get that $A \subset X$ is closed in the weak topology.

The inclusion map of a subcomplex is cellular by definition. Finally, any n-skeleton is union of all the cells of dimension $\leq n$, and the closure of any k-cell is contained in X_k. Thus, $X_k \subset X$ is a subcomplex. $\square$

In general, a closed cell need not be a subcomplex of a CW complex. Indeed, consider the CW complex X obtained from one 0-cell e^0, one 1-cell e^1, and one 2-cell attached to a point p on the 1-cell. In this CW structure, the closure of the 2-cell is the sphere S^2, but this S^2 is *not* a subcomplex of X.



A (non-regular) CW complex

Definition 8.13: (*Regular CW Complex*)

A CW complex is called *regular* if every closed n-cell e is a homeomorphic image of the closed disc D^n under the characteristic map.

Example 8.14:

The CW structure on S^n with two cells is not regular, whereas the CW structure with two cells in each dimension is a regular CW structure.

In a regular CW complex, we have the following.

Proposition 8.15:

If X is a regular CW complex, then the closure of any cell is a subcomplex of X.

Finally, we have the following result.

Theorem 8.16: (*Cofibration in CW Complexes*)

Given a CW complex X and a subcomplex $A \subset X$, the inclusion map $A \hookrightarrow X$ is a cofibration, i.e, it has homotopy extension property against all spaces.

In fact, one can more generally show that if X is a *locally finite* CW complex (i.e, each point has a neighborhood which intersects only finitely many cells), and $A \subset X$ is a closed subset which is also a CW complex (but not necessarily a subcomplex of X), then $A \hookrightarrow X$ is again a cofibration.