Course notes for

Algebraic Topology I (KSM3E02)

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`CW complex is` T_2

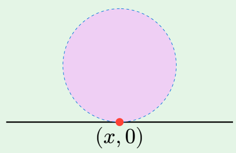
7.1 Hausdorffness of CW Complexes

A desirable property of a CW complex is that it is T_2 (i.e, Hausdorff). In general, attaching spaces need not be T_2, even if we start with T_2-spaces.

Example 7.1:

The attaching space obtained from some T_2-spaces need not always be T_2! Let us justify the hypothesis. Firstly, consider $A = (0,1] \subset [0,1]$, and attach A to a single point via the constant map. One can easily check that the space obtained is not T_2. Indeed, the space is homeomorphic to the two-point set $\{a, b\}$ with topology $\{\emptyset, \{a\}, \{a,b\}\}$, known as the Sierpinski space.

Next, consider the space X known as the Moore plane. As a set, $X = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$. Any point (x,y) with $y > 0$, has usual open discs as a neighborhood basis. For a point $(x,0) \in X$, a basic open set looks like an open disc touching the x -axis at $(x,0)$, along with the point itself. This space known to be T_3 but not normal. We have the subspace $A = \mathbb{R} \times \{0\} \subset X$, which is closed and *discrete*.



Consider the discrete space $Y = \{a,b\}$, and the map $f : A \to Y$ which sends $f(\mathbb{Q}) = a$ and $f(\mathbb{R} \setminus \mathbb{Q}) = b$. Then, one can verify that in the attaching space $X \cup_f Y$, the two points a and b cannot be separated by open sets.

We have the following general result that is applicable while attaching cells to a T_2-space.

Proposition 7.2: (T_2 *Attaching Space*)

Suppose X is a T_3 (i.e, T_2 + regular) and Y is a T_2-space. Let $A \subset X$ be closed and is a neighborhood retract (i.e, there is an open neighborhood $A \subset U \subset X$ and a retraction $r : U \to A$). Then, for any $f : A \to Y$, the attaching space $X \cup_f Y$ is T_2.

Proof : Suppose $\alpha_1, \alpha_2 \in X \cup_f Y$ are distinct points. Consider the quotient map $q : X \sqcup Y \to X \cup_f Y$. Let us do case by case analysis.

- If $\alpha_1, \alpha_2 \in X \cup_f Y \setminus Y$, then $\alpha_i = [x_i]$ for some *unique* $x_i \in X \setminus A$. Since X is T_2, we have open sets $x_i \in V_i \subset X$ such that $V_1 \cap V_2 = \emptyset$. As $A \subset X$ is closed, we have $U_i = V_i \setminus A$ is open, and moreover, U_i is a saturated set. Thus, $q(U_i)$ are open sets, which setparate the points α_i.

- Suppose $\alpha_1 \in X \cup_f Y \setminus Y$ and $\alpha_2 \in Y$. Then, $\alpha_1 = [x]$ for some unique $x \in X \setminus A$ and $\alpha_2 = [y]$ for some $y \in Y$. Since X is regular, we have open sets $x \in V, A \subset W$ such that $V \cap W = \emptyset$. Consider the open set $O = W \sqcup Y$, which is saturated. Then, $q(V)$ and $q(O)$ are disjoint open sets in $X \cup_f Y$, which separates α_1 and α_2.
- Finally, suppose $\alpha_i \in Y$. Then, $\alpha_i = [y_i]$ for some $y_i \in Y$. As Y is T_2, we have open sets $y_i \in V_i \subset Y$ such that $V_1 \cap V_2 = \emptyset$. Consider $W_i = f^{-1}(V_i)$, which is an open set in A. Now, since A is a neighborhood retract, there is an open set $A \subset U \subset X$, and a retraction $r : U \to A$. Consider the sets $U_i = r^{-1}(W_i) = (f \circ r)^{-1}(V_i)$, which are open, and clearly $U_1 \cap U_2 = \emptyset$. We claim that W_i is saturated. Indeed, for any $x \in U_i \setminus A$, we have the equivalence class of x is a singleton. As r is retraction, we have $U_i \cap A = W_i = f^{-1}(V_i)$, which is also saturated. Thus, $U_i \subset X$ is saturated. Consequently, we have open sets $O_i = q(U_i \sqcup V_i) \subset X \cup_f Y$. By construction, $O_1 \cap O_2 = \emptyset$, and $\alpha_i \in O_i$.

Thus, the attaching space $X \cup_f Y$ is T_2. □

In Proposition 7.2, we required that the subspace $A \subset X$, along which we are attaching, be closed and a *neighborhood retract*. This property is immediate for $S^{n-1} \subset D^n$. Indeed, we can consider open shell $S^{n-1} \times \left(\frac{1}{2}, 1\right]$ as a neighborhood, along with the radial projection map as a retraction. This same argument shows that

$$\bigsqcup_{\alpha \in \Lambda} S_\alpha^{n-1} \subset \bigsqcup_{\alpha \in \Lambda} D_\alpha^n$$

is a closed neighborhood retract as well. Clearly, the disjoint union $\sqcup D_\alpha^n$ is a T_3-space. Thus, we can use Proposition 7.2 to show that a (relative) CW complex is T_2.

Theorem 7.3: (*CW Complex is T_2*)

Let A be a T_2-space, and (X, A) be a relative CW complex. Then, X is a T_2-space.

Proof : Since for any $n \geq 1$ we have $\sqcup S_\alpha^{n-1} \subset \sqcup D_\alpha^n$ is a closed neighborhood retract, inductively using Proposition 7.2 it follows that each skeleton $X_n \subset X$ is a T_2-space. The induction base case is $X_{-1} = A$, which is assumed to be T_2. If we have $X = X_N$ for some N large, then we are done, as the weak topology is the same.

Let us consider the general case. Suppose $x, y \in X$ are distinct points. Then, there exists some N large such that $x, y \in X_{n_0}$. But as X_{n_0} is T_2, we have a some open sets $U_{n_0}, V_{n_0} \subset X_{n_0}$ such that $x \in U_{n_0}, y \in V_{n_0}$ and $U_{n_0} \cap V_{n_0} = \emptyset$. Now, the argument of Proposition 7.2 applies again, and we can inductively construct open sets $U_k, V_k \subset X_k$ for $k \geq n_0$ such that $U_k \cap V_k = \emptyset$, and $U_k \cap X_{k-1} = U_{k-1}, V_k \cap X_{k-1} = V_{k-1}$. For notation convenience, set $U_k = X_k \cap U, V_k = X_k \cap V$ for $k < n_0$. Clearly $U_k \subset X_k$ is open in the subspace topology for $k < n_0$. Now, consider the union $U = \cup_k U_k, V = \cup_k V_k$. By construction, $U \cap V = \emptyset$ and $x \in U, y \in V$. Also, in the weak topology on X, we have U, V are open as $U \cap X_k = U_k, V \cap X_k = V_k$ are open for each k. Thus, the CW complex X is T_2. □

Remark 7.4:

In fact one can show that a CW complex X is a T_6-space. That is, in a CW complex X, given two disjoint closed sets $A, B \subset X$, there exists a continuous map $f : X \to [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

As a consequence of Theorem 7.3, we have the following useful fact.

Corollary 7.5:

Given a CW complex X and an n-cell e_α^n, we have the closure $\overline{e_\alpha^n} = \Phi(D_\alpha^n)$, where $\Phi_\alpha : D_\alpha^n \to X_n$ is induced from the characteristic map.

Proof : Recall that $e_\alpha^n = \Phi_{\alpha\left(\mathring{D}_\alpha^n\right)}$ is a homeomorphic copy of the open n-disc. Now, D_α^n is compact, and hence, $\Phi_{\alpha(D_\alpha^n)} \subset X_n$ is also compact. As X is T_2, we have $\Phi_{\alpha(D_\alpha^n)}$ is closed. Thus, $\overline{e_\alpha^n} \subset \Phi_{\alpha(D_\alpha^n)}$. Conversely, let

$$x \in \Phi_\alpha\left(\partial D_\alpha^n \setminus \mathring{D}_\alpha^n\right) = \varphi_\alpha(S_\alpha^{n-1}).$$

For any open neighborhood $x \in U \subset X^n$, we have $\Phi^{-1}(U)$ is open, where $\Phi : \sqcup D_\alpha^n \to X^n$ is obtained from the attaching map. But that would imply $U \cap \Phi\left(\mathring{D}^n\right) \neq \emptyset$ □