Course notes for

Algebraic Topology I (KSM3E02)

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`relative CW complex — weak topology`

6.1 Relative CW Complex

A (relative) CW complex is constructed via subsequently attaching (possibly infinitely many) cells of higher and higher dimension. The first step is to understand how to attach infinitely many n-cells to a space. Suppose we are given a family of maps $\{f_\alpha : S^{n-1}_\alpha \to X\}_{\alpha\in\Lambda}$. Then, we have a canonical map from the disjoint union

$$\bigsqcup_{\alpha\in\Lambda} : \bigsqcup_{\alpha\in\Lambda} S^{n-1}_\alpha \to X.$$

The disjoint union of spheres is naturally a subset of the disjoint union of closed n-discs, i.e, we have $\sqcup S^{n-1}_\alpha \hookrightarrow \sqcup D^n_\alpha$. Hence, we can perform the attaching space construction (Definition 3.8)! The space

$$\left(\bigsqcup_{\alpha\in\Lambda} D^n_\alpha\right) \cup_{\sqcup f_\alpha} X$$

is called *the space obtained from X by attaching n-cells, via the family $\{f_\alpha : S^{n-1}_\alpha \to X\}$*. The attaching map is also called the *characteristic map* for the corresponding n-cell.

Definition 6.1: (*Relative CW Complex*)

A *relative CW decomposition* is the following data

$$A = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \cdots,$$

where

- $X_1 = A$ is a (possibly empty) space, and
- X_n is obtained from X_{n-1} by attaching n-cells via a (possibly empty) collection of maps $\{f_\alpha : S^{n-1}_\alpha \to X_{n-1}\}_{\alpha\in\Lambda}$.

On the union $X = \cup_{n\ge -1} X_n$, we consider *weak topology*, and say (X, A) is a *relative CW complex*. If there is no confusion, we will not mention the underlying CW decomposition. For each n, the subspace X_n is called the *n-skeleton* of the CW complex. If $X_n = X_{n+1} = \cdots$, then we have $X = X_n$, and say $\dim X \le n$. The map $f_\alpha : S^{n-1}_\alpha \to X_{n-1}$ is called the *characteristic map* for the n-cell. It extends to a continuous map $D^n_\alpha \to X_n$, which restricts to a homeomorphism of the interior $\mathring{D}^n_\alpha$ onto its image in X_n, which is called an *n-cell* in the CW complex.

Let us explain what we mean by the weak topology.

Definition 6.2: (*Weak Topology*)

Given an increasing collection of spaces

$$Y_0 \subset Y_1 \subset \cdots,$$

the *weak topology* on the union $Y = \cup_{i \geq 0} Y_i$ is defined as follows: $U \subset Y$ is open if and only if $U \cap Y_i$ is open for each i.

The **W** in CW complex stands for **w**eak topology!

Exercise 6.3:

Show that the weak topology on the union $\cup_{n \geq 1} (-n, n)$ is homeomorphic to $\mathbb{R}$. On the other hand, verify that the weak topology on $\bigcup_{n \geq 1} \left\{0, 1, \frac{1}{2}, \cdots, \frac{1}{n}\right\}$ is *not* homeomorphic to the subspace $K = \{0\} \cup \left\{\frac{1}{n} \mid n \geq 1\right\} \subset \mathbb{R}$.

It is easy to see that if we have $Y_0 \subset Y_1 \subset \cdots Y_n = Y_{n+1} = \cdots$, then $Y = \cup Y_i = Y_n$, and moreover, the weak topology on Y is the same as the topology on Y_n. In particular, for a finite dimensional CW complex, say, $\dim X = n$ the weak topology is the same as that on n^{th} skeleton on X.

Example 6.4: (*Spheres as CW Complexes*)

In Example 5.16 we have seen how to construct the spheres by attaching two cells at each step. As an example, the construction gives S^2 the structure of a 1-dimensional CW complex with two 0-cells, two 1-cells, and two 2-cells. We can continue this way and obtain an increasing chain of spaces

$$S^0 \subset S^1 \subset S^2 \subset \cdots.$$

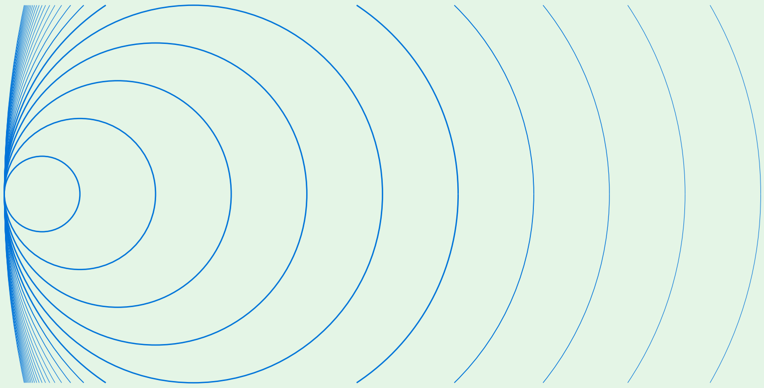
The union is called the infinite-dimensional sphere $S^\infty = \cup_{n \geq 0} S^n$, where we give the weak topology.

Example 6.5: (*Infinite Wedge of Circles*)

Consider $X_0 = \{\star\}$, a singleton. Then, get a countable infinite collection of 1-discs, say, $\{D_i^1\}_{i=1}^{\infty}$. Since the target is a singleton, the attaching maps from the boundaries must be the constant map. Consequently, we get

$$X_1 = \left(\bigsqcup_{i\geq 1} D_i^1\right) \cup_{\text{const}} \{\star\}.$$

We claim that this space is homeomorphic to the subspace



$$Z = \bigcup_{i\geq 1}\{(x,y) \mid (x-i)^2 + y^2 = i^2\} \subset \mathbb{R}^2.$$

Indeed, consider the map that wraps the i^{th} cell around the circle of radius i, and send the 0-cell to the origin $(0,0)$. This map is clearly bijective. The continuity and openness follow from the quotient topology. Thus, the CW complex obtained by infinitely many 1-cells to a point is the infinite wedge of circles. On the other hand, we have seen that this space is not the same as the Hawaiiaan earring.