Course notes for

Algebraic Topology I (KSM3E02)

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`relative homotopy — pointed space — basepoint preserving homotopy — attaching cells`

5.1 Relative Homotopy

Let us now generalize the notion of homotopy in the following way.

Definition 5.1: (*Relative Homotopy*)

Let X, Y be spaces, and $A \subset X$ is a subspace. A homotopy $h : X \times [0, 1] \to Y$ is said to be *relative* to A if h fixes the points of A, i.e, if $h(a, t) = h(a, 0)$ for all $a \in A$ and all $0 \le t \le 1$. If $f = h|_{t=0}$ and $g = h|_{t=1}$, we denote a relative homotopy as $h : f \simeq g \quad (\mathrm{rel}\ A)$.

Example 5.2:

Suppose $A \subset X$ is a retract, $r : X \to A$ be the retraction map. Then, r is *strong* deformation retract precisely when there is a relative homotopy $\iota \circ r \simeq \mathrm{Id}_X \quad (\mathrm{rel}\ A)$.

A special case of relative homotopy is given by basepoint preserving homotopies.

Definition 5.3: (*Pointed Spaces and Maps*)

A *pointed space* (X, x_0) is a space X along with a fixed *basepoint* $x_0 \in X$. A *pointed map* (or *based map*) $f : (X, x_0) \to (Y, y_0)$ between pointed spaces is a continuous map $f : X \to Y$ such that $f(x_0) = y_0$. A *basepoint preserving homotopy* is a homotopy $h : X \times [0, 1]$ such that $h(x_0, t) = y_0$ for all $0 \le t \le 1$. A *basepoint preserving homotopy equivalence* is a pointed map $f : (X, x_0) \to (Y, y_0)$ such that there is another map $g : (Y, y_0) \to (X, x_0)$, and basepoint preserving homotopies $g \circ f \simeq \mathrm{Id}_X$, $f \circ g \simeq \mathrm{Id}_Y$.

It is easy to see that a basepoint preserving homotopy between two pointed maps $f, g : (X, x_0) \to (Y, y_0)$ is precisely a relative homotopy $f \simeq g \quad (\mathrm{rel}\ \{x_0\})$.

So far, all the examples that we have seen, we were not very careful about basepoints! Turns out in many nice situations, we don't need to worry about basepoints. The next few results justify this claim. First, we need a definition.

Definition 5.4: (*Well-pointed Space*)

A pointed space (X, x_0) is called *well-pointed* if $\{x_0\}$ is closed in X, and the inclusion map $\{x_0\} \hookrightarrow X$ is a cofibration, i.e, has homotopy extension property against all spaces.

Lemma 5.5:

Let $(X, x_0), (Y, y_0)$ be pointed spaces. Suppose (X, x_0) is well-pointed, and Y is path connected. Then, given any map $f : X \to Y$, there exists a based map $f' : (X, x_0) \to (Y, y_0)$ such that $f \simeq f'$.

__Proof__ **:** Since Y is path connected, we can get a path $\gamma : [0, 1] \to Y$ such that

$$\gamma(0) = f(x_0), \quad \gamma(1) = y_0.$$

Consider the subspace $Z = (X \times \{0\}) \cup (\{x_0\} \times [0, 1]) \subset X \times [0, 1]$. Since $\{x_0\}$ is closed in X, by pasting lemma, it follows that we have a map $F : Z \to Y$ defined as

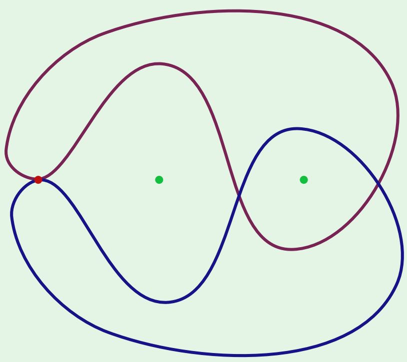
$$F(x, 0) = f(x), \quad F(x_0, t) = \gamma(t).$$

Since $\{x_0\} \hookrightarrow X$ has homotopy extension property against Y, it follows that F can be extended to a map $H : X \times [0, 1] \to Y$. Clearly F is a homotopy from f to a map $f' := F|_{t=1}$. By construction, $f'(x_0) = F(x_0, 1) = \gamma(1) = y_0$. Thus, f' is a based map, and $f \simeq f'$. □

The above result is an example of a *strictification*. One says that an arbitrary map can be *strictified* to a based map, under the appropriate hypothesis. Since a homotopy is also a map, you may try to strictify it in a similar way, but that does not always work!

Example 5.6:

Consider the double-punctured plane $X = \mathbb{R}^2 \setminus \{p = (1, 0), q = (2, 0)\}$, and fix the origin $x_0 = (0, 0)$ as the base point in X.



Then, as in the above picture, we can have two map $\gamma_1, \gamma_2 : S^1 \to X$. Let us fix a basepoint on S^1, say, z_0, and make sure that $\gamma_1(z_0) = x_0 = \gamma_2(z_0)$. It is easy to see that $\gamma_1 \simeq \gamma_2$ if we don't care about basepoints: just slide the two curves avoiding the puncture at p. But γ_1 and γ_2 are *not* based homotopic (as they represent distinct elements in the fundamental group $\pi_1(X, x_0)$).

A homotopy which does not respect basepoints is also known as *free* homotopy. The next result says that under some nice conditions (which are met more often than not!), one can strictify a homotopy equivalent.

Theorem 5.7:

Let $(X, x_0), (Y, y_0)$ be well-pointed, path connected spaces. Suppose $f : X \to Y$ is a homotopy equivalence, with homotopy inverse $g : (Y, y_0) \to (X, x_0)$, and homotopies $\varphi : g \circ f \simeq \mathrm{Id}_X, \psi : f \circ g \simeq \mathrm{Id}_Y$. Then, there exists a basepoint-preserving homotopy equivalence $f' : (X, x_0) \to (Y, y_0)$, with homotopy inverse $g' : (Y, y_0) \to (X, x_0)$, and basepoint preserving homotopies $\varphi' : g' \circ f' \simeq \mathrm{Id}_X, \psi' : f' \circ g' \simeq \mathrm{Id}_Y$, such that $f \simeq f', g \simeq g', \varphi \simeq \varphi', \psi \simeq \psi'$.

In other words, a homotopy equivalence $X \to Y$ can be homotoped in to a basepoint-preserving homotopy equivalence $(X, x_0) \to (Y, y_0)$.

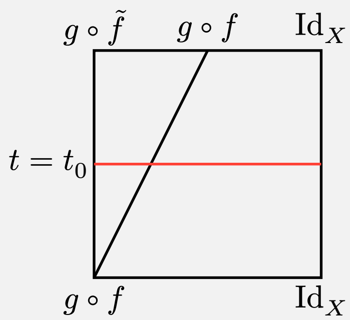
Proof : Applying Lemma 5.5 to f, we can first get a based map $\tilde{f} : (X, x_0) \to (Y, y_0)$, and a homotopy $\zeta : \tilde{f} \simeq f$. We have a composed homotopy $\Phi : g \circ \tilde{f} \simeq g \circ f \simeq \mathrm{Id}_X$, given explicitly as

$$\Phi(x, t) = \begin{cases} g\zeta(x, 2t), & 0 \leq t \leq \frac{1}{2} \\ \varphi(x, 2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

We have a homotopy of homotopies, say, $\chi : \varphi \simeq \Phi$ given explicitly as

$$\chi(x, s, t) = \begin{cases} g\zeta(x, (1-t) + 2st), & 0 \leq s \leq \frac{t}{2} \\ \varphi\left(x, \frac{2s-t}{2-t}\right), & \frac{t}{2} \leq s \leq 1. \end{cases}$$

It is eay to see that χ is indeed continuous, and is the desired homotopy. In order to understand χ, consider the schematic diagram



At some time $t = t_0$, the homotopy χ travels the tail-end of the homotopy $g\zeta$ in double speed, and then follows the homotopy φ in a suitable faster speed; length of the tail grows from 0 to all of it as t moves from 0 to 1. The above formula of χ can be deduced from this diagram.

Next, we homotope g to fix basepoint, but in a controlled way. Consider the path

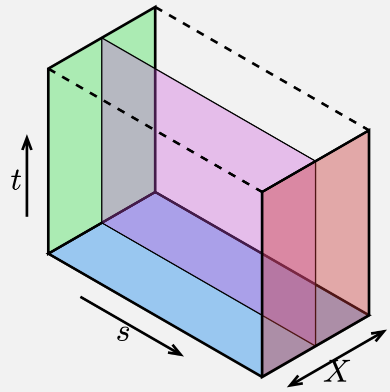
$$\gamma : t \mapsto \chi(x_0, t, 1) = \Phi(x_0, t)$$

in X, which joins $g\tilde{f}(x_0) = g(y_0)$ to x_0. Then, we have map

$$g \cup \gamma : (Y \times \{0\}) \cup (\{y_0\} \times [0, 1]) \to X.$$

Since (Y, y_0) is well-pointed, we have a based map $g' : (Y, y_0) \to (X, x_0)$, and a homotopy, $\xi : g' \simeq g$. Just like above, we have the composed homotopy $\tilde{\Phi} : g' \circ \tilde{f} \simeq g \circ \tilde{f} \simeq \mathrm{Id}_X$. Moreover, the same trick as above, gives a homotopy of homotopies $\Phi \simeq \tilde{\Phi}$; composing we get homotopy $\varphi \simeq \tilde{\Phi}$.

Now, we need to homotope the homotopy $\tilde{\Phi}$ to a basepoint-preserving homotopy. We should do this relative to the boundary $\{0, 1\}$ for the time parameter. Consider the subspace



$$B = (\{x_0\} \times [0,1] \cup X \times \{0,1\}) \times [0,1] \subset X \times [0,1] \times [0,1]$$

Since (X, x_0) is well-pointed, it follows that $B \hookrightarrow X \times [0,1]^2$ is again a cofibration. Now, we have a map defined on B.

- On $X \times [0,1] \times \{0\}$ it is the homotopy $\tilde{\Phi}$.
- On $X \times \{0\} \times [0,1]$ it is the map $g' \circ \tilde{f}$ constantly.
- On $X \times \{1\} \times [0,1]$ it is the map Id_X constantly.
- Finally, we need to define a map on $\{x_0\} \times [0,1] \times [0,1]$. Observe that all the four edges of this square is already prescribed. One the bottom edge $\{x_0\} \times [0,1] \times \{0\}$ it is the curve

$$\begin{aligned}
\eta : t \mapsto \tilde{\Phi}(x_0, t) &= \begin{cases} \xi\Big(\tilde{f}(x_0), 2t\Big), & 0 \le t \le \frac{1}{2} \\ \Phi(x_0, 2t-1), & \frac{1}{2} \le t \le 1 \end{cases} \\
&= \begin{cases} \xi(y_0, 2t), & 0 \le t \le \frac{1}{2} \\ \gamma(2t-1), & \frac{1}{2} \le t \le 1 \end{cases} \\
&= \begin{cases} \gamma(1-2t), & 0 \le t \le \frac{1}{2} \\ \gamma(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}.
\end{aligned}$$

  Thus, the curve γ is traced backward from x_0 to $g(y_0)$, and then back to x_0. We can then fill the square by homotoping this path back to the constant path x_0

Then, using the HEP, we have a homotopy

$$\varphi' : g' \circ \tilde{f} \simeq \mathrm{Id}_X \quad (\mathrm{rel}\ x_0),$$

and moreover, a homotopy of homotopies $\varphi \simeq \tilde{\Phi} \simeq \varphi'$ obtained from composition.

Finally, we fix the homotopy ψ. We already have,

$$\tilde{f} \circ g' \underset{\tilde{f} \circ \xi}{\simeq} \tilde{f} \circ g \underset{\zeta \circ g}{\simeq} f \circ g \underset{\psi}{\simeq} \mathrm{Id}_Y.$$

We perform the previous steps for by switching the roles (f, g) by (g', f). Note that we do not need to modify g' as it is already basepoint preserving. We get a homotopy $f' \simeq \tilde{f}$ such that $f' : (X, x_0) \to (Y, y_0)$ is based, and a homotopy $\psi' : f' \circ g' \simeq \mathrm{Id}_Y \quad (\mathrm{rel}\ \{y_0\})$. Clearly, $\psi \simeq \psi'$ follows from the construction. Also,

$$f' = f' \circ \mathrm{Id}_X \underset{\mathrm{rel}\ \{x_0\}}{\simeq} f' \circ \Big(g' \circ \tilde{f}\Big) = (f' \circ g') \circ \tilde{f} \underset{\mathrm{rel}\ \{x_0\}}{\simeq} \mathrm{Id}_Y \circ \tilde{f} = \tilde{f}.$$

The last homotopy is relative to x_0 since $\tilde{f}(x_0) = y_0$, and the homotopy $f' \circ g' \simeq \mathrm{Id}_Y$ is relative to y_0. But then we have $f' \simeq \tilde{f} \quad \mathrm{rel}\ \{x_0\}$. Composing, we have $f' \circ g' \simeq \tilde{f} \circ g' \simeq \mathrm{Id}_X$, relative to x_0. Clearly, the homotopies themselves are homotopic as well. This completes the claim. □

5.2 Mapping Cone

Let us revisit the construction of mapping cylinder.

Definition 5.8: (*Mapping Cone*)

Given a map $f : X \to Y$, the *mapping cone* C_f (or $C(f)$) is defined as the identification space $M_f / X \times \{1\}$. Explicitly, we have

$$C_f = \frac{(X \times [0,1]) \cup_f Y}{X \times \{1\}},$$

where the attaching map $f : X \to Y$ is on $X \times \{0\}$. We have an inclusion $Y \hookrightarrow C_f$.

The role of 0 and 1 in the definition is just a choice. One could have easily attached the space Y along $X \times \{0\}$, and collapsed $X \times \{1\}$; this would give a naturally homeomorphic copy of the mapping cone.

There are two special examples of mapping cones, which are important on their own.

Definition 5.9: (*Cone and Suspension*)

Given a space, the *cone* on X is denoted by $C(X)$ and is defined as the mapping cone of the identity map Id_X. Explicitly, we have the identification space

$$C(X) := (X \times [0,1])/(X \times \{0\}).$$

The *suspension* of the space X, denoted as ΣX is defined as the mapping cone of the constant map $X \to \{\star\}$. Explicitly, we have

$$\Sigma X := C(X)/(X \times \{1\})$$

Exercise 5.10:

Verify that there are homeomorphisms

$$C(D^n) = D^{n+1}, C(S^n) = D^n, \Sigma(D^n) = D^{n+1}, \Sigma(S^n) = S^{n+1},$$

where D^n is the closed n-disc and S^n is the n-sphere.

Exercise 5.11:

Given any space, show that the cone $C(X)$ is contractible. In fact, prove that it strongly deformation retracts onto the coning point, which is the equivalence class of $[(x,0)]$ for any $x \in X$.

Let us prove a very important property of the mapping cone, which is true for the mapping cylinder as well.

Proposition 5.12: (*Mapping Cone is Homotopy Invariant*)

Suppose $f \simeq g : X \to Y$ are homotopic maps. Then the mapping cones $C_f \simeq C_g$ are homotopy equivalent. Similarly, the mapping cylinders $M_f \simeq M_g$ are also homotopy equivalent.

Proof : Suppose $h : f \simeq g$ is a homotopy. We first construct the map

$$\begin{aligned}\tilde{\varphi} : (X \times [0,1]) \sqcup Y &\to C_g \\ (x,t) &\mapsto \begin{cases} [(x,2t)], & 0 \le t \le \frac{1}{2} \\ [h(x,2-2t)], & \frac{1}{2} \le t \le 1 \end{cases} \\ y &\mapsto [y].\end{aligned}$$

Clearly this is continuous with the disjoint union topology since at $t = \frac{1}{2}$ and $t = 1$ the function definition matches. Intuitively, $\tilde{\varphi}$ maps half of the cylinder $X \times \left[0, \frac{1}{2}\right]$ to the cone part of C_g, and then for the rest of the cylinder $X \times \left[\frac{1}{2}, 1\right]$ it follows the homotopy h back from g to f. Moreover, we have $\tilde{\varphi}(x,0) = [(x,0)] = \{\star\}$ and $\tilde{f}(x,1) = [h(x,0)] = [f(x)] = \tilde{\varphi}(f(x))$. Thus, passing to the quotient, we have a continuous map $\varphi : C_f \to C_g$. Similarly, we have a map $\psi : C_g \to C_f$ induced by the map

$$\begin{aligned}\tilde{\psi} : (X \times [0,1]) \sqcup Y &\to C_f \\ (x,t) &\mapsto \begin{cases} [(x,2t)], & 0 \le t \le \frac{1}{2} \\ [h(x,2t-1)], & \frac{1}{2} \le t \le 1 \end{cases} \\ y &\mapsto [y].\end{aligned}$$

We claim that $\varphi : C_f \to C_g$ is a homotopy equivalence, with homotopy inverse $\psi : C_g \to C_f$. Observe that

$$\begin{aligned}\psi \circ \varphi : C_f &\to C_f \\ [(x,t)] &\mapsto \begin{cases} [(x,4t)], & 0 \le t \le \frac{1}{4} \\ [h(x,4t-1)], & \frac{1}{4} \le t \le \frac{1}{2} \\ [h(x,2-2t)], & \frac{1}{2} \le t \le 1 \end{cases} \\ [y] &\mapsto [y]\end{aligned}$$

Thus, to get a homotopy $\psi \circ \varphi \simeq \mathrm{Id}_{C_f}$ we only need to scale the homotopies suitably. Indeed, consider the map

$$\begin{aligned}\alpha : C_f \times [0,1] &\to C_f \\ ([(x,s)],t) &\mapsto \begin{cases} \left[\left(x, \frac{4s}{3t+1}\right)\right], & 0 \le s \le \frac{3t+1}{4} \\ [h(x,4s-(3t+1))], & \frac{3t+1}{4} \le s \le \frac{t+1}{2} \\ [h(x,2-2s)], & \frac{t+1}{2} \le s \le 1 \end{cases} \\ [y] &\mapsto [y]\end{aligned}$$

The continuity follows from the pasting lemma by checking that the function is well-defined at each boundary values. Finally, we have $\alpha|_{t=0} = \psi \circ \varphi$ and $\alpha|_{t=1} = \mathrm{Id}_{C_f}$. Thus, $\alpha : \psi \circ \varphi \simeq \mathrm{Id}_{C_f}$ is a homotopy. Similarly, we can define a homotopy $\beta : \varphi \circ \psi \simeq \mathrm{Id}_{C_g}$. This proves the claim. $\square$

Exercise 5.13:

Suppose $f : X \to Y$ is null-homotopic. Show that the mapping cone C_f is homotopy equivalent to $\Sigma X \vee Y$

Exercise 5.14:

Suppose $A \subset X$ is a subspace and the inclusion $\iota : A \hookrightarrow X$ is a cofibration. Then, show that the mapping cone C_ι is homotopy equivalent to X/A. Give a counterexample if we drop the cofibration hypothesis.

5.3 Attaching Cells

Given a space X and a map $f : S^{n-1} \to X$, we say that the mapping cone C_f is *obtained from X by attaching an n-cell to X via the map f*. Explicitly, we have

$$C_f = D^n \cup_f X,$$

which justifies the terminology.

Proposition 5.15:

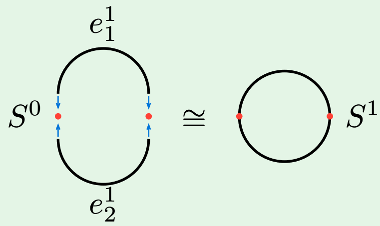
Given $f : S^{n-1} \to X$, there is a homeomorphic copy of the open n-disc in the mapping cone C_f, which is called an *n-cell*.

Proof : Recall that $D^n = C(S^{n-1})$. Indeed, we can *uniquely* write any element $x \in D^n \setminus \{0\}$ as $x = (z, r)$, where $z \in S^{n-1}$ and $r = \|x\|$. Consider the obvious map $g : \mathring{D}^n \to C_f$ given as the inclusion $\mathring{D}^n$ follows by the quotient map $q : D^n \sqcup X \to C_f$. We see that g is a continuous map, which is bijective onto the image. Next, consider an open set $U \subset \mathring{D}^n$. Then, $g(U) \subset C_f$ is a saturated set and $q^{-1}(U) = U$ is open in $D^n \sqcup X$. Hence, g is an open map as well. Consequently, g is a homeomorphism onto its image. □

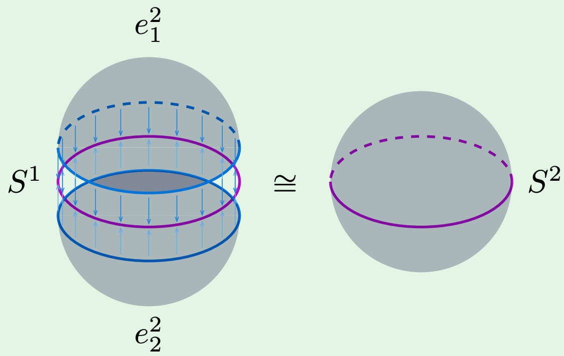
As a consequence of Proposition 5.12, the space obtained from attaching a cell is determined up to homotopy equivalence if we homotope the attaching map. Moreover, we have a canonical inclusion map $X \hookrightarrow C_f$ as the base of the mapping cone.

Example 5.16: (*Attaching cells to get the sphere*)

We can get the n-sphere S^n by attaching an n-cell to a point (via the obvious constant map), i.e, $S^n = D^n \cup_{S^{n-1} \to \star} \{\star\}$. On the other hand, we can get S^n in a different way as well. Let us start with S^0, which is a disjoint union of two points.



Attach **two** 1-cells, say, e_1^1 and e_2^1 as in the picture, and we get S^1. Next, let us begin with S^1.



Attach **two** 2-cells, say, e_1^2 and e_2^2 as the upper and lower hemispheres as in the picture. We then get S^2. Continuing this we can get S^n by attaching exactly *two* cells in each dimensions.