Course notes for

Algebraic Topology I (KSM3E02)

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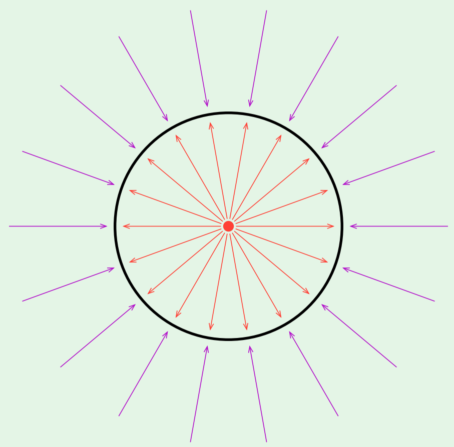
`examples of homotopy equivalence — wedge sum — topologist's sine curve — Hawaiian earring — local contractibility`

4.1 Examples of Homotopy Equivalence

Example 4.1: (*Puncturing the Euclidean Plane*)

Consider the space $X = \mathbb{R}^2$. If you remove a single point, you can show that the space is homotopy equivalent to the circle S^1. In fact, you can get a strong deformation retract to a circle of radius 1 centered at the puncture point.



Example 4.2: (*Puncturing the Euclidean Plane Again*)

Let's say you remove 2-points, say, p, q from $\mathbb{R}^2$. This time, you can again get a strong deformation retract as follows. First, determine the mid point, say, x of the line joining p and q. Say, the distance between p and q is $2d$. Draw the two circles of radii d, centered at p and q respectively; they touches exactly at the point x. Then, push everything inside the two discs away from the punctures to the boundary circles. You get a strong deformation retract onto the plane with two open discs removed. Next, draw the circle of radius $2d$ about the point x and push everything from outside to the boundary circle. You are left with a closed disc, with two open discs removed. Finally, collapse everything from top and bottom to the original circles of radius d. You are left with a figure-8 shape.

Before puncturing more holes, let us define a the notion of wedge sum.

Definition 4.3: (*Wedge Sum*)

Let $\{(X_\alpha, x_\alpha) \mid \alpha \in \Lambda\}$ be a collection of pointed spaces, i.e, a space with a point in it. The *wedge sum*, denoted $\bigvee_{\alpha\in\Lambda} X_\alpha$ is defined as the identification space $\frac{\bigsqcup_{\alpha\in\Lambda} X_\alpha}{\bigsqcup_{\alpha\in\Lambda}\{x_\alpha\}}$. That is, we consider the disjoint union $\bigsqcup_{\alpha\in\Lambda} X_\alpha$, and then identify all the basepoints $\bigsqcup_{\alpha\in\Lambda}\{x_\alpha\}$ to a single point.

The figure-8 shape in Example 4.2 is simply the wedge sum $S^1 \vee S^1$. To be precise, we should fix a basepoint in S^1, but it is easy to see that the wedge sum of path connected spaces are independent of such choice.

Example 4.4: (*Puncturing the Euclidean Plane Again and Again and...*)

Let us now remove n-many points from $\mathbb{R}^2$. Again, by suitably choosing a radius, you can strongly deforme this to $\mathbb{R}^2$ with n-many disjoint open discs removed. The trick is to collapse the outside of the boundary circles to a single point, and thus, we get n-fold wedge of circles. This is no longer a strong deformation retract.

Here is a different way to see this. Firstly, we claim that there is a *homeomorphism* of $\mathbb{R}^2$ to itself which takes any prescribed n-many points to another set of prescribed n-many points. Using this homeomorphism, we can assume that the punctures are arranged symmetrically along the unit circle. Then, draw n-many circles about the punctures which touches the neighboring two circles exactly at one point. Radially perform deformation retract so that you are left with the touching circles and the inside of the disc (which looks like a cookie after taking n-many symmetric bites). Finally, note that the inside is a contractible subspace, and hence you can collapse it. This gives the same space as the n-fold wedge sum of the circles.

Exercise 4.5: (*Neckless of Circles*)

Arrange n-many circles like a necklace: any circle touches exactly two neighboring circle, each at exactly one point. Can you identify this space as a wedge sum?

Exercise 4.6:

Let X be the space obtained from the sphere S^2 by removing $(n+1)$-many points. Can you describe X homotopy equivalent to certain wedge sum?

Before giving more examples of homotopy equivalent spaces, let us state a very useful result that we have already used a few times.

Proposition 4.7: (*Collapsing Contractible Subspaces*)

Let $A \subset X$ be given. Suppose A is contractible and the inclusion $\iota : A \hookrightarrow X$ is a cofibration. Then, the quotient map $q : X \to X/A$ is a homotopy equivalence.

Proof : Since A is contractible, we have a contracting homotopy $h : A \times [0,1] \to A$ which homotopes Id_A to a constant map. Since $\iota : A \hookrightarrow X$ is a cofibration, by HEP we can extend this homotopy to $H : X \times [0,1] \to X$, with $H|_{X\times\{0\}} = \mathrm{Id}_X$. Note that $H_1 := H|_{X\times\{1\}} : X \to X$ maps A to a single point, and hence, induces a map $r : X/A \to X$ such that $r \circ q = H_1$. In particular, H is a homotopy of Id_X with $r \circ q$.

On the other hand, the maps $H_t := H|_{X\times\{t\}}$ maps A into A, since H extends h. Thus, again passing to the quotient, we have a map $\tilde{H} : X/A \times [0,1] \to X/A$. One can show that this map is continuous, and hence, a homotopy of $\mathrm{Id}_{X/A}$. Note that

$$\tilde{H}_1([x]) = \tilde{H}_1(q(x)) = qH_1(x) = qrq(x) = qr([x]).$$

Thus, $\tilde{H}$ is a homotopy of $\mathrm{Id}_{X/A}$ with $q \circ r$.

This proves that $q : X \to X/A$ is a homotopy equivalence, with homotopy inverse $r : X/A \to X$. $\square$

Remark 4.8:

The following is a result due to J. H. C. Whitehead : Suppose $q : X \to Y$ is a quotient map and Z is locally compact. Then, the product $p := q \times \mathrm{Id}_Z : X \times Z \to Y \times Z$ is a quotient map.

Since $[0,1]$ is locally compact, in particular, we have $\tilde{q} : X \times [0,1] \to X/A \times [0,1]$ a quotient map. Now, in Proposition 4.7, the composition $q \circ H : X \times [0,1] \to X/A$ takes $A \times \{t\}$ to a point for all t, and hence induces a map $\tilde{H} : X/A[0,1] \to X/A$. By the property of quotient topology, $\tilde{H}$ is a continuous map.

Example 4.9: (*Pinching a Sphere*)

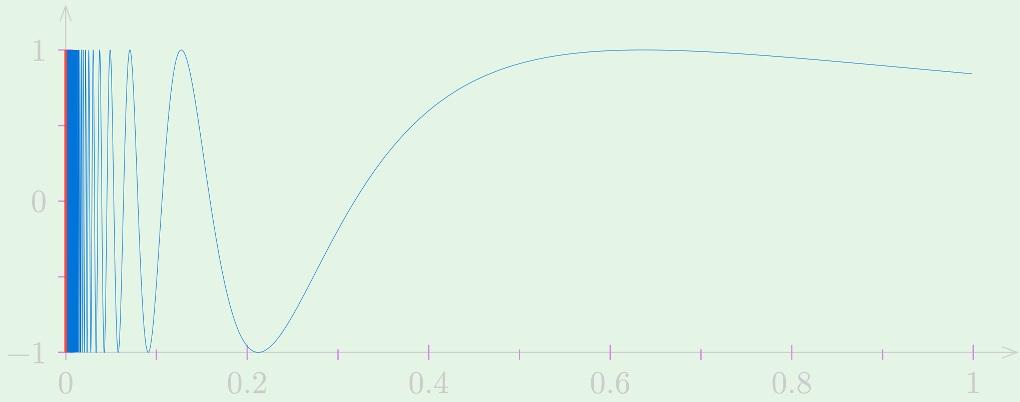
Consider the sphere S^2, along with the line $[-1,1]$ along the z-axis which touches the sphere at the north and south poles. Since the line segment is contractible, we can collapse it. This spaces is known as the *pinched sphere,* which is obtained by identifying two distinct points on the sphere. We claim that this is also homotopy equivalent to a wedge sum, namely, $S^2 \vee S^1$. To see this slide one end of the line segment to the other end and thus getting a circle. To justify this in another way, draw an arc of a great circle passing through the two points. This arc is contractible, and hence, collapsing it will give a homotopy equivalence. In particular, we have that the pinched sphere is homotopy equivalent to $S^2 \vee S^1$.

Caution 4.10:

We have been using Proposition 4.7 without checking whether the inclusion of the subspace is a cofibration. They are in fact cofibrations, since there are CW complex structures adapted to these inclusions. There are examples where this is not the case!

Example 4.11: (*Topologist's Sine Curve*)

Consider the following topologist's sine curve

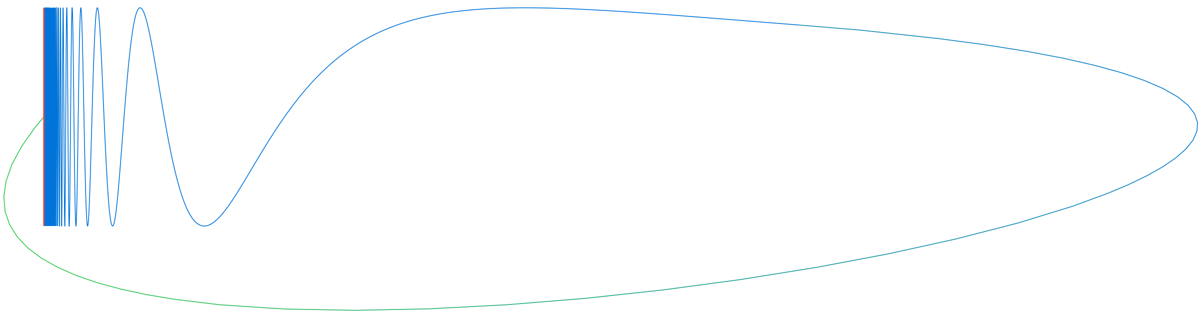


$$X = \{0\} \times [0,1] \cup \left\{ \left(x, \sin \frac{1}{x} \right) \mid 0 < x \leq 1 \right\}.$$

This space connected, but not path connected! Indeed, one can see that X is the closure of the graph of $\sin \frac{1}{x}$ over the segment $(0,1]$, the points on the vertical segment $A := \{0\} \times [0,1]$ being the limit points. Since the graph is homeomorphic to $(0,1]$ (why?!), it is clearly connected, and hence the closure X is also connected. On the other hand, one can show that X has two path components: one is the graph of $\sin \frac{1}{x}$ and the other path component is the vertical segment. Thus, X is not contractible.

Now, the subspace A is contractible being an interval. We claim that X/A is homeomorphic to $[0,1]$, hence contractible, and consequently, cannot be homotopy equivalent to X. Consider the projection map $\pi : X \to [0,1]$. Observe that π maps A to the single point $\{0\}$, and thus passes to the quotient $\tilde{\pi} : X/A \to [0,1]$. We have $\tilde{\pi}$ is bijective. Also, X/A is compact (being the quotient of the compact space X) and $[0,1]$ is Hausdorff. Hence, $\tilde{\pi}$ is a homeomorphism.

Continuing with the previous example, let us construct the *Warsaw circle*.



This is obtained from the topologist's sine curve, by adding a curve starting at $(1, \sin(1))$ and ending at the origin. The Warsaw circle is path connected, but not locally path connected at any of the points of vertical interval.

Proposition 4.12:

Let W denote the Warsaw circle as above. Suppose K is a sequentially compact, locally path connected space. Then, any map $K \to W$ is null-homotopic, i.e, homotopic to a constant map.

Proof : Let us breakdown the Warsaw circle as follows.

- Denote the closed arc attached to the topologist's sine curve by C. Clearly, C is homeomorphic to $[0,1]$.
- Denote the vertical segment $V := \{0\} \times [0,1]$
- Denote the segment $S = \left\{\left(x, \sin\frac{1}{x}\right) \mid 0 < x \le 1\right\}$, which is homeomorphic to $(0,1]$.

We have, $W = V \cup S \cup C$. Now, for each $0 < \varepsilon \le 1$, denote the subspaces

$$S_\varepsilon = \left\{\left(x, \sin\frac{1}{x}\right) \middle| \varepsilon \le x \le 1\right\}, \quad W_\varepsilon = V \cup S_\varepsilon \cup C.$$

Observe that W_ε is clearly contractible, and hence, any map $f : K \to W_\varepsilon$ is null-homotopic. We claim that any map $f : K \to W$ is actually contained in some W_ε. In fact, we show that $f(K) \cap S \subset S_\varepsilon$ for some $\varepsilon > 0$. Suppose not. Let us denote $u(x) = \pi_1(f(x))$, projection of f onto the first component. Then, for each $n \ge 1$, there exists some $x_n \in K$ such that $0 < u(x_n) < \frac{1}{n}$. Since K is sequentially compact, passing to a subsequence, we have $x_n \to x_0$. Since u is continuous, it follows that $u(x_0) = 0$. Thus, $f(x_0)$ is on the vertical segment V. Choose small ball U around $f(x_0)$, such that $U \cap W \not\supset C$. As f is continuous, there is an open neighborhood $x_0 \in V \subset X$ such that $f(V) \subset U \cap W$. Since X is locally path connected, we can assume that V is path connected. Now, for some $n \gg 1$, we have $x_n \in V$. Get a path $\gamma : [0,1] \to V$ joining x_n to x_0. It follows that $f \circ \gamma$ is a path joining $f(x_0) \in V$ to $f(x_n) \in S$. But any such path must traverse the entire curve C. This is a contracdiction, since the path $f \circ \gamma$ is contained in U, and $U \cap W \not\supset C$. This proves the claim. □

Since each sphere S^n is sequentially compact and locally path connected, it follows from Proposition 4.12 that any map $S^n \to W$ is null-homotopic. (This leads to the fact that the Warsaw circle is *weakly contractible*, i.e, every homotopy group of the Warsaw circle is zero with respect to any basepoint).

Exercise 4.13:

If X is contractible, show that any map $f : Y \to X$ is null-homotopic.

Proposition 4.14:

The Warsaw circle is *not* contractible.

Proof : Firstly, observe that the identification space W/V, obtained by collapsing the vertical segment, is homeomorphic to S^1. Thus, we have a map $f : W \to S^1$. We show that this map is *not* null-homotopic. If possible, suppose $h : f \simeq c$ is a homotopy to a constant map $c : W \to S^1$. Consider the map $p : \mathbb{R} \to S^1$, given by $p(\theta) = (\cos\theta, \sin\theta)$. It is well-known (we shall see a prove later) that p is a fibration, i.e, p has homotopy lifting property against all spaces. Since c is a constant map, we can always get a lift $\tilde{c} : W \to \mathbb{R}$ such that $p \circ \tilde{c} = c$. Then, by HLP, we have a lift of h to $\tilde{h} : W \times [0,1] \to \mathbb{R}$. In particular, we have a lift $\tilde{f} : W \to \mathbb{R}$ such that $p \circ \tilde{f} = f$. Now, f is injective everywhere but V, and hence, $\tilde{f}$ is also injective, except possible on V, i.e, $\tilde{f}|_{W \setminus V}$ is injective. Now, $f(V)$ is a point, and $\tilde{f}(V) \subset p^{-1}(f(V)) = \mathbb{Z}$. As V is path connected, the only possibility is that $\tilde{f}(V)$ is also a singleton. Thus, we can pass to the quotient, and get a map $\hat{f} : W/V \to \mathbb{R}$. It follows that $\hat{f}$ is injective. But $W/V \cong S^1$, and (by the intermediate value theorem) there is no injective continuous map $S^1 \to \mathbb{R}$. This is a contradiction. Hence, $f : W \to S^1$ cannot be null-homotopic. This implies that W is not contractible. □

Thus, the Warsaw circle is an example of a weakly contractible space, which is not contractible!

Recall that we can define wedge sum of arbitrary many (pointed) spaces.

Exercise 4.15: (*Countable Infinite Wedge of Circles*)

Consider the space

$$X = \bigcup_{n \geq 1} \{(x, y) \mid (x-n)^2 + y^2 = n^2\} \subset \mathbb{R}^2.$$

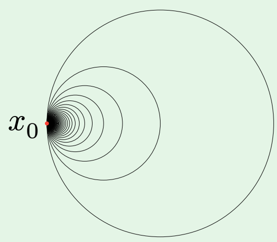
Show that X is homeomorphic to the infinite wedge $\bigvee_{n \geq 1} S^1$. Can you directly show that $\bigvee_{n \geq 1} S^1$ is noncompact? Show that given a compact subspace $K \subset X$, there exists some large N such that

$$K \subset \bigcup_{n=1}^{N} \{(x, y) \mid (x-n)^2 + y^2 = n^2\} \subset X.$$

The next (in)famous example is similar to the infinite wedge of circle, but is vastly different from it!

Example 4.16: (*Hawaiian Earring*)

Consider the space



$$H := \bigcup_{n \geq 1} \left\{ (x, y) \;\middle|\; \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2} \right\} \subset \mathbb{R}^2.$$

This space is known as the Hawaiian earring. Clearly, H is compact (being closed and bounded), path connected, locally path connected space. One can see that the space X/A considered in Caution 4.10 is actually homeomorphic to the Hawaiian earring! We claim that H is not even homotopy equivalent to the countable infinite wedge of circles.

Intuitively, it is clear that H is not contractible, we shall justify this using fundamental group. Let us consider the point $x_0 = (0, 0) \in H$. Any open neighborhood of x_0 will contain infinitely many copies of circles (in fact, a shrunk down copy of the Hawaiian earring itself). Thus, any open neighborhood of x_0 fails to be contractible. But this is not the case for $\bigvee_{n=1}^{\infty} S^1$. This implies that H is not homotopy equivalent to the countable infinite wedge of circles.

Let us give a name to the property discussed above.

Definition 4.17: (*Locally Contractible*)

A space X is called *locally contractible* if given any point $x \in X$ and a neighborhood $x \in U \subset X$, there exists an open set $x \in V \subset U$ such that V is contractible.

We have seen that the Hawaiian earring is *not* locally contractible, but the infinite wedge $\bigvee_{n\geq 1} S^1$ evidently is. Thus, they cannot be homeomorphic. In fact, every CW complex is locally contractible. Hence, Hawaiian earring cannot be given the structure of a CW complex.

Caution 4.18:

In general, local contractibility is not preserved under homotopy equivalence. Indeed, consider the usual comb space

$$X = ([0,1] \times \{0\}) \cup \bigcup_{n\geq 1} \left\{ \left(x, \frac{x}{n}\right) \;\middle|\; 0 \leq x \leq 1 \right\}.$$

X fails to be locally contractible at the point $(1,0)$. But X contractible, i.e, homotopy equivalent to a point, which is tautologically locally contractible!

Remark 4.19:

It is a fact that the Hawaiian earring is not even homotopy equivalent to any CW complex. But to prove this we need more sophisticated argument. More generally, it does not even have the weak homotopy type of a CW complex.