Course notes for

Algebraic Topology I (KSM3E02)

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`contractible spaces — disjoint union — attaching spaces — mapping cylinder — mapping cone`

3.1 A Detour into Contractible Spaces

We have already defined contractible spaces (Definition 2.6). In Example 2.7 we saw that contractible spaces need to *strongly* deformation retract onto any point.

Exercise 3.1:

Given a space X, verify that the following are equivalent.

1. X is contractible.
2. X admits a deformation retract to some $x_0 \in X$.

Next, let us see some nontrivial but important examples of contractible spaces.

Example 3.2: (*Path Space is contractible*)

Given a space X, fix a point $x_0 \in X$. Consider the space of all paths in X starting at x_0

$$\mathcal{P} = \mathcal{P}(X, x_0) = \{\gamma : [0,1] \to X \mid \gamma(0) = x_0\}.$$

Give it the compact-open topology. Recall: the compact-open topology on the space $C(X, Y)$ of continuous maps is generated by the sets

$$\{f : X \to Y \mid f(C) \subset U\}, \quad C \underset{\text{compact}}{\subset} X, \quad U \underset{\text{open}}{\subset} Y.$$

Since $[0,1]$ is locally compact, it follows that the evaluation map

$$\begin{aligned} e : \mathcal{P} &\to X \\ \gamma &\mapsto \gamma(1) \end{aligned}$$

is continuous. We claim that the path space $\mathcal{P}$ is contractible.

Intuitively, the contracting homotopy is given by contracting every path back to the origin x_0, like sucking a bunch of spaghetti! Explicitly, we can consider the map

$$\begin{aligned} H : \mathcal{P} \times [0,1] &\to \mathcal{P} \\ (\gamma, t) &\mapsto (s \mapsto \gamma((1-t)s)). \end{aligned}$$

Clearly, we have $H|_{t=0} = \mathrm{Id}_{\mathcal{P}}$, and $H|_{t=1}$ maps every path γ to the constant path $\gamma_0 : s \mapsto x_0$. Thus, H is a strong deformation retract of $\mathcal{P}$ to the *point* γ_0. The fact that H is continuous is a consequence of the compact-open topology.

Example 3.3: (S^∞ *is contractible*)

Recall that the 0^{th} sphere S^0 is nothing but two points, clearly it is not even connected. We know S^1 is path connected, and later see that it is not *simply connected,* in particular S^1 is not contractible. Similarly, using other machinery (namely, either homology or higher homotopy groups) one can show that the n -sphere S^n is not contractible either.

Let us define the infinite-dimensional sphere S^∞. There are many ways to do so. Recall we have the following inclusions

$$S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow S^3 \hookrightarrow \cdots,$$

where the inclusions are the middle sphere. One defines S^∞ as the colimit of these spaces. Without going into this abstract construction, let us define it in another way.

Consider the space X to be the collection of sequences $\mathrm{x} = (x_0, x_1, x_2, \ldots)$ of real numbers, such that all but finitely many of them are 0 (i.e, for some large enough K, we have $x_n = 0$ for $n \geq K$). This space X is a normed linear space via the ℓ^2-norm

$$\|\mathrm{x}\| = \sqrt{\sum x_i^2}.$$

Note that the sum is a finite sum. One then defines

$$S^\infty = \{\mathrm{x} \in X \mid \|\mathrm{x}\| = 1\}$$

as the unit norm sphere in X.

We claim that S^∞ is contractible. In order to see this, consider the shift map $T : X \to X$ defined as

$$T(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots).$$

One can check that T is continuous and injective. Note that for any $\mathrm{x} \in S^\infty$, we have $T(\mathrm{x})$ is *not* a multiple of x. In particular, the line $t\mathrm{x} + (1-t)T(\mathrm{x})$ is never 0. Consequently, we have a continuous map

$$\begin{aligned} F : S^\infty \times [0,1] &\to S^\infty \\ (\mathrm{x}, t) &\mapsto \frac{t\mathrm{x} + (1-t)T(\mathrm{x})}{\|t\mathrm{x} + (1-t)T(\mathrm{x})\|}. \end{aligned}$$

This is a homotopy between Id and the normalized shift map $f : \mathrm{x} \mapsto \frac{T(\mathrm{x})}{\|T(\mathrm{x})\|}$. Next, note that the point $x_0 = (1, 0, 0, \ldots) \in S^\infty$ is *not* in the image of this normalized shift map. Consequently, the line joining x_0 with any point in the image of f does not pass through the origin, and we have another homotopy

$$\begin{aligned} G : S^\infty \times [0,1] &\to S^\infty \\ (x, t) &\mapsto \frac{tx_0 + (1-t)f(\mathrm{x})}{\|tx_0 + (1-t)f(\mathrm{x})\|}. \end{aligned}$$

This is a homotopy from f to the constant map at x_0. Composing the two homotopies, we get a deformation retract of S^∞ to the point x_0. In particular, S^∞ is contractible.

3.2 Disjoint Union

Let us now define a few important constructions in algebraic topology, which are all special instances of quotient spaces.

Example 3.4: (*Disjoint Union*)

Given two spaces X, Y, consider the *disjoint union* $X \sqcup Y$ as a set. Explicitly, you can consider

$$X \sqcup Y := \{(x, X) \mid x \in X\} \cup \{(y, Y) \mid y \in Y\},$$

and then you can identify X, Y as *subsets* of $X \sqcup Y$ in the obvious way. Next, give it the following topology:

$U \subset X \sqcup Y$ is open if and only if both $U \cap X$ and $U \cap Y$ are open.

Verify that this topology is induced by the following equivalence relation:

$(u, A) \sim (v, B)$ if and only if $(u, A) = (v, B)$.

$X \sqcup Y$ is also known as the *topological sum* of the spaces.

The above description of the disjoint union topology is very cumbersome. The point is that given two subsets $A, B \subset C$, you can construct the union $A \cup B \subset C$, if there are elements repeating in both A and B they are not considered as distinct elements. On the other hand, for disjoint union, you want them to be distinct! That is why, any element $a \in A$ is *tagged* by a unique ID, say, (a, A) and similarly, $b \in B$ is tagged as (b, B). As an explicit example, consider $A = \{1, 2\}, B = \{2, 3\}$, then $A \cup B$ has three elements, namely $\{1, 2, 3\}$, whereas $A \sqcup B$ has 4 elements: $\{(1, A), (2, A), (2, B), (3, B)\}$.

Exercise 3.5:

Consider the spaces $A = [0, \infty)$ and $B = (-\infty, 0]$. Clearly $A \cup B = \mathbb{R}$. Justify that $A \sqcup B$ is *not* homeomorphic to $\mathbb{R}$.

Hint : What are the connected components of $A \sqcup B$?

Exercise 3.6:

Consider $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ as subspaces of $\mathbb{R}$. Is $A \sqcup B$ homeomorphic to $\mathbb{R}$?

The concept of disjoint union easily generalizes to arbitrary collection of spaces. Indeed, given spaces $\{X_\alpha\}_{\alpha\in\Lambda}$, we have the disjoint union $\bigsqcup_{\alpha\in\Lambda} X_\alpha$, where the topology is given as $U \subset \bigsqcup_{\alpha\in\Lambda} X_\alpha$ is open if and only if $U \cap X_\alpha$ is open for each $\alpha \in \Lambda$.

Exercise 3.7: (*Universal Property of Disjoint Union*)

Verify that given any space Z and any continuous maps $f : X \to Z, g : Y \to Z$, there exists a *unique* continuous map $h : X \sqcup Y \to Z$ such that $h|_X = f$ and $h|_Y = g$. Moreover, this property can be used to define the disjoint union:

if there exists a space W such that for any $f : X \to Z, g : Y \to Z$ there is a unique map $h : W \to Z$ satisfying $h|_X = f, h|_Y = g$, then W is (uniquely) homeomorphic to $X \sqcup Y$.

One denotes, $f \sqcup g : X \sqcup Y \to Z$. Do this exercise for an arbitrary collection of spaces.

3.3 Attaching Spaces

Suppose $A \subset X$ is a subspace, and $f : A \to Y$ is a map. We would like to *attach* (or glue) X to Y along this map.

Definition 3.8: (*Attaching Space*)

Given a subspace $A \subset X$ and a map $f : A \to Y$, the *attaching space* $X \cup_f Y$ is defined as the quotient space induced by the following equivalence relation on the disjoint union $X \sqcup Y$: $u \sim v$ if and only if one of the following holds

i. $u = v$.

ii. $u, v \in A$ and $f(u) = f(v)$.

iii. $u \in A$ and $v = f(u)$.

iv. $v \in A$ and $u = f(v)$

The map f is called the *attaching map*.

At first glance, this equivalence relation might seem hard to parse. Instead, we look at the universal property for the attaching spaces. Firstly, note that we have two canonical maps $p : X \to X \cup_f Y, q : Y \to X \cup_f Y$, both of which are continuous. Given an arbitrary space Z, suppose we have maps $\varphi : X \to Z, \psi : Y \to Z$. Then, as discussed earlier, we have a unique map $\varphi \sqcup \psi : X \sqcup Y \to Z$. Now, given a subspace $A \subset X$, and a map $f : A \to Z$, suppose moreover that $\varphi|_A = \psi \circ f$. Then, there exists a *unique* map $h : X \cup_f Y \to Z$ such that $h \circ p = \varphi, h \circ q = \psi$. That is, we have the diagram

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d, "p"] \arrow[ddr, bend left, "\varphi"] & \\
Y \arrow[r, "q"'] \arrow[drr, bend right, "\psi"'] & X \cup_f Y \arrow[dr, dashed, "\exists! h"] & \\
& & Z
\end{tikzcd}$$

Here is an example, we shall see more soon.

Example 3.9:

- X, Y be the closed unit disc in $\mathbb{R}^2$, A be the unit circle, and $f : A \to Y$ be the inclusion map. Then, $X \cup_f Y$ is homeomorphic to the 2-sphere S^2.
- X be the closed unit disc in $\mathbb{R}^2$, Y be the space with a single point, A be the unit circle and $f : A \to Y$ be the (obvious) constant map. Then, $X \cup_f Y$ is again homeomorphic to S^2.

The last example is also known as an *identification space*.

Definition 3.10: (*Identification Space*)

Given a subspace $A \subset X$, the *identification space* X/A is defined as the attaching space $X \cup_f \{\star\}$, where $f : A \to \{\star\}$ is the constant map. Alternatively, X/A is defined as the quotient space induced from the following equivalence relation on X:

$$u \sim v \text{ if and only if either } u, v \in A, \text{ or } u \notin A, v \notin A, u = v.$$

Intuitively, the identification space collapses the subspace $A \subset X$, and identifies it to a single point.

Exercise 3.11:

Given X as the closed unit ball in $\mathbb{R}^n$ and $A \subset X$ as the closed unit sphere S^{n-1}, verify that X/A is homeomorphic to the n-sphere S^n.

Exercise 3.12:

Suppose $A, B \subset X$ are subspaces such that $X = \mathring{A} \cup \mathring{B}$, i.e, the interiors of A and B covers X. Show that the attaching space $A \cup B$ along the intersection $A \cap B$ is homeomorphic to X.

3.4 Mapping Cylinder

The concept of attaching spaces gives rise to many important constructions in algebraic topology.

Definition 3.13: (*Mapping Cylinder*)

Given $f : X \to Y$ a map, the *mapping cylinder* is constructed as the attaching space $M_f := (X \times [0,1]) \cup_f Y$, where we have identified f as the attaching map $f : X \times \{0\} \to Y$. We have an inclusion map

$$\begin{aligned} \iota : X &\hookrightarrow M_f \\ x &\mapsto [(x,1)], \end{aligned}$$

and a retraction map

$$\begin{aligned} r : M_f &\to Y \\ [(x,t)] &\mapsto f(x) \\ [y] &\mapsto y. \end{aligned}$$

Proposition 3.14: (*Mapping Cylinder and Cofibration*)

Given any $f : X \to Y$, the inclusion $\iota : X \hookrightarrow M_f$ in the mapping cylinder is a cofibration, and $r : M_f \to Y$ is a homotopy equivalence. In fact, r is a strong deformation retract.

This construction gives us the following commutative diagram

$$\begin{tikzcd} & M_f \arrow[dr, bend left, two heads, "r"', "s.d.r"] & \\ X \arrow[ur, bend left, hook, "\iota"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

One says that any map $f : X \to Y$ can be replaced (up to homotopy) by a cofibration $\iota : X \to M_f$ into the mapping cylinder.

3.5 Homotopy Lifting Property

The dual concept of homotopy *extension* is homotopy *lifitng*. Given a map $p : E \to B$, we say p has *homotopy lifting property* (or *HLP*) against a spaces X if given any map $f : X \to E$ and any homotopy $h : X \times [0, 1] \to B$ satisfying $h|_{t=0} = p \circ f$, there exists a lift $H : X \times [0, 1] \to E$ such that

$$H(x, 0) = f(x) \quad x \in X, \qquad p \circ H(x, t) = h(x, t) \quad x \in X, 0 \leq t \leq 1.$$

In other words, given a commutative diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

there exists a lift H making everything commute.

Definition 3.15: (*Fibration*)

A map $p : E \to B$ is called a *(Hurewicz) fibration* if it has homotopy lifting property against any space X.

Let us now see few examples of fibrations.

Example 3.16:

Given any space X, we have the constant map $p : X \to \{\star\}$, and it is easy to see a fibration: the lift is given by constant homotopy. More generally, given a product $B \times F$, consider the projection $\pi : B \times F \to B$. Then, π is a fibration. Indeed, given any diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d, hook] & B \times F \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"'] & B \end{tikzcd}$$

define $H(x, t) = f(x)$. It is easy to see that H is a homotopy lifting here.

Remark 3.17:

A more general class of fibrations is given by Serre fibrations: a map $p : E \to B$ is a *Serre fibration* if it has HLP against all CW complexed. Equivalently, one can define Serre fibrations as those maps $p : E \to B$ which has HLP against all the closed n-discs D^n.

One can show that a fiber bundle (which is a *twisted* product space) is a Serre fibration, and it is a Hurewicz fibration if B is a paracompact space. We shall later define *covering spaces,* which are examples of fiber bundles. In particular, they are Serre fibrations.