Course notes for

Algebraic Topology I (KSM3E02)

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Day 2 : 5th August, 2026

`homotopy extension — homotopy lifting`

2.1 Composition and Homtopies

There are two ways to compose homotopies. We can compose them *vertically*

$$
\begin{tikzcd}
X \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{name=G}] \arrow[r, bend right=60, "h"'{name=Hh}] & Y
\arrow[Rightarrow, from=F, to=G, "H"'] \arrow[Rightarrow, from=G, to=Hh, "K"'] \arrow[Rightarrow, from=F, to=Hh, bend left, "L"]
\end{tikzcd}
$$

This is nothing but the transitivity. We can also compose them *horizontally*

$$
\begin{tikzcd}
X \arrow[r, bend left, "f_1"{name=F1}] \arrow[r, bend right, "f_2"'{name=F2}] \arrow[rr, bend left=70, "g_1 \circ f_1"{name=T}] \arrow[rr, bend right=70, "g_2 \circ f_2"'{name=B}] & Y \arrow[r, bend left, "g_1"{name=G1}] \arrow[r, bend right, "g_2"'{name=G2}] & Z
\arrow[Rightarrow, from=F1, to=F2, "H"'] \arrow[Rightarrow, from=G1, to=G2, "K"'] \arrow[Rightarrow, from=T, to=B, bend left, "L"]
\end{tikzcd}
$$

Exercise 2.1:

Construct the horizontal composition of two homotopies.

2.2 Retraction

Definition 2.2: (*Retraction*)

Given a subspace $A \subset X$, a *retraction* is a map $r : X \to A$ such that $r \circ \iota = \mathrm{Id}_A$, where $\iota : A \hookrightarrow X$ is the inclusion. We say A is a *retract* of X. In other words, we have a commuting diagram

$$\begin{tikzcd} A \arrow[r, hook, "\iota"'] & X \arrow[l, bend right, "r"'] \end{tikzcd}$$

As an example, any point in a space is a retract. We shall see that the boundary circle is *not* a retract of the closed disc (and more generally, boundary sphere is not a retract of the closed ball). Try to justify that $A = \{0, 1\} \subset [0, 1]$ is *not* a retract of $[0, 1]$!

Exercise 2.3:

Suppose X is a T_2-space (i.e, Hausdorff). If $A \subset X$ is a retract, verify that A must be a closed subset.

Hint : For a T_2-space X, if we have two maps $f, g : X \to Y$, then the diagonal $\Delta = \{x \in X \mid f(x) = g(x)\}$ is a closed subset.

Let us now specialize homotopy equivalence to the case of retraction, which is easier to visualize.

Definition 2.4: (*Deformation Retract*)

A subspace $A \subset X$ is a *deformation retract* if there is a retraction $r : X \to A$ satisfying $r \circ \iota = \mathrm{Id}_A$, and there is a homotopy $\iota \circ r \simeq \mathrm{Id}_X$. Explicitly, we need a homotopy $H : X \times [0, 1] \to X$ such that

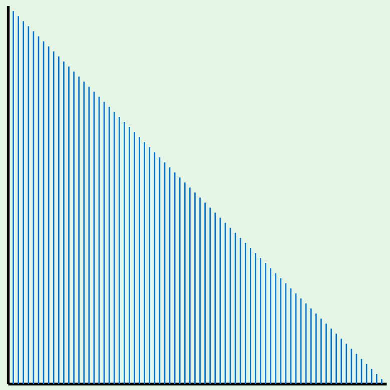
$$H(x, 1) = x, \quad x \in X, \qquad H(x, 0) \in A, \quad x \in X, \qquad H(a, 0) = a, \quad a \in A.$$

We say $A \subset X$ is a *strong deformation retract* if moreover the homotopy restricts to identity on A, i.e, if $H(a, t) = a$ for all $a \in A$ and $0 \leq t \leq 1$.

It is clear from the definition that a deformation retract is always a homotopy equivalence. There are (pathological) examples which are deformation retracts, but *not* strongly.

Example 2.5:

Consider the following comb-like space.



Explicitly, the space X can be identified as the following subspace of $\mathbb{R}^2$

$$X = ([0,1] \times \{0\}) \cup \bigcup_{r \in [0,1] \cap \mathbb{Q}} \{r\} \times [0, 1-r]$$

That is, keep the horizontal interval $[0,1] \times \{0\}$, and then for each rational point r on it add the vertical line of length $1 - r$.

We claim that X strongly deformation retrats onto any point on the horizontal line. Indeed, it is easy to construct a linear homtopy which pushes everything down to the horizontal line, and then collapses the line to a designated point; this designated point is never moved throughout.

On the other hand, for any point *not* on the horizontal line, there is always a deformation retract: just perform the previous deformation to the point, say, $(0,0)$ and then slide this point to any other point, speeding up as necessary. But for such a point, there cannot be a strong deformation retract. To see this intuitively, observe that if there is a strong deformation retract to a point, then in any neighborhood of that point must contain a smaller neighborhood which also collapses to that same point. But this is clearly not the case since you can always find a neighborhood of such a point which cannot be path connected (and hence cannot collapse to a point).

Let us give a name to spaces which are homotopy equivalent to a point.

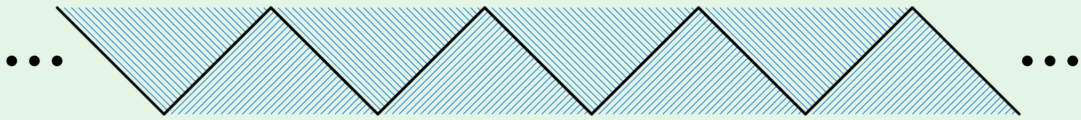
Definition 2.6: (*Contractible Space*)

A space X is said to be *contracitble* if it is homotopy equivalent to a point. We denote it as $X \simeq \star$.

We have seen that any space admits a retraction to any point in it. It is easy to see that if a space deformation retracts to a point, then it is contractible, and vice versa. This does not hold true for *strong* deformation retract.

Example 2.7:

This example builds upon Example 2.5. Consider the following subspace X of $\mathbb{R}^2$



This is obtained from the comb-like space in Example 2.5 by arranging them in the infinite zigzag pattern!

We claim that this space X is *contractible*, but does not *strongly* deformation retract to any point. Intuitively, a strong deformation retract to a point x_0 will imply that there for any neighborhood of x_0, there is a smaller neighborhood which contracts to x_0: this is clearly not the case here (do a case-by-case analysis of different kinds of points).

In order to see why this space is contractible, we do the following. Firstly, for each point x in the space, we have a unit-speed path $r_x : [0, \infty) \to X$ which follows the *bristle* of the comb where x belongs till it hits the bold zigzag line, and then moves along the zigzag to the right. Clearly if x is already on the zigzag, it just starts following it. Consider the following map

$$\begin{aligned} F : X \times [0,1] &\to X \\ (x,t) &\mapsto r_x(t). \end{aligned}$$

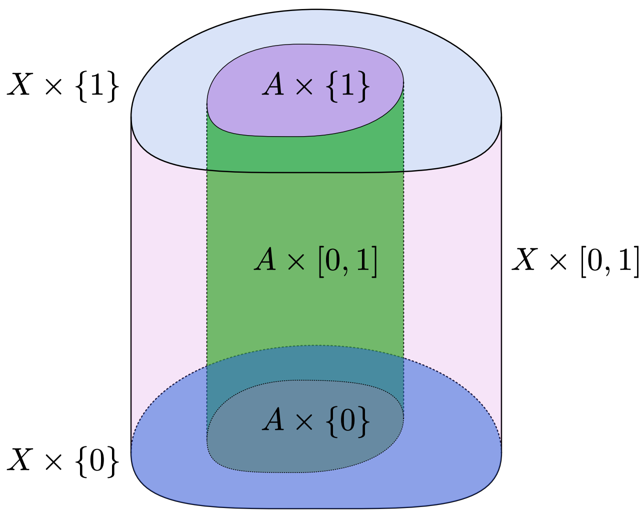
One can check that F is a continuous map, and hence a homotopy between the Id_X to a map which ends in the zigzag. Finally, the zigzag is just a copy of $\mathbb{R}$, and hence contractible. Thus, composing we can get a contracting homotopy. In other words, $X \simeq \star$.

Caution 2.8:

According to Hatcher, a "deformation retract" is precisely what we call a *strong* deformation retract. Hatcher also defines a notion of "deformation retract in the weak sense": this asks for a deformation retraction $H : X \times [0,1] \to X$ such that $H(a,t) \in A$ for all $a \in A$ and for all time $0 \le t \le 1$.

2.3 Homotopy Extension Property

Suppose $A \subset X$ is a subspace. Assume that we are given a map $f : X \to Y$, and a homotopy $h : A \times [0,1] \to Y$ from $f|_A : A \to Y$.



We say, the pair (X, A) has *homotopy extension property* (or *HEP*) with respect to the space Y if there exists a homotopy $H : X \times [0,1] \to Y$ from f, which extends the homotopy h. That is,

$$H(x,0) = f(x) \quad x \in X, \qquad H(a,t) = h(a,t) \quad a \in A, 0 \leq t \leq 1.$$

The schematic picture explains what is going on. We are given a continuous map on the subspace

$$A \times [0,1] \cup X \times \{0\} \subset X \times [0,1],$$

and we wish to extend it to the wholse space $X \times [0,1]$.

Definition 2.9: (*Cofibration*)

The inclusion map $\iota : A \hookrightarrow X$ is called a *(Hurewicz) cofibration* if it has homotopy extension property with respect to every space Y.

Remark 2.10:

More generally, we can define HEP for any map $f : A \to X$, not just inclusion maps. Indeed, this can be formulated via the following diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & Y^{[0,1]} \arrow[d, "e"] \\ X \arrow[r, "g"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

Here, $Y^{[0,1]}$ denotes the space of continuous maps $[0,1] \to Y$, and $e : Y^{[0,1]} \to Y$ is evaluation at 0. A homotopy $h : A \times [0,1] \to Y$ can be interpreted as a map $h : A \to Y^{[0,1]}$. The map $f : A \to X$ is said to have HEP against the space Y if for any given map $g : X \to Y$ and a homotopy $h : A \times [0,1] \to Y$, such that $h|_{A \times \{0\}} = g \circ f$, there exists a homotopy $H : X \times [0,1] \to Y$ which makes the diagram commutative. One then defines $f : A \to X$ to be a *(Hurewicz) cofibration* if it has HEP against all space Y.

Turns out any cofibration $f : A \to X$ is actually injective, and moreover, f is a homeomorphism onto its image. Thus, we are not losing any generality with our definition.

Let us now see a prototypical example of a cofibration!

Example 2.11: ($S^{n-1} \hookrightarrow D^n$ *is a Cofibration*)

Consider the inclusion $\iota : S^{n-1} \hookrightarrow D^n$ of the unit $(n-1)$-sphere S^{n-1} as the boundary of the closed unit disc D^n in $\mathbb{R}^n$. We claim that this has HEP against any space, and hence, is a cofibration. Suppose $f : D^n \to Y$ is a map, and $h : S^{n-1} \times [0,1] \to Y$ is a homotopy of $f|_{S^{n-1}}$. Consider the subspace

$$X := S^{n-1} \times [0,1] \cup D^n \times \{0\} \subset D^n \times [0,1].$$

Clearly, we are given a map $F : X \to Y$, which we intend to extend to $D^n \times [0,1]$. One can imagine X as a hollowed out cylinder (with the bottom intact, basically like a bucket). One can define a *retraction* $r : D^n \times [0,1] \to X$ as follows. Consider the point $z = \mathbf{0} \times \{2\}$, the center of the disc $D^n \times \{2\}$. Given any point p in the solid cylinder $D^n \times [0,1]$, draw a straight line from z which passes through p, and then record the (unique) point q where this line hits the bucket X. Define $r : D^n \times [0,1] \to X$ which maps $p \mapsto q$. It is easy to see that r is continuous, and moreover, r is identity when restricted to the subspace X. Define the homotopy $D^n \times [0,1] \to Y$ as the composition $D^n \times [0,1] \xrightarrow{r} X \xrightarrow{F} Y$. Since r is identity on X, this composition is indeed an extension of F. Thus, $\iota : S^{n-1} \hookrightarrow D^n$ is a cofibration.

Exercise 2.12:

The construction of retraction map in Example 2.11 is also known as the *orthographic projection.* Figure out an explicit formula for the map!