Course notes for

Algebraic Topology I (KSM3E02)

Instructor: Aritra Bhowmick

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`quotient topology — homotopy — homotopy equivalence`

Unless otherwise mentioned, throughout this course, a space means a topological space and a map $f : X \to Y$ between spaces is always assumed to be continuous.

1.1 A Primer on Quotient Topology

Let us recall few important concepts from point-set topology that crops up in algebraic topology all the time!

Definition 1.1: (*Quotient Topology*)

Given a space X, a set Y, and set map $q : X \to Y$, the *quotient topology* on Y induced by q is the *finest* topology (i.e, the *largest* topology) on Y such that q becomes continuous. We say q is a quotient map.

Explicitly, the quotient topology on Y induced by $q : X \to Y$ is given as

$$\{U \subset Y \mid q^{-1}(U) \subset X \text{ is open}\}.$$

It is easy to verify that the quotient topology as defined above is indeed a topology. In order to understand the definition, note that if we give Y the indiscrete topology then any map $f : X \to Y$ is continuous; on the other hand, if Y has a finer topology, a map into it may fail to be continuous. So, we aim for the finest possible topology on Y. A map $f : X \to Y$ is called a quotient map if the following holds:

$$U \subset Y \text{ is open if and only if } f^{-1}(U) \subset X \text{ is open.}$$

An important property of the quotient topology is the following.

Proposition 1.2: (*Universal Property of the Quotient Topology*)

Suppose $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ are given topological spaces, and $f : X \to Y$ is a set function. Then, the following are equivalent.

1. $\mathcal{T}_Y$ is the quotient topology induced by f.
2. $\mathcal{T}_Y$ is the unique topology having the following property: given any space Z and a set map $g : Y \to Z$, we have g is continuous if and only if $g \circ f$ is continuous.

Quotient topology crops up from equivalence relations. Given an equivalence relation $\sim$ on a space X, denote $X/_\sim$ as the set of equivalence classes. Then, we have a set map $q : X \to X/_\sim$ which maps each element to its equivalence class. We equip $X/_\sim$ with the quotient topology induced by this map. As a consequence, if we have a continuous map $f: X \to Z$ such that $a \sim b \Rightarrow f(a) = f(b)$ (i.e, f respects the equivalence relation), then there is an induced map $\tilde{f}: X/_{\sim} \to Z$, which is moreover continuous (Prove it!).

Example 1.3: (*Projective Spaces*)

On $X := \mathbb{R}^{n+1} \setminus \{0\}$ consider the equivalence relation: $\mathrm{u} \sim \mathrm{v}$ if and only if $\mathrm{u} = \lambda \mathrm{v}$ for some nonzero scalar $\lambda \in \mathbb{R}$. The induced quotient space, denoted as $\mathbb{RP}^n$, is known as the n^{th} real projective space. Can you see that $\mathbb{RP}^1$ is nothing but the circle S^1?

Let us now recall some constructions that one can perform on topological spaces, all of which are given quotient topology.

1.2 Why Algebraic Topology?!

In point-set topology, we encounter many properties of a space, and then verify which of them are invariant under homeomorphism. Then, this property can be used to distinguish between spaces. As an example, consider the intervals $[0, 1]$ and $[0, 1)$: they can be homeomorphic since one of them is compact, the other is not, and we know that homeomorphism preserves compactness. Now, if we consider the intervalse $(0, 1)$ and $[0, 1)$, then compactness is not good enough as none of them are compact. To distinguish them we can use the following trick.

> If possible, suppose $f : [0, 1) \to (0, 1)$ is a homeomorhpism. Denote $z = f(0)$. Observe that we have an induced map
>
> $$\tilde{f} : (0, 1) \to (0, 1) \setminus \{z\},$$
>
> which is also a homeomorphism. But this is impossible since deleting a point makes $(0, 1)$ disconnected, and we know that connectedness is preserved under homeomorphism.

This trick can be used to distinguish (up to homeomorphism) between $\mathbb{R}$ and $\mathbb{R}^2$ as well. But the trick fails when you try to distinguish between $\mathbb{R}^2$ and $\mathbb{R}^3$.

The point of algebraic topology is to weak the notion of homeomorphism, and yet be able to distinguish between a spaces. Indeed, we shall define the notion of *simply connectedness* and see that $\mathbb{R}^2$ with a point removed is simply connected, but $\mathbb{R}^3$ with a point removed is not.

More generally, we shall try to meaningfully attach algebraic objects (like groups, rings, chain complexes, etc.) to a space, which are (sometimes) computable. Then, using them we shall try to distinguish between spaces. The first step of course is to define the weakened notion of homeomorphism!

1.3 Homotopy and Homotopy Equivalence

Definition 1.4: (*Homotopy*)

Given maps $f, g : X \to Y$, a *homotopy* between them is a map $h : X \times [0, 1] \to Y$ such that

$$h(x, 0) = f(x), \qquad h(x, 1) = g(x), \qquad\qquad x \in X.$$

We denote this as $h : f \simeq g$, and say f, g are *homotopic* maps.

Intuitively, a homotopy is a continuous way to deform one map to another.

Proposition 1.5: (*Homtopy is an Equivalence Relation*)

Being homotopic is an equivalence relation on the set $C(X,Y)$ of continuous maps $X \to Y$.

__Proof__ **:** We explicitly check the properties.

- **Reflexivity:** Given any $f : X \to Y$, we have a homotopy $h : f \simeq f$ given by $h(x,t) = f(x)$ for $0 \leq t \leq 1$.
- **Symmetry:** Suppose $h : f \simeq g$ is as homotopy. Define the *reversed homotopy* as
$$\begin{aligned} k : X \times [0,1] &\to Y \\ (x,t) &\mapsto h(x,1-t). \end{aligned}$$
It is easy to see that $k : g \simeq f$.
- **Transitivity:** Suppose $\varphi : f \simeq g, \psi : g \simeq h$ are homotopies. Then, define the map
$$\begin{aligned} \zeta : X \times [0,1] &\to Y \\ (x,t) &\mapsto \begin{cases} \varphi(x,2t), & 0 \leq t \leq \frac{1}{2} \\ \psi(x,2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases} \end{aligned}$$
Via the *pasting lemma*, it follows that ζ is continuous, and clearly, $\zeta : f \simeq h$ is a homotopy.

Thus, we have $\simeq$ is an equivalence relation on $C(X,Y)$. $\square$

Definition 1.6: (*Homotopy Equivalence*)

A map $f : X \to Y$ is called a *homotopy equivalence* if there is a map $g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. We say that X and Y are *homotopy equivalent* as spaces, and denote it as $X \simeq Y$.

Intuitively, homotopy equivalence allows you to deform a space: think of a space as a rubber sheet; you are allowed to squish it, stretch it, but you cannot tear it! Explicitly writing down homotopy equivalence is not always feasible as we shall soon see.

Exercise 1.7: (*Homotopy Equivalence is an Equivalence Relation*)

Show that being homotopy equivalence is an equivalence relation between the class of all topological spaces.

__Hint__ **:** Use Exercise 2.1.