Topology Course Notes (KSM1C03)

Day 1 : 12th August, 2025

`basic set theory -- power set -- product of sets -- equivalence relation -- order relation`

1.1 Power set

Given a set X, the *power set* is defined as

$$\mathcal{P}(X) := \{A \mid A \subset X\}.$$

Exercise 1.1

If X is a finite set, prove via induction that $|\mathcal{P}(X)| = 2^{|X|}$, where $|\cdot|$ denotes the cardinality.

Exercise 1.2

For any arbitrary set X, prove that there exists a natural bijection of $\mathcal{P}(X)$ with the set

$$\mathcal{F} := \{f : X \to \{0,1\}\}$$

of all functions from X to the 2-point set $\{0,1\}$.

Hint

How many functions $\{a,b,c\} \to \{0,1\}$ can you define? Look at their inverse images.

Given two sets X, Y denote the set of all functions from X to Y as

$$Y^X := \{f : X \to Y\}.$$

Exercise 1.3

If X and Y are finite sets, then show that $\left|Y^X\right| = |Y|^{|X|}$. Use this to show $|\mathcal{P}(X)| = 2^{|X|}$.

Exercise 1.4: (Set exponential law)

Given three sets X, Y, Z, prove that there is a natural bijection

$$\left(Z^Y\right)^X = Z^{Y\times X}$$

Hint

Write down what the elements look like. Can you see the pattern? This bijection is also known as *Currying*.

1.2 Arbitrary union and intersection

Suppose $\mathcal{A}$ is a collection of sets. Then, we have the *union*

$$\bigcup_{X \in \mathcal{A}} X := \{x \mid x \in X \text{ for } \textbf{some } X \in \mathcal{A}\},$$

and the *intersection*

$$\bigcap_{X \in \mathcal{A}} X := \{x \mid x \in X \text{ for } \textbf{all } X \in \mathcal{A}\}.$$

Exercise 1.5: (Empty union)

Suppose we have an *empty* collection $\mathcal{A}$ of sets. From the definition, prove that

$$\bigcup_{X \in \mathcal{A}} X = \emptyset.$$

Exercise 1.6: (Empty intersection)

Suppose $\mathcal{A}$ is a *nonempty* subset of the power set of some fixed set X. Show that

$$\bigcap_{A \in \mathcal{A}} = \{x \in X \mid x \in A \text{ for all } A \in \mathcal{A}\}.$$

If $\mathcal{A} \subset \mathcal{P}(X)$ is the *empty* collection, justify

$$\bigcap_{A \in \mathcal{A}} A = X$$

1.3 Cartesian product

Given two sets A, B, their *Cartesian product* (or simply, *product*) is defined as the set

$$A \times B := \{(a, b) \mid a \in A, \quad b \in B\}$$

of ordered pairs. We have the two *projections*

$$\begin{aligned} \pi_A : A \times B &\to A \\ (a, b) &\mapsto a, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \pi_B : A \times B &\to B \\ (a, b) &\mapsto b. \end{aligned}$$

Exercise 1.7

Justify $A \times \emptyset = \emptyset$, where $\emptyset$ is the empty set.

Remark 1.8: (A different product?)

Suppose A, B are given. Consider the set

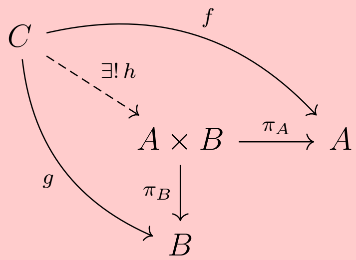
$$C = \{(a, b, a) \mid a \in A, \quad b \in B\}.$$

Clearly there is a natural bijection between C and $A \times B$. Also, we have maps $\pi_A : C \to A$ and $\pi_B : C \to B$.

Exercise 1.9: (Universal property of the product)

Suppose A, B are given sets, and $\pi_A : A \times B \to A, \pi_B : A \times B \to B$ be the projections.

a) Show that given any set C, and functions $f : C \to A, g : C \to B$, there exists a *unique* function $h : C \to A \times B$ such that the diagram commutes.

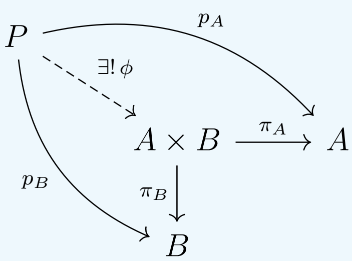


b) Suppose we are given a set P, along with two functions $p_A : P \to A$ and $p_B \to B$, which satisfies the following property : given any set C, and functions $f : C \to A, \ g : C \to B$, there exists a *unique* function $h : C \to P$ satisfying $f = p_A \circ h, \ g = p_B \circ h$.

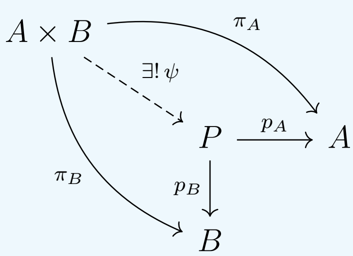
Show that the exists a bijection from $\psi : A \times B \to P$, such that $p_A \circ \psi = \pi_A$ and $p_B \circ \psi = \pi_B$.

Hint

Look at the diagrams



and



Can you show that $\phi \circ \psi = \mathrm{Id}_{A \times B}$ and $\psi \circ \phi = \mathrm{Id}_P$? The uniqueness should be useful.

1.4 Equivalence relation

Definition 1.10: (Relation)

Given a set X, a *relation* on it is a subset $\mathcal{R} \subset X \times X$. We say $\mathcal{R}$ is an *equivalence relation* if the following holds.

a) **(Reflexive)** For each $x \in X$ we have $(x, x) \in \mathcal{R}$.

b) **(Symmetric)** If $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$.

c) **(Transitive)** If $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$.

For any $x \in X$, the *equivalence class* (with respect to the equivalence relation $\mathcal{R}$) is defined as the set

$$[x] \coloneqq \{y \in X \mid (x, y) \in \mathcal{R}\} .$$

We shall denote $x \sim_{\mathcal{R}} y$ (sometimes also denoted $x\mathcal{R}y$, or simply $x \sim y$) whenever $(x, y) \in \mathcal{R}$. The collection of equivalence classes are sometimes denoted as $X/_\sim$.

Exercise 1.11

Given an equivalence relation $\mathcal{R}$ on X, check that any two equivalence classes are either disjoint or equal (i.e., they cannot have nontrivial intersection).

Exercise 1.12

Suppose X is a given set, and $A \subset X$ is a nonempty subset. Define the relation $\mathcal{R} \subset X \times X$ as follows.

$$\mathcal{R} \coloneqq \{(x, x) \mid x \in X \setminus A\} \bigcup \{(a, b) \mid a, b \in A\} .$$

a) Check that $\mathcal{R}$ is an equivalence relation.

b) Identify the equivalence classes. We shall denote the collection of equivalence classes as X/A.

c) What is X/X ?

Exercise 1.13

Suppose G is a group and H is a subgroup. Define a relation

$$\mathcal{C} \coloneqq \left\{(g_1, g_2) \,\middle|\, g_1^{-1} g_2 \in H\right\} \subset G \times G.$$

a) Show that $\mathcal{C}$ is an equivalence relation.

b) Identify the equivalence classes G/H.

Hint

Recall the definition of cosets.

Definition 1.14: Partition

Given a set X, a *partition of X* is a collection of subsets $X_\alpha \subset X$ for some indexing set $\alpha \in \mathcal{I}$, such that the following holds.

- $X_\alpha \cap X_\beta = \emptyset$ for any $\alpha, \beta \in \mathcal{I}$ with $\alpha \neq \beta$.
- $X = \bigcup_{\alpha \in \mathcal{I}} X_\alpha$.

Exercise 1.15: (Partitions and equivalence relations)

Given an equivalence relation $\mathcal{R}$ on a set X, show that the collection of equivalence classes is a partition of X. Conversely, given any partition of X, show that there exists a unique equivalence relation which gives that partition.

1.5 Order relation

Definition 1.16: (Linear order)

A relation $\mathcal{O} \subset X \times X$ on X is called an *order relation* (also known as *linear order* or *simple order*) if the following holds.

a) **(Non-reflexive)** $(x, x) \notin \mathcal{O}$ for all $x \in X$.

b) **(Transitive)** If $(x, y) \in \mathcal{O}$ and $(y, z) \in \mathcal{O}$, then $(x, z) \in \mathcal{O}$.

c) **(Comparable)** For $x, y \in X$ with $x \neq y$, either $(x, y) \in \mathcal{O}$ or $(y, x) \in \mathcal{O}$.

We shall denote $x <_{\mathcal{O}} y$ (or even simply $x < y$) whenever $(x, y) \in \mathcal{O}$. If either $x <_{\mathcal{O}} y$ or $x = y$ holds, then we shall denote $x \leq_{\mathcal{O}} y$ (or $x \leq y$). Given $x, y \in X$, we have the interval

$$(x, y) := \{z \in X \mid x < z \text{ and } z < x\}.$$

Exercise 1.17

Given an ordered set $(X, <)$, define the intervals $[x, y], [x, y), (x, y]$ for some $x, y \in X$. What happens when $x = y$?

Definition 1.18: (Order preserving function)

Given two ordered set $(X_1, <_1)$ and $(X_2, <_2)$, a function $f : X_1 \to X_2$ is said to *order preserving* if

$$x <_1 y \Rightarrow f(x) <_2 f(y), \quad \forall\, x, y \in X_1.$$

Definition 1.19: (Total order)

A relation $\mathcal{O} \subset X \times X$ on a set X is called a *total order* if the following holds.

a) **(Reflexive)** $(x, x) \in \mathcal{O}$ for all $x \in X$.

b) **(Transitive)** If $(x, y) \in \mathcal{O}$ and $(y, z) \in \mathcal{O}$, then $(x, z) \in \mathcal{O}$.

c) **(Total)** For $x, y \in X$ either $(x, y) \in \mathcal{O}$ or $(y, x) \in \mathcal{O}$

d) **(Antisymmetric)** If $(x, y) \in \mathcal{O}$ and $(y, x) \in \mathcal{O}$, then $x = y$.

We shall denote $x \leq_{\mathcal{O}} y$ (or even simply $x \leq y$) whenever $(x, y) \in \mathcal{O}$.

Definition 1.20: (Dictionary order)

Given X, Y two totally ordered sets the *dictionary order* (or *lexicographic order*) on the product $X \times Y$ is defined as

$$(x_1, y_1) < (x_2, y_2) \text{ if and only if } \{x_1 < x_2\} \text{ or } \{x_1 = x_2, \text{and} y_1 < y_2\},$$

for any $(x_1, y_1), (x_2, y_2) \in X \times Y$.

Exercise 1.21

Let X, Y be totally ordered sets.

a) Check that the dictionary order on $X \times Y$ is indeed a total ordering.

b) Check that the projection maps $\pi_X \to X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are order preserving maps.

c) Suppose Z is another totally ordered set. Let $f : Z \to X$ and $g : Z \to Y$ be two order preserving maps. Show that there exists a unique order preserving map $h : Z \to X \times Y$ such that $\pi_X \circ h = f$ and $\pi_Y \circ h = g$.

d) Let us define a new relation $(x_1, y_1) \preceq (x_2, y_2)$ if and only $x_1 \leq x_2$ and $y_1 \leq y_2$. Is $\preceq$ a total order on $X \times Y$?

Day 2 : 13th August, 2025

`metric space -- topological space -- basis -- subbasis`

2.1 Metric Spaces

Definition 2.1: (Metric space)

Given a set X, a *metric* on it is a map $d : X \times X \to [0, \infty)$ such that the following holds.

1) a. $d(x, x) = 0$ for all $x \in X$.

   b. If $x \neq y \in X$, then $d(x, y) > 0$.

2) $d(x, y) = d(y, x)$ for all $x, y \in X$

3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The tuple (X, d) is called a metric space. The open ball of radius r, centered at some $x \in X$ is denoted as

$$B_d(x, r) \coloneqq \{y \in X \mid d(x, y) < r\}.$$

Similarly, the closed ball is defined as

$$\bar{B}_d(x,r) := \{y \in X \mid d(x,y) \le r\}.$$

Definition 2.2: (Open set in metric space)

Given a metric space (X,d), a set $U \subset X$ is called open if

for all $x \in X$, there exists some $r > 0$, such that $B_d(x,r) \subset U$.

Exercise 2.3: (Properties of open sets)

From the definition, verify the following.

i) $\emptyset$ and X are open sets.

ii) Given any collection $\{U_\alpha \subset X\}$ of open sets, the union $\bigcup U_\alpha$ is open in X.

iii) Given a finite collection $\{U_1, \ldots, U_k\}$ of open sets, the intersection $\bigcap_{i=1}^k U_i$ is open in X.

Remark 2.4: (Which properties of metric are needed?!)

You should need $1a$ to show that $x \in B_d(x,r)$, and hence, X is open. You should need 3 to show that

$$B_d(x, \min\{r_1, r_2\}) \subset B_d(x,r_1) \cap B_d(x,r_2),$$

which is needed for the finite intersection.
In particular, $1b$ and 2 are not needed to verify the properties of open sets. Indeed, such general "metric" exists, known as pseud-metric and asymmetric metric.

2.2 Topological Spaces

Definition 2.5: (Topology)

Given a set X, a *topology* on X is a collection $\mathcal{T}$ of subsets of X (i.e., $\mathcal{T} \subset \mathcal{P}(X)$), such that the following holds.

a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

b) $\mathcal{T}$ is closed under arbitrary unions. That is, for any collection of elements $U_\alpha \in \mathcal{T}$ with $\alpha \in \mathcal{I}$, an indexing set, we have $\bigcup_{\alpha \in \mathcal{I}} U_\alpha \in \mathcal{T}$.

c) $\mathcal{T}$ is closed under finite intersections. That is, for any finite collection of elements $U_1, \ldots, U_n \in \mathcal{T}$, we have $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

The tuple $(X, \mathcal{T})$ is called a topological space.

Example 2.6

Given any set X we always have two standard topologies on it.

a) **(Discrete Topology)** $\mathcal{T}_0 = \mathcal{P}(X)$.

b) **(Indiscrete Topology)** $\mathcal{T}_1 = \{\emptyset, X\}$.

They are distinct whenever X has at least 2 points.

Exercise 2.7

Given any set X, verify that both the discrete and the indiscrete topologies are indeed topologies, that is, check that they satisfy the axioms.

Definition 2.8: (Metric topology)

Given a metric space (X, d), the collection of open sets in X form a topology, called the *metric topology* (or the *topology induced by the metric*).

Exercise 2.9: (Metric inducing discrete and indiscrete topology)

Given a set X, can you give a metric on it such that the induced topology on X is the discrete topology? Can you do the same for indiscrete topology?

Exercise 2.10: (Topologies on 3-point set)

Suppose $X = \{a, b, c\}$. Note that

$$|\mathcal{P}(\mathcal{P}(X))| = 2^{|\mathcal{P}(X)|} = 2^{2^{|X|}} = 2^{2^3} = 256.$$

Thus, there are 256 possible collections of subsets of X. How many of them are topologies? How many are distinct if you are allowed to permute the elements $\{a, b, c\}$?

Hint

The answers should be 29 and 9.

Definition 2.11: (Open and closed sets)

Given a topological space $(X, \mathcal{T})$, a subset $U \subset X$ is called an *open set* if $U \in \mathcal{T}$, and a subset $C \subset X$ is called a *closed set* if $X \setminus C \in \mathcal{T}$ (i.e., if $X \setminus C$ is open).

Caution 2.12

Given $(X, \mathcal{T})$, a subset can be both open and closed! Think about the discrete topology. Such subsets are sometimes called *clopen sets*.

Exercise 2.13: (Topology defined by closed sets)

Given X, suppose $\mathcal{C} \subset \mathcal{P}(X)$ is a collection of subsets that satisfy the following.

a) $\emptyset \in \mathcal{C},\ X \in \mathcal{C}$.

b) $\mathcal{C}$ is closed under arbitrary intersections.

c) $\mathcal{C}$ is closed under finite unions.

Define the collection,

$$\mathcal{T} := \{U \subset X \mid X \setminus U \in \mathcal{C}\}.$$

Prove that $\mathcal{T}$ is a topology on X.

Exercise 2.14

On the real line $\mathbb{R}$, consider the collection of subsets

$$\mathcal{T}_{\leftarrow} := \{\emptyset, \mathbb{R}\} \bigcup \{(-\infty, a) \mid a \in \mathbb{R}\}.$$

Show that $\mathcal{T}_{\leftarrow}$ is a topology on $\mathbb{R}$.

2.3 Basis of a topology

Definition 2.15: (Basis of a topology)

Given a topological space $(X, \mathcal{T})$, a *basis* for it is a sub-collection $\mathcal{B} \subset \mathcal{T}$ of open sets such that every open set $U \in \mathcal{T}$ can be written as the union of some elements of $\mathcal{B}$.

Example 2.16: (Usual topology on $\mathbb{R}$)

The collection of all open intervals $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}\}$ is a basis for the usual topology on the real line $\mathbb{R}$.

Proposition 2.17: (Necessary condition for basis)

Suppose $(X, \mathcal{T})$ is a topological space, and consider a basis $\mathcal{B} \subset \mathcal{T}$. Then, the following holds.

(B1) For any $x \in X$, there exists some $U \in \mathcal{B}$ such that $x \in U$.

(B2) For any $U, V \in \mathcal{B}$ and any element $x \in U \cap V$, there exists some $W \in \mathcal{B}$ such that $x \in W \subset U \cap V$.

Proof

Suppose $\mathcal{B}$ is a basis of $(X, \mathcal{T})$. Since X is open in X, we have $X = \bigcup_{O \in \mathcal{B}} O$, which implies (B1). Now, for any $U, V \in \mathcal{B}$, we have $U \cap V$ is open as well. Thus, $U \cap V$ is the union of some elements of $\mathcal{B}$, which implies (B2). □

Example 2.18

Consider the collection

$$\mathcal{B} = \{(a, \infty) \mid a \in \mathbb{R}\}.$$

This is a subcollection of open sets of $\mathbb{R}$ (in the usual topology), and moreover, $\mathcal{B}$ satisfies both $B1$ and $B2$ (Check!). But $\mathcal{B}$ is *not* a basis for the usual topology on $\mathbb{R}$. Thus, $B1$ and $B2$ is not a sufficient condition for $\mathcal{B}$ to be a basis.

Exercise 2.19: (Topology generated by a basis)

Suppose $\mathcal{B} \subset \mathcal{P}(X)$ is a collection of subsets of X satisfying (B1) and (B2). Consider $\mathcal{T}$ to be the collection of all possible unions of elements of $\mathcal{B}$. Show that $\mathcal{T}$ is a topology on X and $\mathcal{B}$ is a basis for it.

Exercise 2.20: (Basis for metric topology)

Suppose (X, d) is a metric space. Consider the collection

$$\mathcal{B} := \{B_r(x) \mid x \in X,\ r > 0\},$$

where $B_r(x) := \{y \mid d(x, y) < r\}$ is the ball of radius r, centered at x. Show that $\mathcal{B}$ is a basis for a topology on X, known as the *metric topology* induced by the metric d.

Exercise 2.21: (Closed discs generate discrete topology)

Let (X, d) be a metric space, and $\bar{B}_r(x) = \{y \in X \mid d(x, y) \le r\}$ be the *closed* ball of radius r centered at x. Show that the collection

$$\mathcal{B} := \left\{\bar{B}_r(x) \;\middle|\; x \in X, r \ge 0\right\}$$

is a basis for the discrete topology on X.

Exercise 2.22: (Usual topology on $\mathbb{R}^2$)

Consider the following collections of subsets of the plane $\mathbb{R}^2$.

a) $\mathcal{B}_1$ be the collection of all open discs with all possible radii and center at any point.

b) $\mathcal{B}_2$ be the collection of all open discs with radius less than 1, and center at any point.

c) $\mathcal{B}_3$ be the collection of all open squares (i.e, only the insides, not the boundary) with sides parallel to the two axes.

Show that all three are bases for the usual topology on $\mathbb{R}^2$.

Hint

Draw pictures!

2.4 Subbasis of a topology

Definition 2.23: (Subbasis of a topology)

Given a topological space $(X, \mathcal{T})$, a *subbasis* is a collection of subsets $\mathcal{S} \subset \mathcal{T}$ such that $\mathcal{T}$ is the smallest topology on X containing $\mathcal{S}$.

Proposition 2.24: (Topology generated by subbasis)

Let X be a set, and $\mathcal{S}$ be any collection of subsets of X (i.e, $\mathcal{S} \subset \mathcal{P}(X)$). Then, $\mathcal{S}$ is a subbasis for a (unique) topology on X (called the *topology generated* $\mathcal{S}$).

Proof

Consider the collection

$$\mathfrak{T} := \{\mathcal{T} \subset \mathcal{P}(X) \mid \mathcal{T} \text{ is a topology and } \mathcal{S} \subset \mathcal{T}\} .$$

Note that it is a nonempty collection, as $\mathcal{P}(X) \in \mathfrak{T}$. Denote $\mathcal{T}_0 = \bigcap_{\mathcal{T} \in \mathfrak{T}} \mathcal{T}$. Then $\mathcal{T}_0$ is a topology, and by definition, it is the smallest one containing $\mathcal{S}$. ☐

Explicitly, an open set of the topology generated by a subbasis $\mathcal{S}$ can be (non-uniquely) written as an arbitrary union of finite intersections of elements of $\mathcal{S}$.

Exercise 2.25: (Trivial subbases)

Given any set X, figure out the topologies generated by the following sub-bases :

$$\mathcal{S}_1 = \emptyset, \quad \mathcal{S}_2 = \{\emptyset\}, \quad \mathcal{S}_3 = \{X\}, \quad \mathcal{S}_4 = \{\emptyset, X\} .$$

Exercise 2.26

Given the plane $\mathbb{R}^2$ consider the collection

$$\mathcal{S} := \left\{B_1(x) \,\middle|\, x \in \mathbb{R}^2\right\},$$

where $B_1(x)$ is the unit open disc centered at x. Show that

a) $\mathcal{S}$ is not a basis for any topology on $\mathbb{R}^2$, but

b) the topology generated by $\mathcal{S}$ is the usual metric topology.

Hint

Place 4 unit discs with centers at the four corners of a square, with side length strictly less than 2. Look at the intersection!

2.5 Fine and coarse topology

Definition 2.27: (Fine and coarse topology)

Given two topologies $\mathcal{T}_1, \mathcal{T}_2$ on a set X, we say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$ (and $\mathcal{T}_2$ is said to be *coarser* than $\mathcal{T}_1$) if $\mathcal{T}_1 \supset \mathcal{T}_2$.

Caution 2.28

One way to remember the terminology is to think of each open set as small pebbles. If you crush each pebble in to finer pebbles, then you get more of it! Thus, the finer collection is larger (has more open sets), and the coarser collection is smaller (has less number of open sets).

Exercise 2.29

Check that the discrete topology on a set X is the finest, i.e., finer than any other topology that can be given on X. Dually, the indiscrete topology is the coarsest topology.

Caution 2.30

Not all topologies on a set are comparable to each other! Can you construct such examples on $\{a, b, c\}$?

Exercise 2.31

Show that the lower limit topology $\mathbb{R}_l$ is stictly finer than the usual topology on $\mathbb{R}$.

Day 3 : 14th August, 2025

`closure -- interior -- boundary -- subspaces -- continuous function`

3.1 Limit points and closure

Definition 3.1: (Limit point)

Given a space X and a subset $A \subset X$, a point $x \in X$ is called a *limit point* (or *cluster point*, or *point of accumulation*) of A if for any open set $U \subset X$, with $x \in U$, we have $A \cap U$ contains a point other than x.

Exercise 3.2

Show that if A is a closed set of X, then A contains all of its limit points. Give an example of a space X and a subset $A \subset X$, such that

a) there is a limit point x of A which is not an element of A, and

b) there is an element $a \in A$ which is not a limit point of A.

Definition 3.3: (Adherent and isolated points)

Given a subset $A \subset X$, a point $x \in X$ is called an *adherent point* (or *points of closure*) if every open neighborhood of x intersects A. An adherent point which is *not* a limit point is called an *isolated point* of A (which is then necessarily an element of A).

Definition 3.4: (Closure of a set)

Given $A \subset X$, the *closure* of A, denoted $\bar{A}$ (or $\mathrm{cl}\, A$), is the smallest closed set of X that contains A.

Exercise 3.5

Show that $A \subset X$ is closed if and only if $A = \bar{A}$.

Exercise 3.6

For any $A \subset X$, show that $\bar{A}$ is the intersection of all closed sets of X containing A. In particular, $A \subset \bar{A}$.

Proposition 3.7

Given $A \subset X$, we have

$$\bar{A} = \{x \in X \mid x \text{ is an adherent point of } A\}.$$

Proof

Suppose $x \in X$ is an adherent point of A. Let $C \subset X$ be a closed set containing A. If possibly, say $x \notin C \Rightarrow x \in X \setminus C$. Now, $X \setminus C$ is an open set, and $A \cap (X \setminus C) = \emptyset$. This contradicts that x is an adherent point of A. Thus, $x \in C$. Since C was arbitrary, we get $x \in \bar{A}$. Thus, $\bar{A}$ contains all the adherent points of A.

Conversely, suppose $x \in \bar{A}$. If possible, suppose x is not an adherent point of A. Then, there exists some open set U such that $x \in U$ and $U \cap A = \emptyset$. Now, $A \subset (X \setminus U)$, and $X \setminus U$ is a closed set. So, $\bar{A} \subset X \setminus U \Rightarrow \bar{A} \cap U = \emptyset$. This means, $x \notin \bar{A}$, a contradiction. Thus, x must be an adherent point of A. This concludes the claim. □

Exercise 3.8

Suppose $A = \{x_n\} \subset \mathbb{R}$ is an infinite set.

a) If $x = \lim_n x_n$ exists, then show that x is a limit point of A.

b) If $x \in \mathbb{R}$ is a limit point of A, then show that there is a subsequence $\{x_{n_k}\}$ with $x = \lim_k x_{n_k}$.

Suppose,

$$x_n = \begin{cases} 1 - \frac{1}{k}, & n = 2k, \\ 2 + \frac{1}{k}, & n = 2k + 1. \end{cases}$$

What are the limit points of $A = \{x_n \mid n \in \mathbb{N}\}$?

Definition 3.9: (Locally finite)

Given any collection $\mathcal{A}$ of subsets of a space X, we say $\mathcal{A}$ is a *locally finite* collection if for each $x \in X$, there exists an open neighborhood $x \in U$, such that U intersects only finitely many subsets from $\mathcal{A}$

Proposition 3.10: (Closure of locally finite collection)

Suppose $\mathcal{A} = \{A_\alpha\}_{\alpha \in \mathcal{I}}$ is a locally finite collection of subsets of X. Then, $\overline{\bigcup_\alpha A_\alpha} = \bigcup_\alpha \overline{A_\alpha}$.

Proof

We only show $\overline{\bigcup_\alpha A_\alpha} \subset \bigcup_\alpha \overline{A_\alpha}$. If possible, suppose $x \in \overline{\bigcup_\alpha A_\alpha}$ and $x \notin \bigcup \overline{A_\alpha}$. By local finiteness, we have some open neighborhood U of x, which only intersects, say, $A_{\alpha_1}, \ldots, A_{\alpha_n} \in \mathcal{A}$ (the list can be empty as well). Now, consider the set $V = U \setminus \bigcup_{i=1}^n \overline{A_{\alpha_i}}$, which is open (check). Clearly $x \in V$. But $V \cap (\bigcup A_\alpha)$. This contradicts the fact that x is a closure point. □

3.2 Interior

Definition 3.11: (Interior of a set)

Given $A \subset X$, the *interior* of A, denoted $\mathring{A}$ (or $\operatorname{int} A$), is the largest open set contained in A. A point $x \in \mathring{A}$ is called an *interior point* of A.

Exercise 3.12: (Interior of open sets)

For any $A \subset X$ show that $\mathring{A}$ is the union of all open sets contained in A. In particular, show that $A \subset X$ is open if and only if $A = \mathring{A}$.

Exercise 3.13: (Interior point)

Given $A \subset X$, show that a point $x \in X$ is an interior point of A if and only if there exists some open set $U \subset X$ such that $x \in U \subset A$.

3.3 Boundary

Definition 3.14: (Boudary of a set)

Given $A \subset X$, the *boundary* of A, denoted ∂A (or $\operatorname{bd} A$), is defined as

$$\partial A = \bar{A} \cap \overline{(X \setminus A)}.$$

Clearly boundary of any set is always a closed set. Also, observe the following. Given any $A \subset X$, a point $x \in X$ can satisfy exactly one of the following.

a) There exists an open set U with $x \in U \subset A$ (whence x is an interior point of A).

b) There exists an open set U with $x \in U \subset X \setminus A$ (whence x is an interior point of $X \setminus A$).

c) For any open set U with $x \in U$, we have $U \cap A \neq \emptyset$ and $U \cap (X \setminus A) = \emptyset$ (whence x is a boundary point of A).

Exercise 3.15

Given $A \subset X$, show that

$$\partial A = \{x \in X \mid \text{for any } U \subset X \text{ open, with } x \in U\text{, we have } U \cap A \neq \emptyset \neq U \cap (X \setminus A)\}$$

Exercise 3.16

Find out the boundaries of A, when

a) $A = \{(x, y) \mid x^2 + y^2 < 1\} \subset \mathbb{R}^2$, and

b) $A = \{(x, y, z) \mid x^2 + y^2 < 1,\ z = 0\} \subset \mathbb{R}^3$.

Caution 3.17

The above exercise shows that our intuitive notion of boundary of a disc may be misleading! In order to justify our intuition that "the boundary of a disc is the circle", one needs to treat it as a 'manifold with boundary'.

3.4 Subspaces

Definition 3.18: (Subspace topology)

Given a topological space $(X, \mathcal{T})$ and a subset $A \subset X$, the *subspace topology* on A is defined as the collection

$$\mathcal{T}_A := \{U \subset A \mid U = A \cap O \text{ for some } O \in \mathcal{T}\}.$$

We say $(A, \mathcal{T}_A)$ is a subspace of $(X, \mathcal{T})$.

Exercise 3.19

Suppose $U \subset X$ is an open set. What are the open subsets of U in the subspace topology? What are the closed sets?

Proposition 3.20: (Closure in subspace)

Let $Y \subset X$ be a subspace. Then, a subset of Y is closed in Y if and only if it is the intersection of Y with a closed set of X. Consequently, for any $A \subset Y$, the closure of A in the subspace topology is given as $\bar{A}^Y = \bar{A} \cap Y$.

Proof

For any $C \subset Y$, we have

$$\begin{aligned} C \text{ is closed in } Y &\Leftrightarrow Y \setminus C \text{ is open in } Y \text{ (by definition of closed set)} \\ &\Leftrightarrow Y \setminus C = Y \cap U, \text{ for some } U \subset X \text{ open (by definition of subspace topology).} \end{aligned}$$

Then,

$$C = Y \setminus (Y \setminus C) = Y \setminus (Y \cap U) = Y \setminus U = Y \cap \underbrace{(X \setminus U)}_{\text{closed in } X}.$$

On the other hand, for any closed set $F \subset X$, we have

$$Y \setminus (Y \cap F) = Y \setminus F = Y \cap \underbrace{(X \setminus F)}_{\text{open in } X},$$

which implies $Y \setminus (Y \cap F)$ is open in F. But then $Y \cap F$ is closed in Y.

Now,

$$\bar{A}^Y = \bigcap_{\substack{C \subset Y \text{ closed} \\ A \subset C}} C = \bigcap_{\substack{C \subset X \text{ closed} \\ A \subset C}} (Y \cap C) = Y \cap \left(\bigcap_{\substack{C \subset X \text{ closed} \\ A \subset C}} C \right) = Y \cap \bar{A}.$$

This concludes the proof. □

Exercise 3.21: (Interior and subspace)

Prove or disprove : Let $Y \subset X$ be a subspace, and $A \subset Y$. Then, the interior of A in Y (with respect the subspace topology) is $\mathring{A} \cap Y$.

Exercise 3.22: (Metric topology and subspace)

Suppose (X, d) is a metric space. Given any $A \subset X$, show that d restricts to a metric on A. Show that the subspace topology on any $A \subset X$ is the same as the metric topology for the induced metric space (A, d).

3.5 Continuous function

Definition 3.23: (Continuous function)

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be *continuous* if $f^{-1}(U) \in \mathcal{T}_X$ for any $U \in \mathcal{T}_Y$ (i.e., pre-image of open sets are open).

Exercise 3.24: (Pre-image of closed set)

Show that $f : X \to Y$ is continuous if and only if preimage of closed sets of Y is closed in X.

Exercise 3.25: (Continuity of the identity)

Suppose X is equipped given topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Show that $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if and only if $\mathrm{Id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous.

Definition 3.26: (Open map)

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be *open* if $f(U) \in \mathcal{T}_Y$ for any $U \in \mathcal{T}_X$ (i.e, image of opens sets are open).

Exercise 3.27: (Openness of the identity)

Suppose X is equipped given topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Show that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ if and only if $\mathrm{Id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is open.

Exercise 3.28: (Openness of bijection)

Suppose $f : X \to Y$ is a bijection. Show that f is open if and only if f^{-1} is continuous.

Definition 3.29: (Homeomorphism)

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be a *homeomorphism* if the following holds.

a) f is bijective, with inverse $f^{-1} : Y \to X$.

b) f is continuous.

c) f is open (or equivalently, f^{-1} is continuous).

Exercise 3.30: (Continuous bijective map)

For $0 \leq t < 1$, consider $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Check that $f : [0, 1) \to \mathbb{R}^2$ is a continuous, injective map. Draw the image. Is it a homeomorphism onto the image (with the corresponding subspace topologies)?

Caution 3.31: (Invariance of domain)

In general, a continuous bijection need not be a homeomorphism. However, there is a special situation known as the *Invariance of domain*. Suppose $U \subset \mathbb{R}^n$ is an open set. Consider a continuous injective map $f : U \to \mathbb{R}^n$. Denote $V := f(U)$. Clearly, $f : U \to V$ is a continuous bijection.
It is a very important theorem in topology that states : V is open and $f : U \to V$ is a homeomorphism.

Definition 3.32: (Closed map)

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be *closed* if $f(C)$ is closed in Y for any closed set $C \subset X$.

Exercise 3.33: (Open and closed map)

Give examples of continuous maps which are :

a) open, but not closed,

b) closed, but not open,

c) neither open nor closed,

d) both open and closed.

Hint

Consider $f_1(x, y) = x$, $f_2(x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$, $f_3(x) = \sin(x)$, and $f_4(x) = x$.

Exercise 3.34: (Continuity is local)

Suppose $X = \bigcup U_\alpha$, for some open sets U_α. Show that $f : X \to Y$ is continuous if and only if $f|_{U_\alpha} \to Y$ is continuous for all α.

Theorem 3.35: (Pasting lemma)

Suppose $X = A \cup B$, for some closed sets $A, B \subset X$. Let $f : A \to Y, g : B \to Y$ be given continuous maps, such that $f(x) = g(x)$ for any $x \in A \cap B$. Then, there exists a (unique) continuous map $h : X \to Y$ such that $h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B. \end{cases}$

Proof

Clearly, h is a well-defined function, and it is uniquely defined. We show that h is continuous. Let $C \subset Y$ be a closed set. Then,

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C).$$

Now, $f^{-1}(C) \subset A$ and $g^{-1}(C) \subset B$ are closed sets (in the subspace topology). But then they are closed in X, since A, B are closed. Then, $h^{-1}(C)$ is closed. Since C was arbitrary, we have h is continuous. □

Exercise 3.36: (Pasting lemma for finite collection)

Suppose $X = \bigcup_{i=1}^n C_i$ for some closed sets $C_i \subset X$. Let $f_i : C_i \to Y$ be continuous functions such that

$$f_i(x) = f_j(x), \quad x \in C_i \cap C_j, \quad 1 \le i < j \le n.$$

Show that there exists a (unique) continuous function $h : X \to Y$ such that $h(x) = f_i(x)$ whenever $x \in C_i$.

Caution 3.37: (Pasting lemma for infinite collection)

Pasting lemma need not hold true for infinite collection! Consider X to be the integers $\mathbb{Z}$ equipped with the cofinite topology (i.e., open sets are either $\emptyset$ or complements of finite subsets). Check that $\{n\} \subset X$ is closed, and the inclusion map $\iota : X \hookrightarrow \mathbb{R}$ is continuous on each $\{n\}$. Finally, check that ι is not continuous itself.

Day 4 : 20th August, 2025

`product spaces`

4.1 Product space

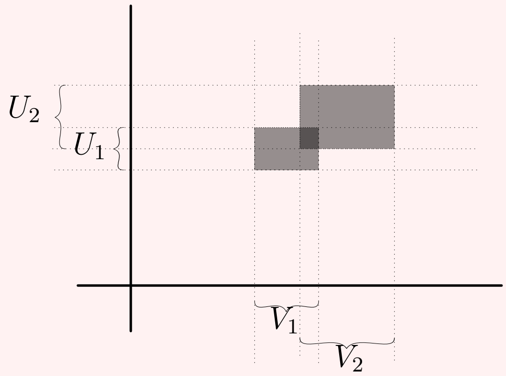
Definition 4.1: (Finite product)

Given $X_1, \ldots, X_n$, the *product space* is the Cartesian product $X = X_1 \times \cdots \times X_n$, equipped with the topology generated by the basis

$$\mathcal{B} := \{U_1 \times \cdots \times U_n \mid U_i \subset X_i \text{ is open for all } 1 \leq i \leq n\}.$$

Caution 4.2: (Product topology and basis)

Note that the product topology on $X \times Y$ is *generated by the basis* $\{U \times V \mid U \subset X, V \subset Y \text{ are open}\}$. In particular, not all open sets look like a product.



An open set $(U_1 \times V_1) \cup (U_2 \cup V_2)$

Exercise 4.3: (Finite product induced by projection)

Show that the product topology on $X := X_1 \times \cdots \times X_n$ is induced by the collection of projection maps $\{\pi_i : X \to X_i\}_{i=1}^n$.

Motivated by this, let us define the product of arbitrary many spaces.

Definition 4.4: (Product topology)

Let $\{X_\alpha\}_{\alpha \in \mathcal{I}}$ be an arbitrary collection topological spaces, indexed by the set $\mathcal{I}$. Denote the product as the set of tuples

$$X := \Pi_{\alpha \in \mathcal{I}} X_\alpha = \{(x_\alpha) \mid x_\alpha \in X_\alpha \text{ for all } \alpha \in \mathcal{I}\}.$$

Then, the *product topology* (or the *Tychonoff topology*) on X is defined as the topology induced by the collection of projection maps $\{\pi_\alpha : X \to X_\alpha\}_{\alpha \in \mathcal{I}}$

Proposition 4.5: (Product topology basis)

The product topology is generated by the basis

$$\mathcal{B} := \{\Pi_\alpha U_\alpha \mid U_\alpha \subseteq X_\alpha \text{ is open, and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in \mathcal{I}\}.$$

Proof

It is easy to see that $\mathcal{B}$ is a basis. Indeed, elements of $\mathcal{B}$ are of the form

$$\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_k}^{-1}(U_{\alpha_k}),$$

for some open sets $U_{\alpha_1} \subset X_{\alpha_1}, \ldots, U_{\alpha_k} \subset X_{\alpha_k}$. The claim follows. □

Definition 4.6: (Box topology)

Given a collection $\{X_\alpha\}$ of spaces, the *box topology* on $X = \Pi X_\alpha$ is generated by the subbasis

$$\mathcal{S} := \{\Pi U_\alpha \mid U_\alpha \subset X_\alpha \text{ is open}\}.$$

Clearly, the box topology is *finer* than the product topology. In particular, the projection maps are continuous with respect to the box topology as well.

Exercise 4.7: (Box and product topology)

Show that for a finite product $X_1 \times \cdots \times X_n$ of spaces, the box and the product topology agree. Moreover, show that for an infinite product, the box topology is always strictly finer than the product topology.

Caution 4.8: (Product topology always means Tychonoff topology)

Unless explicitly mentioned, always assume that a product space is given the Tychonoff topology. The box topology is usually too fine (i.e, has too many open sets), and is useful in constructing counter-examples.

Theorem 4.9: (Universal property of the product topology)

Let $\{X_\alpha\}_{\alpha\in\mathcal{I}}$ be a collection of topological spaces. For a space $(Z, \mathcal{T})$, and a collection of continuous maps $g_\alpha : Z \to X_\alpha$, consider the following property.

P(Z, g_α) : Given a space Y and any collection of continuous maps $f_\alpha : Y \to X_\alpha$, there exists a unique continuous map $h : Y \to Z$, such that $f_\alpha = g_\alpha \circ h$.

Then, the following holds.

a) The product space $X = \Pi X_\alpha$ with the product topology, and the projection maps $\pi_\alpha : X \to X_\alpha$ satisfies the property P(X, π_α)

b) If (Z, g_α) is any other tuple satisfying the property P(Z, g_α), then there is a homeomorphism $\Phi : Z \to X$ such that $\pi_\alpha \circ \Phi = g_\alpha$

Proof

Given any $f_\alpha : Y \to X_\alpha$, define $h : Y \to X = \Pi X_\alpha$ by

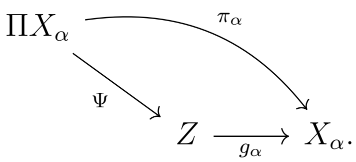
$$h(y) = (f_\alpha(y)),$$

which clearly satisfies $\pi_\alpha \circ h = f_\alpha$, and hence, is unique. Let us show h is continuous. We only need to check continuity for subbasic open sets, which are of the form $\pi_{\alpha_0}^{-1}(U)$ for some $U \subset X_{\alpha_0}$ open. Now,

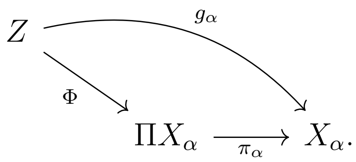
$$h^{-1}\left(\pi_{\alpha_0}(U)\right) = \left(\pi_{\alpha_0} \circ h\right)^{-1}(U) = f_{\alpha_0}^{-1}(U),$$

which is open as f_{α_0} is continuous. Thus, the property $\mathsf{P}(X, \pi_\alpha)$ holds.

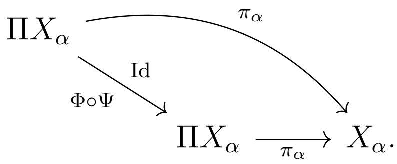
The second part is a standard diagram chasing argument. Suppose (Z, γ_α) is a tuple satisfying $\mathsf{P}(Z, \gamma_\alpha)$. Then, consider the collection of commutative diagrams.



The existence of (unique) $\Psi : \Pi X_\alpha \to Z$ is justified by $\mathsf{P}(Z, g_\alpha)$. Next, consider the collection of commutative diagrams



Again, existence of (unique) Φ is justified by $\mathsf{P}(\Pi X_\alpha, \pi_\alpha)$. Now, consider the following case.



Let us observe that

$$\pi_\alpha \circ (\Phi \circ \Psi) = (\pi_\alpha \circ \Phi) \circ \Psi = g_\alpha \circ \Psi = \pi_\alpha,$$

which follows from the previous two diagrams. Also, clearly

$$\pi_\alpha \circ \mathrm{Id} = \pi_\alpha.$$

Hence, by the *uniqueness* in $\mathsf{P}(\Pi X_\alpha, \pi_\alpha)$, we must have $\Phi \circ \Psi = \mathrm{Id}_{\Pi X_\alpha}$. By a similar argument, we get $\Psi \circ \Phi = \mathrm{Id}_Z$. Hence, Φ is a homeomorphism, with inverse given by Ψ. $\square$

Exercise 4.10: (Map into box topology)

Suppose $X = \mathbb{R}^{\mathbb{N}}$, equipped with the box topology. Show that the map $f : \mathbb{R} \to X$ defined by $f(t) = (t, t, \ldots)$ is not continuous.

Hint

Consider the open set $U = \Pi(-\frac{1}{n}, \frac{1}{n}) = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \cdots \subset X$.

Day 5 : 21th August, 2025

`Hausdorff axiom -- ` T_2, T_1, T_0 ` -- convergence of sequence -- sequential continuity -- quotient space`

5.1 Hausdorff Axiom

Definition 5.1: (Hausdorff space)

A space X is called *Hausdorff* (or a *T_2-space*) if for any $x, y \in X$ with $x \neq y$, there exists open neighborhoods $x \in U_x \subset X, y \in U_y \subset X$, such that $U_x \cap U_y = \emptyset$. In other words, any two points of a Hausdorff space can be separated by open sets.

Exercise 5.2: (Product of T_2-spaces)

Suppose $\{X_\alpha\}$ is a collection of T_2-spaces. Show that $X = \Pi X_\alpha$ is T_2 with respect to the product topology (and hence, with respect to the box topology as well).

Being Hausdorff is a very desirable property of a space.

Exercise 5.3: (Metric spaces are Hausdorff)

If (X, d) is a metric space, then show that the metric topology is Hausdorff.

Proposition 5.4: (Points are closed in Hausdorff space)

Suppose X is a Hausdorff space. Then, $\{x\}$ is a closed subset of X for any $x \in X$.

Proof

Suppose $y \neq x$. Then, by Hausdorff property, we have some open sets U and V such that $x \in U, y \in V$ and $U \cap V = \emptyset$. In particular, y is not a closure point of $\{x\}$. Thus, $\{x\}$ is closed. □

Note that in the proof, the full strength of the Hausdorff property is not used.

Definition 5.5: (T_1 space)

A space X is called a *T_1-space* (or a *Fréchet space*) if for any $x \in X$, the subset $\{x\}$ is a closed set.

Exercise 5.6: (T_1 but not T_2 space)

Given an example of a space X which is T_1 but not T_2.

Exercise 5.7: (T_1-space equivalent definition)

Let X be a space. Show that the following are equivalent.

a) X is a T_1 space.

b) For any $x, y \in X$ with $x \neq y$, there exists open neighborhoods $x \in U_x \subset X$ and $y \in U_y \subset X$ such that $y \notin U_x$ and $x \notin U_y$.

c) Any $A \subset X$ is the intersection of all open sets containing A.

d) For any $A \subset X$ and $x \in X$, we have x is a limit point of A if and only every open neighborhood of x contains infinitely many points of A. (What happens when X is finite?!)

Definition 5.8: (T_0-space)

A space X is called a *T_0-space* (or a *Kolmogorov space*) if for any two points $x \neq y \in X$, there exists an open set $U \subset X$ which contains exactly one of x and y.

Remark 5.9: (Topolgoically distinguishable and separable)

Suppose $x, y \in X$ are two points. Note the following hierarchy.

- **(Distinct)** If $x \neq y$, we say x, y are distinct.
- **(Topologically distinguishable)** If there is at least one open set that contains exactly one of x and y, we say x, y are topologically distinguishable.
- **(Separable)** If there are two neighborhoods U_x, U_y of x, y respectively, which does not contain the other, we say x, y are topologically separable.
- **(Separated by opens)** If there are two neighborhoods U_x, U_y of x, y respectively, such that $U_x \cap U_y = \emptyset$, we say x, y are separated by open sets.

Later, we shall see how this continues to points and closed sets as well.

Exercise 5.10: (T_0 but not T_1 space)

Given an example of a space X which is T_0 but not T_1. What about

Exercise 5.11: (Zariski topology)

Suppose $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Give it the topology $\mathcal{T} = \{\emptyset, \mathbb{F}^\times, \mathbb{F}\}$, where $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$. Consider the family of polynomial functions $\mathcal{F} := \{p : \mathbb{F}^n \to \mathbb{F}\}$. The topology induced by $\mathcal{F}$ on $\mathbb{F}^n$ is known as the *Zariski topology*. Determine whether it is T_0, T_1 or T_2.

5.2 Convergence of sequence

Definition 5.12: (Convergence of sequence)

Suppose $\{x_n\}_{n\geq 1}$ is a sequence of points in a space X (i.e, $x : \mathbb{N} \to X$ is a function). We say $\{x_n\}$ converges to a limit $x \in X$ if for any open neighborhood U of x, there is a natural number N_U such that $x_n \in U$ for all $n \geq N_U$.

Exercise 5.13: (Convergence in metric)

Check that the notion of convergence in a metric space is equivalent to the usual notion (i.e, $x_n \to x$ if and only if $d(x_n, x) \to 0$). In particular, they are the same from real analysis.

Example 5.14

Suppose X is an indiscrete space, with at least two distinct points $x, y \in X$. Consider the sequence

$$x_n = \begin{cases} x, & n \text{ is odd}, \\ y, & n \text{ is even}. \end{cases}$$

Observe that the sequence converges to both x and y. In fact, any sequence in X converges to every point in the space X. Note that an indiscrete space is not even T_0.

Example 5.15

Suppose $X = \{0, 1\}$, with topology $\mathcal{T} = \{\emptyset, \{0\}, \{0, 1\}\}$. This space $(X, \mathcal{T})$ is known as *Sierpiński space*. Clearly it is T_0, but not T_1 since $\{0\}$ is not closed. Now, consider the sequence $x_n = 0$ for all $n \geq 1$. Then, $\{x_n\}$ converges to both 0 and 1.

Proposition 5.16: (Convergence in T_2)

Suppose $\{x_n\}$ is a sequence in a T_2-space X. Then, $\{x_n\}$ can converge to at most one point in X.

Proof

If possible, suppose $\{x_n\}$ converges to distinct point $x \neq y$. By Hausdorff property, we have two open neighborhoods U_x, U_y of x, y respectively, such that $U_x \cap U_y = \emptyset$. We also have two natural numbers N_1, N_2 such that $x_n \in U_x$ for all $n \geq N_1$ and $x_n \in U_y$ for all $n \geq N_2$. Set $N = \max\{N_1, N_2\}$. Then,

$$x_n \in U_x \cap U_y, \quad \text{for all } n \geq N.$$

This is a contradiction. Thus, any sequence can converge to at most one point. □

5.3 Sequential Continuity

Definition 5.17: (Sequenttial continuity)

A function $f : X \to Y$ is said to be *sequentially continuous* if for any converging sequence $x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y.

Proposition 5.18: (Continuous functions are sequentially continuous)

Suppose $f : X \to Y$ is a continuous map. Then f is sequentially continuous.

Proof

Suppose $x_n \to x$ is a converging sequence in X. Let $f(x) \in U \subset Y$ be an arbitrary open neighborhood. Then, it follows from continuity of f that $f^{-1}(U) \subset X$ is open. Clearly $x \in f^{-1}(U)$. Hence, there is some $N \geq 1$ such that $x_n \in f^{-1}(U)$ for all $n \geq N$. This implies $f(x_n) \in U$ for all $n \geq N$. Since U was arbitrary, we see that $f(x_n) \to f(x)$. But this means f is sequentially convergent. □

Proposition 5.19: (Sequential continuity in metric spaces)

Suppose (X, d) is a metric space with the metric topology, and Y be any space. Then, any sequentially continuous map $f : X \to Y$ is a continuous map.

Proof

Let $U \subset Y$ be open. In order to show $f^{-1}(U) \subset X$ is open, we show that any $x \in f^{-1}(U)$ is an interior point of $f^{-1}(U)$. Consider the metric balls $B_n := B_d\left(x, \frac{1}{n}\right) \subset X$. If possible, suppose $B_n \not\subset f^{-1}(U)$ for any n. Pick points $x_n \in f^{-1}(U) \setminus B_n$, and observe that $x_n \to x$ (Check!). Then, we have $f(x_n) \to f(x)$. Since U is an open neighborhood of $f(x)$, we have some $N \geq 1$ such that $f(x_n) \in U$ for all $n \geq N$. But then $x_n \in f^{-1}(U)$ for $n \geq N$, which is a contradiction. Hence, we must have that for some $N_0 \geq 1$ the metric ball $B_{N_0} \subset f^{-1}(U)$. Thus, x is an interior point. Since x is arbitrary, we get $f^{-1}(U)$ is open. Consequently, f is continuous. □

Caution 5.20: (Sequential conitinuity may not imply continuity)

In general, sequential continuity may not imply continuity! Consider X to be a space equipped with the cocountable topology. Then, any convergent sequence in X is eventually constant. That is, if $x_n \to x$ in X, then for some $N \geq 1$, we have $x_n = x$ for all $n \geq N$. But then any function $f : X \to Y$ is sequentially continuous (Why?). Assume X is uncountable, so that the cocountable topology is not the discrete topology. Then, there are non-continuous maps on X. For example, consider $Y = X$ equipped with the discrete topology, and then look at the identity map $\mathrm{Id} : X \to Y$.

5.4 Quotient space

Definition 5.21: (Quotient map)

Given a space $(X, \mathcal{T})$ and a function $f : X \to Y$ to a set Y, the *quotient topology* on Y is defined as

$$\mathcal{T}_f := \left\{ U \mid f^{-1}(U) \in \mathcal{T} \right\}.$$

The map $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_f)$ is called a *quotient map*. In other words, f is a quotient map if $U \subset Y$ is open if and only if $f^{-1}(U) \subset X$ is open.

Proposition 5.22: (Quotient topology is topology)

The quotient topology $\mathcal{T}_f$ is indeed a topology on Y, and $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_f)$ is continuous.

Proof

We check the axioms.

i) $\emptyset \in \mathcal{T}_f$ since $\emptyset = f^{-1}(\emptyset) \in \mathcal{T}$.

ii) $Y \in \mathcal{T}_f$ since $X = f^{-1}(Y) \in \mathcal{T}$.

iii) For any collection $\{U_\alpha \in \mathcal{T}_f\}$, we have $f^{-1}(\bigcup U_\alpha) = \bigcup f^{-1}(U_\alpha) \in \mathcal{T}$. Thus, $\mathcal{T}_f$ is cloes under arbitrary union.

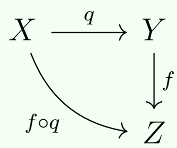
iv) For a finite collection $\{U_i\}_{i=1}^k$, we have $f^{-1}\left(\bigcap U_i\right) = \bigcap f^{-1}(U_i) \in \mathcal{T}$. Thus, $\mathcal{T}_f$ is closed under finite intersection.

Hence, $\mathcal{T}_f$ is a topology. By construction, f is then continuous. □

Theorem 5.23: (Universal property of quotient topology)

Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are given. Then, for any set function, $q : X \to Y$, the following are equivalent.

1. $\mathcal{T}_Y$ is the quotient topology induced by q (in other words, q is a quotient map).
2. $\mathcal{T}_Y$ is the finest (i.e, largest) topology for which q is continuous.
3. $\mathcal{T}_Y$ is the unique topology having the following property :



for any space $(Z, \mathcal{T}_Z)$ and any set map $f : Y \to Z$, we have f is continuous if and only if $f \circ q$ is continuous

Proof

Suppose q is a quotient map. If possible, there is some topology $\mathcal{S}_Y$ on Y such that $\mathcal{T}_Y \subsetneq \mathcal{S}_Y$ and $q : (X, \mathcal{T}_X) \to (Y, \mathcal{S}_Y)$ is continuous. Since $\mathcal{S}_Y$ is strictly finer than $\mathcal{T}_Y$, there is some set $U \in \mathcal{S}_Y \setminus \mathcal{T}_Y$. But then $q^{-1}(U) \in \mathcal{T}_X$, as q is continuous. This implies $U \in calT_Y$, a contradiction. Hence, the quotient topology is the finest topology on Y making q continuous.

Conversely, suppose $\mathcal{T}_Y$ is the finest topology so that q is continuous. Recall the quotient topology is

$$\mathcal{T}_q = \{U \mid q^{-1}(U) \in \mathcal{T}_X\}$$

Since q is continuous, for each $U \in \mathcal{T}_Y$ we have $q^{-1}(U) \in \mathcal{T}_X$. In particular, $\mathcal{T}_Y \subset \mathcal{T}_q$. Also, $q : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_q)$ is continuous. Since $\mathcal{T}_Y$ is the finest such topology, we must have $\mathcal{T}_Y = \mathcal{T}_q$.

Next, suppose $\mathcal{T}_Y$ is the quotient topology. Let us choose some space $(Z, \mathcal{T}_Z)$ and set map $f : Y \to Z$. If f is continuous, then we have $f^{-1}(U) \in \mathcal{T}_Y$ for all $U \in \mathcal{T}_Z$. Then,

$$(f \circ q)^{-1}(U) = q^{-1}\left(f^{-1}(U)\right) \in \mathcal{T}_X,$$

by the definition of quotient topology. Thus, $f \circ q$ is continuous. On the other hand, suppose $f \circ q$ is continuous. Then, for any $U \circ \mathcal{T}_Z$, we have $q^{-1}(f^{-1}(U)) \in \mathcal{T}_X$. But then again by the definition of quotient topology, we have $f^{-1}(U) \in \mathcal{T}_Y$, which shows that f is continuous. Thus, $\mathcal{T}_Y$ satisfies the property. If possible, suppose $\mathcal{S}_Y$ is another topology on Y satisfying the property. Let us take $Z = (Y, \mathcal{T}_Y)$ and $f = \mathrm{Id}_Y : (Y, \mathcal{S}_Y) \to (Y, \mathcal{T}_Y)$. Then, we have f is continuous if and only if $f \circ q$ is continuous. But, $f \circ q = q : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, which is continuous being the quotient map. Hence, f is continuous. This implies $\mathcal{T}_Y \subset \mathcal{S}_Y$. But $\mathcal{T}_Y$ is the finest topology for which q is continuous, and hence, $\mathcal{T}_Y = \mathcal{S}_Y$. This proves the uniqueness.

Finally, suppose $\mathcal{T}_Y$ is the unique topology satisfying the property above. We show that the quotient topology $\mathcal{T}_q$ satisfies the property. Suppose $(Z, \mathcal{T}_Z)$ is some space, and $f : Y \to Z$ is a set map. If $f : (Y, \mathcal{T}_q) \to (Z, \mathcal{T}_Z)$ is continuous, then for any $U \in \mathcal{T}_Z$ we have

$$(f \circ q)^{-1}(U) = q^{-1}(f^{-1}(U)) \in \mathcal{T}_X,$$

since $f^{-1}(U) \in \mathcal{T}_q$. On the other hand, if $f \circ q$ is continuous, then for any $U \in \mathcal{T}_Z$ we have $q^{-1}(f^{-1}(U)) \in \mathcal{T}_X$, which implies, $f^{-1}(U) \in \mathcal{T}_q$. Thus, f is continuous. In particular, $\mathcal{T}_q$ satisfies the property, and hence, $\mathcal{T}_Y$ is the quotient topology by uniqueness.

This concludes the proof. □

Remark 5.24: (Quotient map and surjectivity)

Suppose $f : X \to Y$ is a quotient map. Assume that f is *not* surjective. Then, for any $y \in Y \setminus f(X)$ we have $f^{-1}(y) = \emptyset \subset X$ open, and hence, $\{y\}$ is open in Y. In other words, $Y \setminus f(X)$ has the discrete topology. Also, $f(X) \subset Y$ is both an open and closed set. Hence, the open and closed sets of $f(X)$ in the subspace topology are precisely the same in the actual (quotient) topology on Y. For these reasons, we can (and usually we do) assume that a quotient map is surjective.

Remark 5.25: (Surjective map and equivalence relation)

Suppose $f : X \to Y$ is a surjective map. Then, the collection $\bigsqcup_{y \in Y} f^{-1}(y)$ is a partition on X, and hence, induces an equivalence relation. Indeed, we can define $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$. Conversely, given any equivalence relation $\sim$ on X, we see that $q : X \to X/_\sim$, is a surjective map, where $X/_\sim$ is the collection of all equivalence classes under the relation $\sim$.

Given a set map $f : X \to Y$, a subset $S \subset X$ is called *saturated* (or *f-saturated*) if $S = f^{-1}(f(S))$ holds.

Exercise 5.26: (Saturated open set)

Given a quotient map $q : X \to Y$, a set $U \subset X$ is q-saturated if and only if it is the union of the equivalence classes of its elements (i.e, $U = \bigcup_{x \in U} [x]$).

Definition 5.27: (Identification topology)

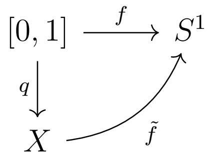
Given an equivalence relation $\sim$ on a space X, the *identification topology* on the set $Y = X/_\sim$ of all equivalence classes is the quotient topology induced by the map $q : X \to Y$, which sends $x \mapsto [x]$. The quotient map q is called the *identification map*.

Proposition 5.28: $[0,1]/0,1$ is S^1

Consider $\{0,1\} \subset [0,1]$, and let $X = [0,1]/_{\{0,1\}}$ be the identification space. Then, X is homeomorphic to the circle $S^1 := \{(x,y) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$.

Proof

Consider the map $f : [0,1] \to S^1$ given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Clearly, f is continuous and surjective. Also, $f(0) = (1,0) = f(1)$.



Passing to the quotient $X = [0,1]/\{0,1\}$, we get a map $\tilde{f} : X \to S^1$ defined by $\tilde{f}([x]) = f(x)$. It is easy to see that $\tilde{f}$ is well-defined, and hence, by the property of the quotient topology, $\tilde{f}$ is continuous. Now, $\tilde{f}$ is surjective (as f was), and moreover, it is injective.
In order to show $\tilde{f}$ is open, we consider the two cases.

i) Suppose $V \subset X$ is an open set, such that $[0] = [1] = \{0,1\} \notin V$. Then, $q^{-1}(V) \subset [0,1]$ is an open set, which is actually contained in $(0,1)$. In particular, $q^{-1}(V)$ is a union of open intervals. Observe that (by drawing picture or otherwise) f maps such open intervals to open arcs of S^1 (which are open in S^1). Then, $\tilde{f}(V) = f\left(q^{-1}(V)\right)$ is open.

ii) Suppose $V \subset X$ is an open set, such that $[0] = [1] = \{0,1\} \in V$. Then, $q^{-1}(V)$ is the union of open intervals of $(0,1)$, as well as, $[0,\epsilon_1) \cup (1-\epsilon_2, 1]$ for some $\epsilon_1, \epsilon_2 > 0$. We have already seen that any open intervals get mapped to open arcs. Also, $f\left([0,\epsilon_1) \cup (1-\epsilon_2, 2]\right)$ is another open arc in S^1 containing the point $(0,1)$. Thus, $\tilde{f}(V) = f\left(q^{-1}(V)\right)$ is open in S^1.

Hence, $\tilde{f} : X \to S^1$ is a homeomorphism. □

Exercise 5.29: ($\mathbb{R}/\mathbb{Z}$ is S^1)

Consider the quotient space $X = \mathbb{R}/\mathbb{Z}$, where the equivalence relation is given as $a \sim b$ if and only $a - b \in \mathbb{Z}$. Show that X is homeomorphic to the circle S^1.

Day 6 : 27th August, 2025

`connectedness -- components`

6.1 Connectedness

Definition 6.1: (Connected space)

A space X is called *connected* if the only clopen sets (i.e., simultaneously open and closed sets) of X are $\emptyset$ and X itself. If there is a nontrivial clopen set $\emptyset \subsetneq U \subsetneq X$, then X is called *disconnected*.

Proposition 6.2: (Disconnected space)

For a space X, the following are equivalent.

1) X is disconnected.

2) X can be written as the disjoint union of two open sets $X = U \sqcup V$, such that, $\emptyset \subsetneq U \subsetneq X$ and $\emptyset \subsetneq V \subsetneq X$.

3) X can be written as the disjoint union of two closed sets $X = F \sqcup G$, such that, $\emptyset \subsetneq F \subsetneq X$ and $\emptyset \subsetneq G \subsetneq X$.

4) There is a surjective continuous map $X \to \{0,1\}$, where $\{0,1\}$ is given the discrete topology.

Proof

The equivalence of $1, 2, 3$ follows from the definition. Suppose $f : X \to \{0,1\}$ is a surjective continuous map. Then, X can be written as the disjoint union $X = f^{-1}(0) \sqcup f^{-1}(1)$, each of which are non-trivial open sets. Conversely, if $X = U \sqcup V$ for some nontrivial open sets, then $f : X \to \{0,1\}$ defined by $f(U) = 0$ and $f(V) = 1$ is a surjective continuous map. □

Theorem 6.3: (Image of connected set)

Suppose $f : X \to Y$ is a continuous map. Then, for any connected $A \subset X$, we have $f(A) \subset Y$ is connected. In particular, if X is connected, then so is $f(X)$.

Proof

Suppose $f(A) \subset Y$ is disconnected. Then, there is a surjective continuous map $g : f(A) \to \{0,1\}$. But then, $h := g \circ f : A \to \{0,1\}$ is a surjective continuous map, a contradiction. Hence, $f(A)$ is connected. □

Definition 6.4: (Connected component)

Given $x \in X$, the *connected component* of X containing x is the largest possible connected subset containing x.

Proposition 6.5: (Existence of connected component)

Given $x \in X$, the connected component of X containing X is defined as the

$$\mathcal{C}(x) := \bigcup \{A \mid x \in A \subset X, A \text{ is connected}\}.$$

Proof

Observe that $\{x\}$ is a connected set, and hence, the family is non-empty. Let us check $\mathcal{C}(x)$ is connected. If not, then there exists open sets $U, V \subset X$ such that

- $\emptyset \subsetneq \mathcal{C}(x) \cap U \subsetneq \mathcal{C}(x)$,
- $\emptyset \subsetneq \mathcal{C}(x) \cap V \subsetneq \mathcal{C}(x)$, and
- $\mathcal{C}(x) = (\mathcal{C}(x) \cap U) \sqcup (\mathcal{C}(x) \cap V)$.

Now, for any connected set A containing x, we have

$$A = (A \cap U) \sqcup (A \cap V).$$

Then, both

$$\emptyset \subsetneq A \cap U \subsetneq A, \quad \text{and} \quad \emptyset \subsetneq A \cap V \subsetneq A$$

cannot appear simultaneously. Hence, either $A \subset U$ or $A \subset V$. Thus, we can define the two collections

$$\mathcal{U} := \{A \mid x \in A \subset X,\ A \text{ is connected},\ A \subset U\}, \mathcal{V} := \{A \mid x \in A \subset X,\ A \text{ is connected},\ A \subset V\}.$$

Since $x \in A$ for all such A, we must have either $\mathcal{U} = \emptyset$ or $\mathcal{V} = \emptyset$. Without loss of generality, assume $\mathcal{V} = \emptyset$. But then, $\mathcal{C}(x) \cap V = \emptyset$, a contradiction. Hence, $\mathcal{C}(x)$ is connected. By construction, it is the largest such connected set which contains x. Thus, $\mathcal{C}(x)$ is the connected component containing x. □

Exercise 6.6: (Hyperbola and axes)

Suppose

$$A = \{(x, y) \mid xy = 1\} \cup \{(x, y) \mid xy = 0\} \subset \mathbb{R}.$$

Show that A has three connected components.

Theorem 6.7: (Closure is connected)

If $A \subset X$ is a connected set, then for any subset B satisfying $A \subset B \subset \bar{A}$, we have B is connected. In particular, $\bar{A}$ is connected.

Proof

Suppose, we have $B = U \sqcup V$ for some open sets $\emptyset \subsetneq U, V \subsetneq B$. Since $A \subset B$, we have $A \subset U$ or $A \subset V$ (otherwise, $A = (A \cap U) \sqcup (A \cap V)$ will be a separation of A). Without loss of generality, say, $A \subset U \Rightarrow \bar{A}^B \subset \bar{U}^B$. Now, $U \subset B$ is closed (in B), as $B \setminus U = V$ is open (in B). In particular, $\bar{U}^B = U$. On the other hand, $\bar{A}^B = \bar{A} \cap B \supset B \Rightarrow B \subset \bar{A}^B \subset \bar{U}^B = U$. This contradicts that $\emptyset \subsetneq V \subsetneq B$. Hence, B is connected. □

Example 6.8: (Discrete space)

In a discrete space X, every singleton $\{x\}$ is a connected component. Any subset with at least two elements is then disconnected.

Definition 6.9: (Totally disconnected space)

A space X is called *totally disconnected* if the only connected components of x are precisely the singletons.

Note that totally disconnected spaces need not be discrete.

Day 7 : 29th August, 2025

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product of connected spaces -- interval connected
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7.1 Connectedness (cont.)

Theorem 7.1: (Product of connected spaces is connected)

Suppose $\{X_\alpha\}_{\alpha \in I}$ is a collection of connected spaces. Let $X = \Pi X_\alpha$ be the product space. Then, X is connected.

Proof

For finite product $X \times Y$, fix a point $y_0 \in Y$, and observe that

$$X \times Y = \bigcup_{x \in X} \Bigg(\underbrace{\{x\} \times Y \cup X \times \{y_0\}}_{C_x} \Bigg) .$$

Note that C_x is connected since it is the union of two connected sets $\{x\} \times Y \cong Y$ and $X \times \{y_0\} \cong X$ (check!), and they have a common point (x, y_0). But then $X \times Y$ is connected, as $\bigcap_{x \in X} C_x = X \times \{y_0\} \neq \emptyset$. This can be generalized to any finite product.

As for the infinite product, fix a point $z = (z_\alpha) \in X$ (*If you don't want to assume axiom of choice, then X could be empty, which is still a connected set*). Consider the subset

$$A := \{(x_\alpha) \in X \mid \text{all but finitely many } x_\alpha = z_\alpha\} .$$

Since $X = \bar{A}$, it is enough to show that A is connected. Firstly, for any finite $J \subset I$, define

$$A_J := \{(x_\alpha) \in X \mid x_\alpha = z_\alpha \text{ for any } \alpha \in I \setminus J\} .$$

Observe that $A_J \cong \Pi_{\alpha \in J} X_\alpha$ (check!), and hence, connected. Next, observe that $A = \bigcup_{\text{J} \subset \text{I finite}} A_J$, and $\bigcap_{\text{J} \subset \text{I finite}} A_J = \{z\}$. Thus, A is connected as well. But then $X = \bar{A}$ is connected. □

Exercise 7.2: (Box topology may not be connected)

Consider $X = \mathbb{R}^{\mathbb{N}}$ equipped with the box topology, where $\mathbb{R}$ has the usual topology. Check that the following sets are nontrivial clopen sets of X.

a) $U_0 := \{(x_n) \mid \lim x_n = 0 \text{ in } \mathbb{R}\}$.

b) $U_1 := \{(x_n) \mid \{x_n\} \text{ is a bounded sequence in } \mathbb{R} \}$.

Theorem 7.3: (Closed interval is connected)

The closed interval $[a, b] \subset \mathbb{R}$ for some $a < b$ is connected.

Proof

Suppose $[a, b] = A \sqcup B$ for some open (and hence closed) nontrivial subsets $\emptyset \subsetneq A, B \subset [a, b]$. Without loss of generality, assume that $a \in A$. Consider the set

$$C := \{c \mid [a, c] \subset A\} .$$

Observe that since $a \in C$ since $\{a\} = [a, a] \subset A$. Clearly b is an upper bound for C. Then, there is a least upper bound, say, $L := \sup C$.

As A is open, there is some $\epsilon_0 > 0$ such that $[a, a + \epsilon_0) \subset A$, and thus $L \geq a + \epsilon_0 > a$. Let us show that $L \in C$. Firstly, note that for any $0 < \epsilon \leq \epsilon_0$, we have some $L - \epsilon \leq c_0 \in C$, and thus, $[a, L - \epsilon] \subset [a, c_0] \subset A$. In other words, $(L - \epsilon, L + \epsilon) \cap A \neq \emptyset$. But then, L is a closure point of A (in the subspace topology of $[0, 1]$). Since A is closed, we have $L \in A$. As A is open as well, we have some $\epsilon_1 \leq \epsilon_0$ such that $(L - \epsilon_1, L + \epsilon_1) \cap [a, b] \subset A$. But then,

$$[a, L] = [a, L - \epsilon_1] \cup (L - \epsilon_1, L] \subset A,$$

which shows that $L \in C$.

Now, $L \leq b$, as b is an upper bound of C. If possible, suppose $L < b$. Then, for some $\epsilon > 0$ small, we have $[L - \epsilon, L + \epsilon] \subset [a, b]$. Choosing ϵ smaller, and using the openness of A, we have $[a, L + \epsilon] = [a, L] \cup (L, L + \epsilon] \subset A$, which implies $L + \epsilon \in C$, contradicting $L = \sup C$. Hence, $L = b$. But then, $[a, L] = [a, b] \subset A$, contradicting that $B \neq \emptyset$.

Thus, $[a, b]$ is connected. □

Proposition 7.4: (All intervals are connected)

Any finite or infinite interval, whether open, closed or semi-open, of $\mathbb{R}$ is connected. In particular, $\mathbb{R}$ is connected

Proof

Let us show that $\mathbb{R}$ is connected. If not, then $\mathbb{R} = U \sqcup V$ is a separation by open sets. Pick some $a \in U$ and $b \in V$. Then, $[a, b] = ([a, b] \cap U) \sqcup ([a, b] \cap V)$ is a separation of $[a, b]$. This is a contradiction as $[a, b]$ is connected. Hence, $\mathbb{R}$ is connected.
Similar argument works for the other cases. □

Exercise 7.5: (Intermediate value property)

Suppose $f : [a, b] \to \mathbb{R}$ is a continuous map. If $f(a) < f(b)$, then for any $f(a) < x < f(b)$ there exists some $a < c < b$ such that $f(c) = x$.

Day 8 : 9^{th} September, 2025

`path connectedness`

8.1 Path connectedness

Definition 8.1: (Path connected space)

A space X is called *path connected* if for any $x, y \in X$, there exists a continuous map $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$. Such an f is called a *path* joining x to y. A subset $P \subset X$ is called a *path connected set* if P is path connected in the subspace topology.

Exercise 8.2: (Path connected set)

Check that $P \subset X$ is a path connected set if and only if for any $x, y \in P$, there exists a path $\gamma : [0, 1] \to X$ joining $x = \gamma(0)$ to $y = \gamma(1)$, such that γ is contained in P.

Exercise 8.3: (Star connected spaces are path connected)

Given a space X and fixed point $x_0 \in X$, suppose for any $x \in X$ there exists a path in X joining x_0 to x. Show that X is path connected. What about the converse?

Proposition 8.4: (Path connected spaces are connected)

If X is a path connected space, then X is connected.

Proof

Suppose not. Then, there is a continuous surjection $g : X \to \{0, 1\}$. Pick $x \in g^{-1}(0)$ and $y \in g^{-1}(1)$. Get a path $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. Then, $h := g \circ f : [0, 1] \to 0, 1$ is a continuous surjection, which contradicts the connectivity of $[0, 1]$. Hence, X is connected. □

Proposition 8.5: (Connected open sets of $\mathbb{R}^n$ are path connected)

Connected open sets of $\mathbb{R}^n$ are path connected.

Proof

Let U be a connected open subset of $\mathbb{R}^n$. If $U = \emptyset$, there is nothing to show. Fix some $x \in u$. Consider the subset

$$A = \{y \in U \mid \text{there is path in } U \text{ from } x \text{ to } y\}.$$

Clearly $A \neq \emptyset$ as $x \in A$.

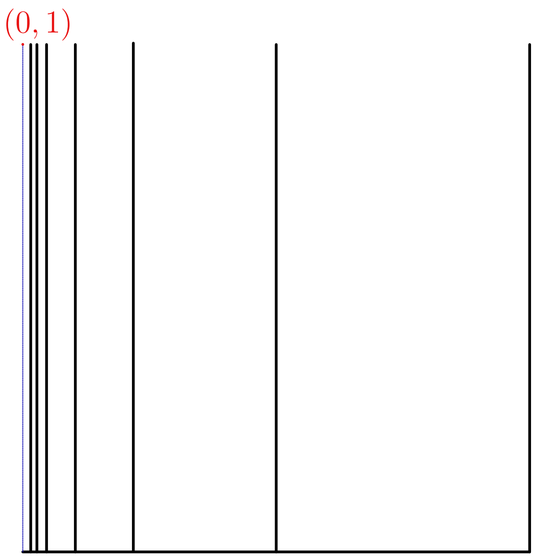
Let us show A is open. Say, $y \in A$. Then, there exists a Euclidean ball $y \in B(y, \epsilon) \subset U$. Now, it is clear that for any $z \in B(y, \epsilon)$ the radial line joining y to z is a path, contained in $B(y, \epsilon)$, and hence, in U. Thus, by concatenating, we get a path from x to any $z \in B(y, \epsilon)$, showing $B(y, \epsilon) \subset A$. Thus, A is open.

Next, we show that A is closed. Let $y \in U$ be an adherent point of A. As U is open, we get some ball $y \in B(y, \epsilon) \subset U$. Now, $B(y, \epsilon) \cap A \neq \emptyset$. Say, $z \in B(y, \epsilon) \cap A$. Then, we can get a path from x to y by first getting a path to z (which exists, since $z \in A$), and then considering the radial line from z to y. Clearly, this path is contained in U. Thus, $y \in A$. Hence, A is closed.

But U is connected. Hence, the only non-empty clopen set of U is U. That is, $A = U$. But then clearly U is path connected. □

In general, connected spaces need not be path connected! Here is one such example. Consider $K_0 := \left\{\frac{1}{n} \mid n \geq 1\right\}$, and the set

$$C := ([0, 1] \times \{0\}) \cup (K_0 \times [0, 1]) \subset \mathbb{R}.$$



Comb space. Removing the dotted blue line $\{0\} \times (0,1)$*, we get the deleted comb space.*

In the picture, this is the collection of vertical black lines, along with the 'spine' $[0,1]$ along the x-axis. It is easy to see that C is path connected, and hence, connected. Indeed, any point can be joined by a path to the origin $(0,0)$. The closure of C in $\mathbb{R}^2$ is called the *comb space*. One can easily see that

$$\bar{C} := C \cup (\{0\} \times [0,1]) .$$

The *deleted comb space* D is obtained by removing the segment $\{0\} \times (0,1)$ from the comb space.

Theorem 8.6: (Deleted comb space is connected but not path connected)

The deleted comb space is connected, but not path connected.

Proof

Since C is connected, and $C \subset D \subset \bar{C}$, we have both the comb space and the deleted comb space are connected.

Intuitively, it is clear that there cannot be a path from $p = (0,1) \in D$ to any other point of D. Let us prove this formally. If possible, suppose $f : [0,1] \to D$ is a path from p to some point in D. Consider the set

$$A := \{t \mid f(t) = p\} = f^{-1}(p).$$

Clearly, A is closed in $[0,1]$, and it is non-empty as $0 \in A$.

Let us show that A is open. Let $t_0 \in A$. Since f is continuous, there exist some $\epsilon > 0$ such that for any $t \in [0,1]$ with $|t - t_0| < \epsilon$, we have $\|f(t) - f(t_0)\| < \frac{1}{2}$. In particular, such $f(t)$ does not intersect the x-axis. Consider $B = \left\{x \in \mathbb{R}^2 \;\middle|\; \|x - p\| < \frac{1}{2}\right\} \cap \bar{C}$, and denote the interval

$$J = (t_0 - \epsilon, t_0 + \epsilon) \cap [0,1].$$

Consider the first-component projection map $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$, which is continuous. Observe that π_1 restricts to the continuous map $\pi : B \to K_0 \cup \{0\}$ (this is where we are using the fact B does not intersect the x-axis). Now, $h := \pi \circ f|_J : J \to K_0 \cup \{0\}$ is a continuous map. We have $K_0 \cup \{0\}$ is totally disconnected, i.e, the only components are singletons. Now, $h(t_0) = \pi(f(t_0)) = \pi(p) = 0$. Hence, we must have $h(t) = 0$ for all $t \in J$, as J is connected and continuous image of a connected set is again connected. But then, $f(t) \in \pi^{-1}(0) = \{p\}$

for any $t \in J$, i.e, $f(t) = p$ for all $t \in J$. This shows that t_0 is an interior point of A. Thus, A is open.

Since $[0, 1]$ is connected, we must have $A = [0, 1]$, as it is a nonempty clopen set. But then the original path f is constant at p. Since f was an arbitrary path from p, we see that D is not path connected. □

Remark 8.7

The above argument is a very common method of proving many statements in analysis and topology. So try to understand it thoroughly!

Day 9 : 10^{th} September, 2025

`path connectedness -- path component -- locally connected -- locally path connected -- compactness`

9.1 Path connectedness (cont.)

Proposition 9.1: (Image of path connected set)

Let $f : X \to Y$ be continuous. Then, for any path connected subset $A \subset X$, we have $f(A) \subset Y$ path connected. In particular, if X is path connected, then so is $f(X)$.

Proof

Pick $x, y \in f(A)$. Then, $x = f(a)$ and $y = f(b)$ for some $a, b \in A$. Get a path $\gamma : [0, 1] \to A$ joining a to b. Then, $h = f \circ \gamma : [0, 1] \to f(A)$ is a path in $f(A)$ joining x to y. Thus, $f(A)$ is path connected. □

Exercise 9.2: (Product of path connected)

Let $\{X_\alpha\}$ be a family of path connected spaces. Show that the product space $X = \Pi X_\alpha$ is path connected. Give an example to show that X may not be path connected equipped with the box topology.

Definition 9.3: (Path component)

Given $x \in X$, the *path component* of X containing x is the largest possible path connected set of X containing x.

Proposition 9.4: (Existence of path component)

Given $x \in X$, the path component of X can be defined as

$$\mathcal{P}(x) := \{y \in X \mid \text{there is a path } f : [0, 1] \to X \text{ with } f(0) = x \text{ and } f(1) = y\}.$$

Equivalently,

$$\mathcal{P}(x) := \bigcup \{P \subset X \mid x \in P,\ P \text{ is path connected}\}.$$

Proof

Let us check the first part. Firstly, note that $\mathcal{P}(x)$ is path connected. Indeed, given any two $y, z \in \mathcal{P}(x)$, we have two paths $f : [0,1] \to \mathcal{P}(x)$ and $g : [0,1] \to \mathcal{P}(x)$ joining, respectively, x to y and x to z. We can construct the concatenated path h as follows

$$h(t) = \begin{cases} f(1-2t), & 0 \le t \le \frac{1}{2}, \\ g(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Check that h is continuous! Clearly, h is a then a path connecting y to z. Thus, $\mathcal{P}(x)$ is path connected.

Now, suppose A is the union of all path connected sets of X containing x. For any $y, z \in A$, we have $y \in P$ and $z \in Q$ for some path connected sets $x \in P, Q \subset X$. Then, we can get a path joining y to x and then from x to z, which is in $P \cup Q \subset A$. Thus, A is path connected, which is clearly the larges such set containing x. Hence, the second definition of $\mathcal{P}(x)$ is also true. □

Exercise 9.5: (Path component equivalence relation)

Define a relation $x \sim y$ if and only if x, y are in the same path component. Check that $\sim$ is an equivalence relation, and the equivalence classes are precisely the path components of X.

9.2 Locally connected and locally path connected spaces

Definition 9.6: (Locally connected)

A space X is called *locally connected at $x \in X$* if given any open neighborhood $x \in U$, there exists a (possibly smaller) open neighborhood $x \in V \subset U$, such that V is connected. The space is called *locally connected* if it is locally connected at every point $x \in X$.

Theorem 9.7

A space X is locally connected if and only if for all open set $U \subset X$, all the components of U are open.

Proof

Suppose X is locally connected. Pick some $U \subset X$ open, and a component $C \subset U$. Now, for any $x \in C \subset U$, by local connectedness, there is a connected open set $x \in V \subset U$. Since $x \in V \cap C$, we see that $V \cup C$ is connected. But C is the larges connected set containing x. Thus, $x \in V \subset C$, proving that $x \in \mathring{C}$. Thus, C is open.

Conversely, suppose for any open $U \subset X$, each component of U is open. Fix some x and some open neighborhood $x \in U$. Consider the component of x in U to be C. Then, C is open. Hence, X is locally connected. □

Definition 9.8: (Locally path connected)

A space X is called *locally path connected at $x \in X$* if given any open neighborhood $x \in U$, there exists a (possibly smaller) open neighborhood $x \in V \subset U$, such that V is path connected. The space is called *locally path connected* if it is locally path connected at every point $x \in X$.

Theorem 9.9

A space X is locally path connected if and only if for all open set $U \subset X$, all the path components of U are open.

Theorem 9.10

The path components of X lies in a single component. If X is locally path connected, then the path components and the components coincide.

Proof

Suppose P is a path component, which is path connected, and hence, connected. But then P can only lie in a single component.

Suppose X is locally path connected. Then, every path components of X is open. Let C be a component. For some $x \in C$, consider P to be the path component of x. Then, $x \in P \subset C$. If $P \neq C$, then consider Q to be the union of every other path components of points of $C \setminus P$. Again, we have $Q \subset C$. Now, we have a separation $C = P \sqcup Q$ by nontrivial open sets, which contradicts the fact that C is connected. Hence, $P = C$. Thus, path components of X coincide with the components. □

9.3 Compactness

Definition 9.11: (Covering)

Given a set X, a collection $\mathcal{A} \subset \mathcal{P}(X)$ of subsets of X is called a *covering* of X if we have $X = \bigcup_{A \in \mathcal{A}} A$. Given a topological space $(X, \mathcal{T})$, we say $\mathcal{A}$ is an *open cover* (of X) $\mathcal{A}$ is a covering of X and if each $A \in \mathcal{A}$ is an open set. A *sub-cover* of $\mathcal{A}$ is a sub-collection $\mathcal{B} \subset \mathcal{A}$, which is again a covering, i.e, $X = \bigcup_{B \in \mathcal{B}} B$.

Definition 9.12: (Compact space)

A space X is called *compact* if every open cover of X has a finite sub-cover. A subset $C \subset X$ is called compact if C is compact as a subspace.

Example 9.13: (Finite space is compact)

Any finite topological space is compact, since there can be at most finitely many open sets in X. An infinite discrete space is not compact.

Proposition 9.14: (Compact subspace)

A subset $C \subset X$ is compact if and only if given any collection $\mathcal{A} = \{A_\alpha\}$ of open sets of X, with $C \subset \bigcup A_\alpha$, we have a finite sub-collection $\{A_{\alpha_1}, \ldots, A_{\alpha_k}\}$ such that $C \subset \bigcup_{i=1}^k A_{\alpha_i}$.

Proof

Suppose C is compact (as a subspace). Consider a cover $\mathcal{A} = \{A_\alpha\}$ of C by opens of X. Then, $\mathcal{A}' = \{A_\alpha \cap C\}$ is an open cover of C in the subspace topology. Since C is compact, we have a finite sub-cover, say, $\{A_{\alpha_1} \cap C, \ldots, A_{\alpha_k} \cap C\}$. But then $C \subset \bigcup_{i=1}^k \mathcal{A}_{\alpha_i}$.

Conversely, suppose given any cover of C by open sets of X, we have a finite sub-cover. Choose any open cover of C (in the subspace topology), say, $\mathcal{U} = \{U_\alpha \subset C\}$. Now, each $U_\alpha = C \cap V_\alpha$ for some open $V_\alpha \subset X$. Then, $C \subset \bigcup V_\alpha$ is a cover, which has finite sub-cover, $C \subset \bigcup_{i=1}^k V_{\alpha_i}$. Clearly, $C = \bigcup_{i=1}^k C \cap V_{\alpha_i} = \bigcup_{i=1}^k U_{\alpha_i}$. Thus, C is compact. $\square$

Exercise 9.15: (Compactness is independent of subspace)

Let $Y \subset X$ be a subspace. A subset $C \subset Y$ is compact if and only if C is compact as a subspace of X.

Proposition 9.16: (Closed in compact is compact)

Suppose X is a compact space, and $C \subset X$ is closed. Then, C is compact.

Proof

Fix some cover $\{U_\alpha\}$ of C by open sets $U_\alpha \subset X$. Now, C being closed, we have $V := X \setminus C$ is open. We have, $X = V \cup \bigcup U_\alpha$. Since X is compact, there is a finite subcover. Without loss of generality, $X = V \cup \bigcup_{i=1}^k U_{\alpha_i}$. Then, $C \subset \bigcup_{i=1}^k U_{\alpha_i}$. Hence, C is compact. $\square$

Example 9.17: (Compact need not be closed)

Let X be an indiscrete space. Then, any subset is compact, but there are non-closed subsets.

Proposition 9.18: (Compact in T_2 is closed)

Let X be a T_2 space. Then, any compact $C \subset X$ is closed.

Proof

If $C = X$, then there is nothing to show. Otherwise, we show that any $y \in X \setminus C$ is an interior point. For each $c \in C$, by T_2, there is some open neighborhoods $y \in U_c, c \in V_c$, such that $U_c \cap V_c = \emptyset$. Now, $C \subset \bigcup_{c \in C} V_c$. Since C is compact, there are finitely many points, $c_1, \ldots, c_k$, such that

$$C \subset \bigcup_{i=1}^k V_{c_i}.$$

Let us consider $U := \bigcap_{i=1}^k U_{c_i}$, which is an open neighborhood of y. Also, $U \cap \left(\bigcup_{i=1}^k V_{c_i}\right) = \emptyset \Rightarrow U \cap C = \emptyset \Rightarrow U \subset X \setminus C$. Thus, $y \in \operatorname{int}(X \setminus C)$. Since y was arbitrary, C is closed. $\square$

Example 9.19: (Compact is not closed in T_1)

Let X be an infinite set, equipped with the cofinite topology. Then, X is T_1, but not T_2.
Let $C = X \setminus \{x_0\}$ for some $x_0 \in X$, which is clearly not closed.
Suppose $C \subset \bigcup_{\alpha \in I} U_\alpha$ is some open covering. Choose some U_{α_0}. Now, $U_{\alpha_0} = X \setminus \{x_1, \dots, x_k\}$ (if $U_{\alpha_0} = X$, then there is nothing to show). For each $1 \leq i \leq k$ with $x_i \in C$, choose some U_{α_i} such that $x_i \in U_{\alpha_i}$. If $x_i \notin C$, choose U_{α_i} arbitrary. Then, $C \subset \bigcup_{i=0}^k U_{\alpha_i}$. Thus, C is compact, but not closed.

Day 10 : 11th September, 2025

`compactness -- finite product of compact`

10.1 Compactness (cont.)

Theorem 10.1: (Image of compact space)

$f : X \to Y$ be a continuous map. If X is compact, then $f(X)$ is compact.

Proof

Consider an open cover $\mathcal{V} = \{V_\alpha\}$ of $f(X)$ by opens of Y. Then, $\mathcal{U} = \{U_\alpha := f^{-1}(V_\alpha)\}$ is an open cover of X. Since X is compact, there is a finite subcover, say $X = \bigcup_{i=1}^k U_{\alpha_k} = \bigcup_{i=1}^k f^{-1}(V_{\alpha_i})$. But that, $f(X) \subset \bigcup_{i=1}^k V_{\alpha_i}$. Thus, $f(X)$ is compact. □

Theorem 10.2: (Maps from compact space to T_2)

Let $f : X \to Y$ be a surjective continuous map. Suppose X is compact, and Y is T_2. Then, f is an open map.

Proof

Let $U \subset X$ be an open set. Then, $C = X \setminus U$ is closed, and hence, compact. Since f is continuous, $f(C) \subset Y$ is compact. As Y is T_2, we have $f(C)$ is closed in Y. Finally, as f is surjective, we have $f(U) = Y \setminus f(X \setminus U) = Y \setminus f(C)$, which is then open. Thus, f is an open map. □

Remark 10.3: (Non-surjective map from compact to T_2)

Consider the inclusion map of the point $\{0\}$ in $\mathbb{R}$. Clearly, $\{0\}$ is compact, but the inclusion map is not open!

Exercise 10.4: (Compact to T_2 is closed)

Suppose X is compact, Y is T_2, and $f : X \to Y$ is a continuous map (not necessarily surjective). Then, show that f is a closed map.

Theorem 10.5: (Compactness of closed interval)

The closed interval $[a, b] \subset \mathbb{R}$ is compact (in the usual topology).

Proof

Suppose $\mathcal{A} = \{U_\alpha\}$ is a collection open sets of $\mathbb{R}$ covering $[a, b]$. Consider the set

$$C = \{c \in [a, b] \mid [a, c] \text{ is covered by a finite number of opens from } \mathcal{A}\}.$$

Note that $C \neq \emptyset$, since $[a, a] = \{a\}$ is clearly contained in some U_α. Let $L = \sum C$ be the least upper bound. Observe that $a \in U_\alpha \Rightarrow [a, a + \epsilon) \subset U_\alpha$ for some $e > 0$. Thus, $a < L \leq b$. Now, there is some U_β such that $L \in U_\beta$. Then, there is some $\epsilon > 0$ such that $a < L - \epsilon < L$ and $(L - \epsilon, L] \subset U_\beta$. Also, L being the least upper bound, there is some $c \in C$ such that $L - \epsilon < c < L$. Thus, $[a, c]$ is covered by finitely many opens, say, $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$. But then $[a, L] = [a, c] \cup [L - \epsilon, L]$ is covered by a finite collection $\{U_{\alpha_1}, \ldots, U_{\alpha_k}, U_\beta\}$. Thus, $L \in C$.
Now, if $L < b$, then, there is some $\epsilon > 0$ such that $L < L + \epsilon < b$, and $[L, L + \epsilon] \subset U_\beta$. By a similar argument, it follows that $[a, L + \epsilon]$ is covered by finitely many opens of $\mathcal{A}$. This contradicts L be the least upper bound. Hence, $L = b$.
Thus, $[a, b]$ is covered by a finitely many sub-collection of $\mathcal{A}$. Since $\mathcal{A}$ is arbitrary, it follows that $[a, b]$ is compact. □

Exercise 10.6: (Real line is noncompact)

Show that $\mathbb{R}$ is not compact.

10.2 Product of compacts

Lemma 10.7: (Tube lemma)

Suppose Y is a compact space. Fix a point $x_0 \in X$, and suppose $W \subset X \times Y$ is an open set such that $\{x_0\} \times Y \subset$. Then, there exists an open set $x_0 \in U \subset X$ such that $\{x_0\} \times Y \subset U \times Y \subset W$.

Proof

For each $y \in Y$, consider a basic open set $(x_0, y) \in U_y \times V_y \subset W$. Now, $\{x_0\} \times Y \subset \bigcup_{y \in Y} U_y \times V_y$. Since Y, and hence $\{x_0\} \times Y$, is compact, we have a finite cover, say, $\{x_0\} \times Y \subset \bigcup_{i=1}^{k} U_{y_i} \times V_{y_i}$. Now, set $U = \bigcap_{i=1}^{k} U_{y_i}$, which is an open set with $x_0 \in U$. Clearly $\{x_0\} \times Y \subset U \times Y$. Now, for any $(x, y) \in U \times Y$, we have $(x_0, y) \in U_{y_{i_0}} \times V_{y_{i_0}}$ for some i_0. Then, $y \in V_{y_{i_0}}$. Also, $x \in U \subset U_{y_{i_0}}$. Thus, $(x, y) \in U_{y_{i_0}} \times V_{y_{i_0}}$. In other words, we have

$$\{x_0\} \times Y \subset U \times Y \subset \bigcup_{i=1}^{k} U_i \times V_i \subset W.$$

□

Theorem 10.8: (Finite product of compacts are compact)

If X, Y are compact, then so is $X \times Y$.

Proof

Suppose $\mathcal{W}$ is an open cover of $X \times Y$. For each $x \in X$, the space $\{x\} \times Y$ is compact, and hence, can be covered by a finite collection, say

$$\{x\} \times Y \subset \bigcup_{i=1}^{k_x} W_{x,i},$$

for $W_{x,i} \in \mathcal{W}$. Then, by the tube lemma, there exists some $x \in U_x \subset X$ such that

$$\{x\} \times Y \subset U_x \times Y \subset \bigcup_{i=1}^{k_x} W_{x,i}.$$

Now, $\{U_x\}$ is an open cover of X, which is also compact. Hence, we have a finite cover, say, $X = \bigcup_{i=1}^{n} U_{x_i}$. Then, clearly,

$$X \times Y = \bigcup_{i=1}^{n} U_{x_i} \times Y \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_{x_i}} W_{x_i,j}.$$

Thus, $X \times Y$ can be covered by finitely many elements of $\mathcal{W}$. Hence, $X \times Y$ is compact. ☐

Day 11 : 16th September, 2025

`sequential compactness -- limit point compactness -- first countability`

11.1 Sequential and limit point compactness

Definition 11.1: (Sequentially compact)

A space X is called *sequentially compact* if every sequence $\{x_n\}$ has a convergent subsequence. A subset $Y \subset X$ is sequentially compact if every sequence $\{y_n\}$ in Y has a subsequence, that converges to some $y \in Y$.

Theorem 11.2: (Sequentially compact is equivalent to compact in metric space)

Suppose (X, d) is a metric space. Then, $Y \subset X$ is sequentially compact if and only if Y is compact.

Proof

Suppose Y is compact. Then, Y is closed and bounded. Consider a sequence $\{x_n\}$ in Y. If possible, suppose $\{x_n\}$ has no convergent subsequence in Y. Then, $\{x_n\}$ is an infinite sequence (i.e., there are infinitely many distinct elements). Now, for each $y \in Y$, there exists a ball $y \in B_y = B_d(y, \delta_y) \subset X$ such that B_y contains at most finitely many $\{x_n\}$ (as no subsequence of $\{x_n\}$ converge to y). We have $Y \subset \bigcup_{y \in Y} B_y$, which admits a finite subcover, say, $Y \subset \bigcup_{i=1}^{n} B_{y_i}$. But this implies Y contains at most finitely many $\{x_n\}$, which is a contradiction.

Conversely, suppose every sequence in Y has a subsequence converging in Y. Consider an open cover $\mathcal{U} = \{U_\alpha\}$ of Y by opens of X.

- Let us first show that for any $\delta > 0$, the collection $\{B_d(a,\delta) \mid a \in A\}$ has a finite sub-cover. Suppose not. Then, there is $x_1 \in A$ such that $A \not\subset B_d(x_1,\delta)$. Pick $x_2 \in A \setminus B_d(x_1,\delta)$. Then, $A \not\subset B_d(x_1,\delta) \cup B_d(x_2,\delta)$. Inductively, we have a sequence $\{x_n\}$ in A. Now, by construction, $d(x_i,x_j) \geq \delta$ for all $i \neq j$. Consequently, $\{x_n\}$ has no convergent subsequence, a contradiction. Indeed, if $x_{n_k} \to x \in A$, then $d(x_{n_k},x) < \frac{\delta}{2}$ for all $k \geq N$. But then, $d(x_{n_{k_1}},x_{n_{k_2}}) < \delta$ for any $k_1 \neq k_2 \geq N$.

- Next we claim that there exists a $\delta > 0$ such that for any $y \in Y$, we have $B_d(y,\delta) \subset U_\alpha$ for some α. Suppose not. Then, for each $n \geq 1$, there exists some $y_n \in Y$ such that $B_d(y_n,\frac{1}{n}) \not\subset U_\alpha$ for each α. Passing to a subsequence, we have $y_n \to y_0 \in A$. Now, $y_0 \in V_\alpha$ for some α, and so, $y_0 \in B_d(y_0,\epsilon) \subset V_\alpha$. There exists some $N_1 \geq 1$ such that $y_n \in B_d(y_0,\frac{\epsilon}{2})$ for all $n \geq N_1$. Also, there is $N_2 \geq 1$ such that $\frac{1}{N_2} < \frac{\epsilon}{2}$. Then, for any $n \geq \max\{N_1,N_2\}$, and for any $d(y_n,y) < \frac{1}{n}$ we have,

$$d(y_0,y) \leq d(y_0,y_n) + d(y_n,y) < \epsilon.$$

  Thus, $B_d(y_n,\frac{1}{n}) \subset B_d(y_0,\epsilon) \subset V_\alpha$ for all $n \geq \max\{N_1,N_2\}$, a contradiction.

- Finally, pick the δ from the last step. Then, we have a cover $A \subset \bigcup_{i=1}^n B_d(x_i,\delta)$ with $x_i \in A$. But each of these balls are contained in some V_{α_i}. So, we have $A \subset \bigcup_{i=1}^n V_{\alpha_i}$.

□

Definition 11.3: (Limit point compactness)

A space X is called *limit point compact* (or *weakly countably compact*) if every infinite set $A \subset X$ has a limit point in X

Exercise 11.4: (Sequential compact implies limit point compact)

Show that a sequentially compact space is limit point compact.

Proposition 11.5: (Compact implies limit point compact)

A compact space is limit point compact.

Proof

Suppose X is a compact space which is not limit point compact. Then, there exists an infinite set A which has no limit point. In particular, A is closed, as it contains all of its limit points (which are none). Also, for every $x \in X$, there is an open set $x \in U_x \subset X$ such that $A \cap (U_x \setminus \{x\}) = \emptyset$. Observe that we have a covering $X = (X \setminus A) \cup \bigcup_{x \in A} U_x$, which admits a finite subcover, say, $X = (X \setminus A) \cup \bigcup_{i=1}^n U_{x_i}$. Now, $A \subset \bigcup_{i=1}^n U_{x_i}$. But this implies A is finite, as $A \cap U_{x_i} \setminus \{x_i\} = \emptyset$. This is a contradiction. □

Example 11.6: (Limit point comact but neither compact nor sequentially compact)

Consider the space $X = \mathbb{N} \times \{0,1\}$, where give $\mathbb{N}$ the discrete topology, and $\{0,1\}$ the indiscrete topology. Consider the sequence $x_n = (n,0)$. Then, it does not have a convergent subsequence (otherwise, the first component projection will give convergent subsequence, as continuity implies sequential continuity). Also, X is not compact either, as the open cover $U_n = \{(n,0),(n,1)\}$ has no finite subcover. On the other hand, X is limit point compact. Indeed, say $A \subset X$ is infinite, and, without loss of generality, pick some $(a,0) \in A$. Then, check that $(a,1)$ is a limit point of A. Indeed, any open set containing $(a,1)$ contains the open set $\{(a,0),(a,1)\}$, which obviously intersects A in a different point $(a,0)$.

Definition 11.7: (First countable)

Given $x \in X$, a *neighborhood basis* is a collection $\{U_\alpha\}$ of open neighborhoods of x such that given any open neighborhood $x \in U \subset X$, there exists some U_α such that $x \in U_\alpha \subset U$. We say X is *first countable at x* if there exists a countable neighborhood basis $\{U_i\}$ of x. The space X is called *first countable* if it is first countable at every point.

Remark 11.8: (Decreasing neighborhood basis)

Suppose $\{U_i\}$ is a countable neighborhood basis of $x \in X$. Set $V_1 = U_1, V_2 = U_1 \cap U_2, \ldots, V_j = V_{j-1} \cap U_j = \bigcap_{i=1}^{j} U_j$. Clearly, we have

$$V_1 \supset V_2 \supset \cdots \ni x.$$

We claim that $\{V_j\}$ is a neighborhood basis of x as well. Let $x \in U \subset X$ be an open neighborhood. Then, there is some $x \in U_j \subset U$. But then $x \in V_j \subset U_j \subset U$ as well. Thus, we can always assume that a countable neighborhood basis is decreasing. Note : in a discrete space $\{U_n = \{x\}\}$ is a non-strictly decreasing countable neighborhood basis of x.

Example 11.9: (Metric space is first countable)

Any metric space (X,d) is first countable. The converse is evidently not true, as any indiscrete space is also first countable.

Proposition 11.10: (Compact first countable is sequentially compact)

Suppose X is a first countable compact space. Then X is sequentially compact.

Proof

Let $\{x_n\}$ be a sequence in X with no convergent subsequence. Then $\{x_n\}$ must be an infinite set. Without loss of generality, assume each x_n are distinct (just extract such a subsequence). For each $x \in X$, fix some neighborhood basis $\mathcal{U}_x$. Now, since no subsequence of $\{x_n\}$ converges to x, there must be some $U_x \in \mathcal{U}_i^x$ such that only finitely many $\{x_n\}$ is contained in U_x. Otherwise, using the countability of $\mathcal{U}_x$, we can extract a subsequence converging to x. Now, we have a cover $X = \cup_{x \in X} U_x$, which admits a finite subcover, say, $X = \cup_{i=1}^{n} U_{x_i}$. But this implies the sequence $\{x_n\}$ is finite, a contradiction. $\square$

Day 12 : 17th September, 2025

`sequential compactness -- limit point compactness -- second countable -- Lindelöf`

12.1 Sequential Compactness (Cont.)

Definition 12.1: (Countably compact)

A space X is called *countably compact* if every countable open cover admits a finite sub-cover.

Proposition 12.2: (Limit point compact T_1 is countably compact)

A limit point compact T_1-space is countably compact.

Proof

Let $X = \bigcup U_i$ be a countable cover. If possibly, suppose there is no finite subcover. In particular, $X \setminus \bigcup_{i=1}^n U_i \neq \emptyset$ for each $n \geq 1$. Moreover, $X \setminus \bigcup_{i=1}^n U_i \neq \emptyset$ must be infinite, otherwise we can readily get a finite sub-cover. Inductively choose $x_n \notin \bigcup_{i=1}^n U_i \cup \{x_1, \ldots, x_{n-1}\}$. Thus, we have an infinite set $A = \{x_i\}$, which admits a limit point, say, x. Since X is T_1, it follows that for any open nbd $x \in U \subset X$, we must have $A \cap (U \setminus \{x\})$ is infinite (Check!). Now, we have $x \in U_{i_0}$ for some i_0. But by construction, U_{i_0} contains at most finitely many x_i, a contradiction. Hence, we must have a finite subcover. Thus, X is countably compact. □

Proposition 12.3: (Countably compact first countable is sequentially compact)

A first countable, countably compact space is sequentially compact.

Proof

Suppose, $\{x_n\}$ is a sequence. WLOG, assume element is distinct. If possible, suppose $A = \{x_n\}$ has no convergent subsequence.

If possible, $A = \{x_n\}$ has no convergent subsequence. Since X is first countable, for any $x \in X$, we must have some open set $x \in U_x \subset X$ such that $U_x \cap A$ is finite (Check!). Now, for any finite subset, $F \subset A$, consider the open set

$$\mathcal{O}_F := \bigcup \{U_x \mid U_x \cap A = F\}.$$

Since A is countable, there are countable finite subsets of F. Thus, $\mathcal{O} := \{\mathcal{O}_F \mid F \subset A \text{ is finite}\}$ is a countable collection, which is clearly an open cover. By countable compactness, we have a finite subcover $X = \bigcup_{i=1}^k \mathcal{O}_{F_i}$. Consider $F = \cup_{i=1}^k F_i$, which is again finite. Pick some $x_{i_0} \in A \setminus F$. Now, $\mathcal{O}_{F_i} \cap A = F_i \Rightarrow x_{i_0} \notin \cup_{i=1}^k F_i = \bigcup_{i=1}^k \mathcal{O}_{F_i} \cap A = X \cap A = A$, a contradiction. Hence, $\{x_n\}$ must have a convergent subsequence. Thus, X is sequentially compact. □

Proposition 12.4: (Limit point compact, T_1, first countable is sequentially compact)

Suppose X is a first countable, T_1, limit point compact space. Then X is sequentially compact.

Proof

Since X is limit point compact and T_1, we have X is countably compact. Since X is countably compact and first countable, we have X is sequentially compact. □

Example 12.5: (Necessity of T_1)

Recall the topolgoy $\mathcal{T}_\to = \{\emptyset, \mathbb{R}\} \cup \{(a,\infty) \mid a \in \mathbb{R}\}$ on $\mathbb{R}$. For any infinite subset $A \subset \mathbb{R}$, choose any x such that $x < a$ for some $a \in A$. Then, x is a limit point of A. Also, for any $x \in \mathbb{R}$, we have a countable neighborhood basis $\{U_i = (x - \frac{1}{n}, \infty) \mid n \geq 1\}$. We have seen that $(\mathbb{R}, \mathcal{T}_\to)$ is not T_1. Finally, observe that the sequence $\{x_n = -n\}$ has no convergent subsequence.

Definition 12.6: (Second countable)

A space X is called *second countable* if it admits a countable basis.

Definition 12.7: (Lindelöf)

A space X is called *Lindelöf* if every open cover admits a countable sub-cover.

Proposition 12.8: (Second countable is Lindelöf)

A second countable space is Lindelöf.

Proof

Suppose $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ is an open cover. Fix a countable base $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$. Suppose $J \subset \mathbb{N}$ is the subset of indices for which B_i is contained in some $U_\alpha \in \mathcal{U}$. For each B_j with $j \in J$, fix some $U_{\alpha_j} \in \mathcal{U}$ with $B_j \subset U_{\alpha_j}$. Clearly $\{U_{\alpha_j}\}_{j \in J}$ is a countable collection. For any $x \in X$, we have $x \in U_\alpha$ for some $U_\alpha \in \mathcal{U}$. Now, there is some basic open set $x \in B_{i_0} \subset U_\alpha$. But then $x \in B_{i_0} \subset U_{\alpha_{i_0}}$. Thus, $\{U_{\alpha_j}\}_{j \in J}$ is a countable open cover, showing that X is Lindelöf. $\square$

Proposition 12.9: (Limit point compact, Lindelöf, T_1 is compact)

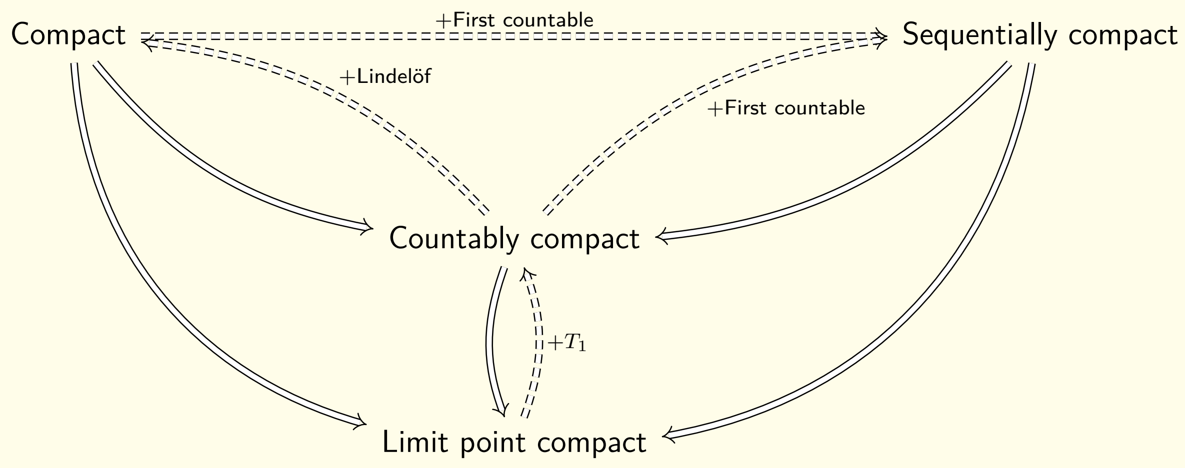
A limit point compact, T_1, Lindelöf space is compact.

Proof

A limit point compact T_1 space is countably compact. A countably compact Lindelöf space is compact. $\square$

Remark 12.10

We have observed the implications



Day 13 : 18th September, 2025

`order topology -- compact interval -- well-ordereing -- uncountable ordinal`

13.1 Order topology and compactness

Definition 13.1: (Order topology)

Given any totally ordered set $(X, \leq)$, the *order topology* on X is defined as the topology generated by the subbasis consisting of rays $\{x \in X \mid x < a\}$ and $\{x \in X \mid a < x\}$ for all $a \in X$.

Exercise 13.2: (Order topology basis)

Given a total order $(X, \leq)$ (with at least two points), check that the following collection

$$\mathcal{B} := \{(a, b) \mid a, b \in X,\ a < b\},$$

is a basis for the order topology. Here, the intervals are defined as $(a, b) := \{x \in X \mid a < x < b\}$.

Proposition 13.3: (Order topology is T_2)

Let $(X, \leq)$ be a totally ordered set equipped with the order topology. Then, X is T_2.

Proof

Let $a \neq b \in X$. Without loss of generality, $a < b$. There are two possibilities. Suppose there is some c such that $a < c < b$. Then, consider $U = \{x \in X \mid x < c\}$ and $V = \{x \in X \mid c < x\}$. Clearly, $a \in U, b \in V$ and $U \cap V = \emptyset$. If no such c exists, take $U = \{x \mid x < b\}$ and $V = \{x \mid a < x\}$. $\square$

Theorem 13.4: (Compact sets in ordered topology)

Suppose X is a totally ordered space, with the least upper bound property : any upper bounded set $A \subset X$ has a least upper bound. Then, for any $, b \in X$ with $a < b$, the interval $[a, b] = \{c \in X \mid a \leq c \leq b\}$ is compact.

Proof

Suppose $\mathcal{U} = \{U_\alpha\}$ be an open cover of $[a, b]$.

For any $x \in [a, b)$, we first observe that there is some $y \in (x, b]$ such that $[x, y]$ is covered by at most two elements of $\mathcal{U}$. If x has an immediate successor in X, let $y = x + 1$. Then, $y \in (x, b]$, and $[x, y]$ contains exactly two points. Clearly, $[x, y]$ can be covered by at most two open sets of $\mathcal{U}$. If there is no immediate successor, get $x \in U_\alpha$, and some $x < c \leq b$ such that $[x, c) \subset U_\alpha$. Since x has no immediate successor, we have some $x < y < c$ so that $[x, y] \subset [x, c) \subset U_\alpha$.

Now, consider the collection

$$\mathcal{A} := \{c \in [a, b] \mid [a, c] \text{ is covered by finitely many } U_\alpha.\}$$

Observe that for a, we have some $a < y \leq b$ such that $[a, y]$ is covered by at most two open sets of $\mathcal{U}$. Thus, $y \in \mathcal{A}$. Clearly $\mathcal{A}$ is upper bounded by b. Let c be the least upper bound of $\mathcal{A}$. We then have, $a < c \leq b$.

We show that $c \in \mathcal{A}$. We have $c \in U_\alpha$ for some α. Then, there is some c' such that $(c', c] \subset U_\alpha$. Now, being the least upper bound, we must have some $z \in \mathcal{A}$ such that $c' < z \leq c$. Then, $[a, z]$ lies in finitely many opens of $\mathcal{U}$. Adding U_α to that finite collection, we get a finite cover of $[a, c] = [a, z] \cup [z, c]$. Thus, $c \in \mathcal{A}$.
Finally, we claim that $c = b$. If not, then there is some $c < y \leq b$ such that $[c, y]$ is covered by at most two opens from $\mathcal{U}$. This implies that $[a, y] = [a, c] \cup [c, y]$ admits a finite sub-cover, and hence, $y \in \mathcal{A}$. But this contradicts c is an upper bound. Thus, $c = b$.
In other words, $[a, b]$ is covered by finitely many open sets of $\mathcal{U}$. □

Corollary 13.5: (Intervals are compact)

For any real numbers $a < b$, the interval $[a, b]$ is compact in the usual topology of real line.

Proof

It is clear that $\mathbb{R}$ is a totally ordered set, equipped with the order topology. Also, $\mathbb{R}$ has the least upper bound property. Hence, $[a, b]$ is compact. □

13.2 Well-ordering

Definition 13.6: (Well-order)

A *well-ordering* on a set X is a total order, such that every non-empty subset has a least element. Explicitly, it is a relation $\mathcal{R} \subset X \times X$, denote, $a \leq b$ if and only if $(a, b) \in \mathcal{R}$, such that the following hold.

a) **(Reflexivity)** $x \leq x$ for all $x \in X$.

b) **(Transitivity)** If $x \leq y$ and $y \leq z$, then $x \leq z$.

c) **(Totality)** For $x, y \in X$ either $x \leq y$ or $y \leq x$.

d) **(Antisymmetric)** If $x \leq y$ and $y \leq x$, then $x = y$.

e) For any $\emptyset \neq A \subset X$, there exists $a_0 \in A$ such that for all $a \in A$ we have $a_0 \leq a$. We call it *the least element* of A (which is unique, by antisymmetry)

Given a well-ordered set $(X, \leq)$, and a point $x \in X$, the *section* (or *initial segment*) is defined as $S_x := \{y \in X \mid y < x\}$.

Proposition 13.7: (Successor in well-order)

Given a well-ordering $(X, \leq)$, each $x \in X$ (except possibly the greatest element) has an immediate successor, denoted, $x + 1$. That is, $x < x + 1$, and there is no $y \in X$ such that $x < y < x + 1$.

Proof

For any $x \in X$, consider the set

$$U_x := \{y \in X \mid x < y\} .$$

If x is not the greatest element of X, then $U_x \neq \emptyset$, and hence, has a least element. This least element is the successor (Check!). □

Theorem 13.8: (Well-ordering principle)

Every set admits a well-ordering.

Remark 13.9: (Construction of uncountable well-order)

The well-ordering principle (also known as *Zermelo's theorem* named after Ernst Zermelo) is equivalent to the axiom of choice. On the other hand, explicitly constructing an uncountable well-order is possible without using the (full strength of) axiom of choice!

Theorem 13.10: (Construction of an uncountable well-order)

There exists an uncountable well-ordered set.

Proof

Consider $\mathbb{N}$ with the usual order, and observe that any subset $A \subset \mathbb{N}$ is a well-ordering with this ordering. Consider the set

$$\mathcal{A} := \{(A, \prec) \mid A \in \mathcal{P}(\mathbb{N}), \prec \text{ is a strict well-order on } A\}\,.$$

Since $\mathcal{P}(\mathbb{N})$ is uncountable, and since every subset admits at least one well-order, clearly, $\mathcal{A}$ is uncountable. Let us define a relation

$$(A, \prec_A) \sim (B, \prec_B) \Leftrightarrow ((A, \prec_A)) \text{ is order-isomorphic to } (B, \prec_B).$$

Then, $\sim$ is an equivalence relation on $\mathcal{A}$ (check!). On the equivalence classes, define a new relation

$$[A, \prec_A] \ll [B, \prec_B] \Leftrightarrow (A, \prec_A) \text{ is order-isomorphic to some section of } (B, \prec_B).$$

Then, $\ll$ is a well-defined (strict) well-ordering on $\Omega := \mathcal{A}/_\sim$ (Check! (It is tricky!)). □

Proposition 13.11: (Construction of S_Ω)

There exists a well-ordering, denoted S_Ω (or, ω_1, known as the *first uncountable ordinal*), such that

i) S_Ω is uncountable, and

ii) for each $x \in S_\Omega$ the section $S_x := \{y \in S_\Omega \mid y < x\}$ is countable.

Proof

Suppose $(A, \leq)$ is an uncountable well-ordered set. Then, on $B = A \times \{0, 1\}$, the dictionary order is again a well-ordering (check!). Observe that for any $x = (a, 1)$, the section $S_x = \{y \in B \mid y < x\}$ is uncountable. Consider the set

$$S := \{x \in B \mid S_x \text{ is uncountable}\}\,.$$

This is non-empty, and hence, admits a least element $\Omega \in S$. Denote

$$S_\Omega := \{x \in B \mid x < \Omega\}.$$

Clearly S_Ω itself is uncountable, as $\Omega \in S$. But that for any $x \in S_\Omega$, we have the section S_x is countable. Since S_Ω is a section of a well-ordering, it is itself well-ordered (check!). □

We shall denote

$$\bar{S}_\Omega := S_\Omega \cup \{\Omega\},$$

and give it the obvious ordering : for any $x \in S_\Omega$ set $x < \Omega$. Note that S_Ω is a section in $\bar{S}_\Omega$, so that the notation is consistent.

Theorem 13.12: ($\bar{S}_\Omega$ is compact)

The space $\bar{S}_\Omega = S_\Omega \cup \{\Omega\}$ is compact.

Proof

Let m_0 be the least element of S_Ω. On $\bar{S}_\Omega = S_\Omega \cup \{\Omega\}$, extend the ordering by setting $x < \Omega$ for all $x \in S_\Omega$. Observe that this is a total order. And moreover, $\bar{S}_\Omega = [m_0, \Omega]$ is a closed interval. Let us check the least upper bound property. Say $A \subset \bar{S}_\Omega$. If $\Omega \in A$, then clearly, Ω is the least upper bound of A. WLOG, assume $\Omega \notin A$, that is, $A \subset S_\Omega$. We have two possibilities. If A is bounded in S_Ω, consider the set

$$X = \{b \in S_\Omega \mid b \text{ is an upper bound of } A\}.$$

As X is nonempty, there exists a least element, say, $b_0 \in X$. By definition, it is the least upper bound of A. Suppose A is unbounded in S_Ω. Clearly, Ω is an upper bound of A. We claim that Ω is the least upper bound. If not, then there is some upper bound $x < \Omega$, which implies A is bounded by $x \in S_\Omega$, a contradiction. Thus, $\bar{S}_\Omega$ has the least upper bound property. So, $\bar{S}_\Omega$ is compact. □

Day 14 : 19th September, 2025

`uncountable ordinal -- filter -- ultrafilter lemma -- Tychonoff's theorem`

14.1 Properties of S_Ω

Proposition 14.1: (Properties of S_Ω)

Suppose S_Ω is given the order topology.

a) For any set $A \subset S_\Omega$, the union $\bigcup_{a \in A} S_a$ is either a section (and hence countable), or all of S_Ω.

b) Any countable set of S_Ω is bounded

c) S_Ω is sequentially compact.

d) S_Ω is limit point compact.

e) S_Ω is not compact.

f) S_Ω is first countable.

Proof

a) If A admits an upper bound, then it admits a least upper bound, say, b. We claim that $\bigcup_{a\in A} S_a = S_b$. Indeed, for any $x < a \in A$, we have $x < a \leq b$ and so $x \in S_b$. On the other hand, for any $x < b$, we have x is not an upper bound of A, and so, $x < a \leq b$ for some $a \in A$. Then, $x \in S_a$.

Otherwise, assume A is not bounded. Suppose $\bigcup_{a\in A} S_a$ is not all of S_Ω. Pick some $b \in S_\Omega \setminus \bigcup_{a\in A} S_a$. Now, b is not an upper bound of A (as A is not upper bounded). So, $b < a \in A$. But then $b \in S_a$, a contradiction.

b) For a countable set $A \subset S_\Omega$, the subset $\bigcup_{a\in A} S_{a+1}$ is countable, and hence, not all of S_Ω. Then, $A \subset \bigcup_{a\in A} S_{a+1} = S_b$ for some b. Clearly, b is an upper bound of A.

c) WLOG, suppose $\{x_n\}$ be a sequence of distinct elements in S_Ω. Consider

$$x_{n_k} = \min \{x_n \mid n \geq k\}.$$

Then, clearly $x_{n_1} < x_{n_2} < \dots$. Now, $\{x_{n_k}\}$ being countable set, is bounded, and hence admits a least upper bound, say b. Clearly $b \notin \{x_{n_k}\}$, as the subsequence is strictly increasing. For any open set $b \in U \subset S_\Omega$, we have $b \in (x, b] \subset U$. Now, x is not an upper bound of $\{x_{n_k}\}$, and hence, $a < x_{n_{k_0}} < b$ for some k_0. But then $a < x_{n_l} < b$ for any $l \geq k_0$. In other words, $x_{n_l} \in U$ for all $l \geq k_0$. Thus, $x_{n_k} \to b$.

d) Since S_Ω is sequentially compact, it is limit point compact.

e) For each $x \in S_\Omega$, consider the open sections $S_{x+1} := \{y \in X \mid y < x+1\}$, which are open. Here $x+1$ is the successor of x. Clearly, $S_\Omega = \bigcup_{x\in S_\Omega} S_{x+1}$. If possible, suppose, there is a finite subcover, $S_\Omega = \bigcup_{i=1}^n S_{x_i+1}$. But the right-hand side is a finite union of countable sets, and hence countable, whereas S_Ω is uncountable. This is a contradiction.

f) For any $x \in S_\Omega$, we have the section $S_x = \{a \mid a < x\}$ is countable. Consider the open sets $\{U_a = (a, x+1) \mid a < x\}$, which are all open neighborhoods of x. It is clear that this is a countable basis at x (Check!).

□

Proposition 14.2: ($\bar{S}_\Omega$ is not first countable)

The space $\bar{S}_\Omega = S_\Omega \cup \{\Omega\}$ is not first countable at Ω.

Proof

Observe that the basic open sets containing Ω are of the form $(x, \Omega]$ for $x \in S_\Omega$. If possible, suppose, there is countable neighborhood basis at Ω, say, $\{U_i\}$. We then have $\Omega \subset (x_i, \Omega] \subset U_i$ for some $x_i \in S_\Omega$. Now, $\bigcup S_{x_i} = S_b$ for some $b \in S_\Omega$. Consider the basic open set $(b+1, \Omega]$. There is some $\Omega \in (x_i, \Omega] \subset U_i \subset (b+1, \Omega]$. But then $b+1 \leq x_i$, a contradiction. Hence, $\bar{S}_\Omega$ is not first countable at Ω. ☐

14.2 (Ultra)Filters

Definition 14.3: (Filter and ultrafilter)

Given a set X, a *filter* on it is a collection $\mathcal{F} \subset \mathcal{P}(X)$ of subsets such that the following holds.

a) $\emptyset \notin \mathcal{F}$.

b) For any $A, B \subset X$, we have $A \cap B \in \mathcal{F}$ if and only if $A, B \in \mathcal{F}$.

A filter $\mathcal{F}$ on a set X, is called an *ultrafilter* if for any $A \subset X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Exercise 14.4: (Filter equivalent definition)

Given any collection $\mathcal{F} \subset \mathcal{P}(X)$ of subsets, the following are equivalent.

a) For any $A, B \subset X$, we have $A \cap B \in \mathcal{F}$ if and only if $A, B \in \mathcal{F}$

b) $\mathcal{F}$ satisfies the following.

   i) $\mathcal{F}$ is closed under finite intersection, i.e, $F_1, \ldots, F_n \in \mathcal{F}$ implies $\cap_{i=1}^n F_n \in \mathcal{F}$.

   ii) $\mathcal{F}$ is closed under supersets, i.e, if $A \in \mathcal{F}$, then $B \in \mathcal{F}$ whenver $B \supset A$.

Example 14.5: (Principal ultrafilter)

For any $x \in X$ fixed, consider the collection

$$\mathcal{F} = \{A \subset X \mid x \in A\}.$$

It is easy to see that $\mathcal{F}$ is an ultrafilter on X, Such ultrafilters are called the *principal ultrafilter*. Any ultrafilter which is not principal, is called a *free ultrafilter*.

Theorem 14.6: (Ultrafilter lemma)

Every filter on a set X is contained in an ultrafilter.

Proof

Let $\mathcal{F}$ be a filter on X. Consider the collection

$$\mathfrak{F} := \{\mathcal{G} \mid \mathcal{G} \text{ is a filter on } X, \text{ and } \mathcal{F} \subset \mathcal{G}\}.$$

It follows that every chain (ordered by inclusion) in $\mathfrak{F}$ admits a maximal element, given by the union. Then, by Zorn's lemma, $\mathfrak{F}$ admits a maximal element, say, $\overline{\mathcal{F}}$. Since $\overline{\mathcal{F}}$ is a maximl filter, it is an ultrafilter, which contains $\mathcal{F}$ by construction. ☐

Definition 14.7: (Convergence of filter)

Given a filter $\mathcal{U}$ on a space X, we say $\mathcal{U}$ converges to a point $x \in X$, if for any open neighborhood $x \in U$, we have $U \in \mathcal{U}$.

Theorem 14.8: (Ultrafilter and compactness)

A space X is compact if and only if every ultrafilter on X converges to at least one point.

Proof

Suppose X is a compact space. Let $\mathcal{U}$ be an ultrafilter on X. If possible, suppose $\mathcal{U}$ does not converge to any point in X. Then, for each $x \in X$, there exists an open nbd $x \in U_x$ such that $U_x \notin \mathcal{U}$. Since $\mathcal{U}$ is ultrafilter, this means $X \setminus U_x \in \mathcal{U}$. Now, $X = \bigcup_{x \in X} U_x$ admits a finite sub-cover, say, $X = \bigcup_{i=1}^k U_{x_i}$. This, means

$$\emptyset = X \setminus X = \bigcap_{i=1}^k (X \setminus U_{x_i}) \in \mathcal{U},$$

as $\mathcal{U}$ is closed under finite intersection. This is a contradiction as $\emptyset \notin \mathcal{U}$.
Conversely, suppose X is not compact. Then, there exists an open cover, $\mathcal{U} = \{U_\alpha\}$ such that there is no finite sub-cover. Consider the collection

$$\mathcal{F} := \{F_\alpha = X \setminus U_\alpha\}.$$

Note that for any finite collection, we have $\cap_{i=1}^k F_{\alpha_i} = X \setminus \bigcup_{i=1}^k U_{\alpha_i} \neq \emptyset$. In other words, $\mathcal{F}$ has finite intersection property. Then, we can close $\mathcal{F}$ under finite intersections, and then under supersets, to get a filter, say, $\mathfrak{F} \supset \mathcal{F}$. But $\mathfrak{F}$ is contained in some ultrafilter, say $\mathfrak{U} \supset \mathfrak{F}$. Now, for any $x \in X$, we have $X \in U_\alpha$ for some α. Then, $F_\alpha = X \setminus U_\alpha \in \mathfrak{U} \Rightarrow U_\alpha \notin \mathfrak{U}$. Thus, $\mathfrak{U}$ does not converge to any $x \in X$, a contradiction. □

14.3 Tychonoff's Theorem

Theorem 14.9: (Tychonoff's Theorem)

Given a collection $\{X_\alpha\}$ of compact spaces, the product $X = \Pi X_\alpha$, with the product topology, is a compact space.

Proof

Suppose $\mathcal{U}$ is an ultrafilter on X. For the projection map $\pi_\alpha : X \to X_\alpha$, we have the ultrafilter

$$\mathcal{U}_\alpha := (\pi_\alpha)_* \mathcal{U} = \left\{ A \subset X_\alpha \,\middle|\, (\pi_\alpha)^{-1}(A) \in \mathcal{U} \right\}$$

on X_α. Since X_α is compact, $\mathcal{U}_\alpha$ converges to some point in X_α. By the axiom of choice, we have some $x = (x_\alpha) \in X$ such that $\mathcal{U}_\alpha$ converges to x_α for each α. Let us show that $\mathcal{U}$ converges to x. Observe that for any open neighborhood $x \in U \subset X$, we have U is generated by the sub-basic open sets of the form $\{\pi_\alpha^{-1}(V) \mid V \subset X_\alpha\}$. Since a filter is closed under finite intersection and supersets, if we are able to show that any sub-basic open neighborhood of x is an element of $\mathcal{U}$, we are done. But for any $V \subset X_\alpha$ open, with $x \in \pi_\alpha^{-1}(V)$ precisely when $x_\alpha \in V$. Since $\mathcal{U}_\alpha$ converges to x_α, we have $V \in \mathcal{U}_\alpha \Rightarrow \pi_\alpha^{-1}(V) \in \mathcal{U}$. Hence, $\mathcal{U}$ converges to x. Since $\mathcal{U}$ is an arbitrary ultrafilter, we have X is compact. $\square$

Proposition 14.10: (Axiom of choice from Tychonoff)

Suppose Tychonoff's theorem is true. Then, axiom of choice holds.

Proof

Let $\{X_\alpha\}$ be an arbitrary collection nonempty sets. Since a set cannot be an element of itself, we have new sets $Y_\alpha = X_\alpha \sqcup \{X_\alpha\}$. For simplicity, denote $p_\alpha = \{X_\alpha\} \in Y_\alpha$. Now, give a topology on Y_α as

$$\mathcal{T}_\alpha = \{\emptyset, \{p_\alpha\}, X_\alpha, Y_\alpha\}$$

. Clearly $(Y_\alpha, \mathcal{T}_\alpha)$ is a compact space, having only finitely many open sets. Consider the product $Y = \Pi_\alpha Y_\alpha$. Now, for each α, we have the sub-basic open set

$$U_\alpha := \{y \in Y \mid \pi_\alpha(y) = p_\alpha\} = \pi_\alpha^{-1}(p_\alpha),$$

since $\{p_\alpha\}$ is open in Y_α. We claim that $\{U_\alpha\}$ has not finite sub-cover. If possible, suppose, $Y = \bigcup_{i=1}^n U_{\alpha_i}$. Then, make finitely many choices : $x_i \in X_{\alpha_i}$, and define x by setting $\pi_\alpha(x) = p_\alpha$ for $\alpha \notin \{a_1, \dots, \alpha_n\}$ and $\pi_{\alpha_i}(x) = x_i$ for $1 \le i \le n$. Then, clearly $x \notin \bigcup_{i=1}^n U_{\alpha_i}$, a contradiction. Thus, the collection $\{U_\alpha\}$ admits no finite sub-cover. By Tychonoff's theorem, Y is compact. Hence, $\{U_\alpha\}$ is not a covering of Y. So, there exists some $y \in Y \setminus \bigcup_\alpha U_\alpha$. Observe that $\pi_\alpha(y) \in X_\alpha$, as $y_\alpha \neq p_\alpha$. Thus, $y \in \Pi X_\alpha$. This is precisely the axiom of choice. $\square$

Proposition 14.11: (Compact but not sequentlly compact)

The product space $X = [0,1]^{[0,1]} = \Pi_{0 \le t \le 1}[0,1]$ is compact, but not sequentially compact.

Proof

It follows from Tychonoff's theorem that the product space $X = [0,1]^{[0,1]}$ is compact, since each $[0,1]$ is so. For each $n \ge 1$, consider the function $\alpha_n : [0,1] \to \{0,1\}$ defined by

$$\alpha_n(x) = \text{the } n^{\text{th}} \text{ digit in the binary expansion of } x.$$

Clearly, $\{\alpha_n\}$ is a sequence in X. If possible, suppose, $\alpha_{n_k} \to \alpha \in X$. Then, for each $x \in [0,1]$, we must have $\alpha_{n_k}(x) \to \alpha(x)$. Consider any point x such that $\alpha_{n_k}(x)$ is 0 or 1 according as k is even or odd. Clearly the sequence $\alpha_{n_k}(x)$ cannot converge, a contradiction. Thus, X is not sequentially compact. $\square$

Day 15 : 25th September, 2025

`Zorn's lemma -- well-ordering principle -- ultrafilter lemma`

15.1 A digression : Zorn's Lemma and applications

Definition 15.1: (Partial ordering)

A relation $\leq$ on a set X is called a *partial order* if it satisfies the following.

1. $x \leq x$ for all $x \in X$.
2. $x \leq y, y \leq z \Rightarrow x \leq z$
3. $x \leq y, y \leq x \Rightarrow x = y$

The tuple $(X, \leq)$ is called a *partially ordered set* (or a *poset*). A point $x \in X$ is called a *maximal element* if for any $y \in X$ with $x \leq y$, we have $x = y$.

Definition 15.2: (Chain)

A subset C of a poset $(X, \leq)$ is called a *chain* if C is totally ordered with respect to $\leq$, i.e, for any $c_1, c_2 \in C$, either $c_1 \leq c_2$ or $c_2 \leq c_1$ holds.

Lemma 15.3: (Zorn's lemma)

Given a non-empty poset $(X, \leq)$, suppose every chain has an upper bound in X. Then, X has a maximal element.

Theorem 15.4: (Basis of a vector space)

Given a field $\mathbb{K}$, any non-zero vector space V over $\mathbb{K}$ admits a basis.

Proof

Consider the collection

$$\mathcal{B} := \{B \subset V \mid B \text{ is linearly independent over } \mathbb{K}\}.$$

Note that $\mathcal{B} \neq \emptyset$, since for any $0 \neq v \in V$, we have $B = \{v\} \in \mathcal{B}$. Define

$$B_1 \leq B_2 \Leftrightarrow B_1 \subset B_2, \qquad B_1, B_2 \in \mathcal{B}$$

which is clearly a partial order. Let us consider a chain $\mathcal{C} = \{B_i\}_{i \in I}$ in $(\mathcal{B}, \leq)$. Consider the set $B = \bigcup_{i \in I} B_i$. We check that B is linearly independent. Say, $b_1, \ldots, b_k \in B$. Since $\mathcal{C}$ is a chain, without loss of generality, we have $b_1, \ldots, b_k \in B_{i_0}$ for some $i_0 \in I$. But then clearly $\{b_1, \ldots, b_k\}$ is linearly independent. Hence, $B \in \mathcal{B}$. By construction, we have $B_i \leq B$ for all $i \in I$. Thus, B is an upper bound of $\mathcal{C}$. Then, we have a maximal element, say, $\mathfrak{B} \in \mathcal{B}$. We claim that $\mathfrak{B}$ is a basis of V. If not, then $\mathfrak{B}$ fails to span V. Thus, we must have some

$$v_0 \in V \setminus \operatorname{Span} \langle \mathfrak{B} \rangle.$$

Consider the set $\mathfrak{B}_0 = \mathfrak{B} \sqcup \{v_0\}$. Clearly, $\mathfrak{B}_0$ is linearly independent, and $\mathfrak{B} \subsetneq \mathfrak{B}_0$. Thus contradicts the maximality of $\mathfrak{B}$. Hence, $V = \operatorname{Span} \langle \mathfrak{B} \rangle$. Thus, V admits a basis. □

Theorem 15.5: (Well-ordering principle)

Every nonempty set S admits a well-ordering.

Proof

Consider the collection

$$\mathcal{W} = \{(W, \leq_W) \mid \emptyset \neq W \subset S, \text{ and } \leq_W \text{ is a well-ordering on } W\}.$$

Clearly $\mathcal{W} \neq \emptyset$, since for any $x \in S$, we have the singleton set $\{x\}$ is trivially well-ordered. Let us define $(A, \leq_A) \preceq (B, \leq_B)$ if and only if

i) $A \subset B$,

ii) $\leq_A$ is the restriction of $\leq_B$ (i.e, $a_1 \leq_A a_2$ if and only if $a_1 \leq_B a_2$), and

iii) for any $b \in B \setminus A$ we have $b >_B a$ for all $a \in A$.

It is easy to see that $\preceq$ is a total order on $\mathcal{W}$ (Check!). Suppose $\mathcal{C} = \{(W_\alpha, \leq_\alpha)\}_{\alpha \in I}$ is a chain in $(\mathcal{W}, \preceq)$. Consider

$$W = \bigcup_{\alpha \in I} W_\alpha.$$

Let us define $\leq_W$ as follows. For any $w_1, w_2 \in W$, using the chain condition, we have $w_1, w_2 \in W_{\alpha_0}$ for some $\alpha_0 \in I$. Then, define

$$w_1 \leq_W w_2 \Leftrightarrow w_1 \leq_{\alpha_0} w_2.$$

Again from the chain condition, it follows that $\leq_W$ is well-defined (Check!). Moreover, it is easy to see that $\leq_W$ is a total order (Check!). Let us show that $\leq_W$ is actually a well-order. Say, $\emptyset \neq A \subset W$ is given. Then, $A \cap W_\alpha \neq \emptyset$ for some $\alpha \in I$. Now, $(W_\alpha, \leq_\alpha)$ being a well-order, we have a least element $m_0 = \min A \cap W_\alpha$. We claim that m_0 is the least element of A in the order $\leq_W$. If not, then there is some $a \in A$, with $a <_W m_0$. Now, $a \in W_\beta$ for some $\beta \in I$. From the chain condition, we have two cases.

1. If $W_\beta \leq W_\alpha$, then we have $a \in W_\beta \subset W_\alpha$. But then $a \in W_\alpha \cap A \Rightarrow m_0 \leq_\alpha a \Rightarrow m_0 \leq_W a$, a contradiction.

2. Say, $W_\alpha \leq W_\beta$. We again have two possibilities.

   (a) Say, $a \in W_\beta \setminus W_\alpha$. Then, by the definition of $\preceq$, we have $a \geq_\beta x$ for all $x \in W_\alpha$. In particular, $a \geq_\beta m_0 \Rightarrow a \geq_W m_0$, a contradiction.

   (b) Say, $a \in W_\alpha$. But then $m_0 \leq_\alpha a \Rightarrow m_0 \leq_W a$, again a contradiction.

Thus, it follows that $m_0 = \min A$ in the order $\leq_W$. Thus, $(W, \leq_W) \in \mathcal{W}$. Clearly, it is an upper bound of the chain $\mathcal{C}$ (Check!). Now, by Zorn's lemma, there exists a maximal element, say, $(\mathfrak{W}, \leq_{\mathfrak{W}}) \in \mathcal{W}$. We claim that $\mathfrak{W} = S$. If not, then there exists $x \in S \setminus \mathfrak{W}$. Consider

$$\mathfrak{W}_0 = \mathfrak{W} \sqcup \{x\}.$$

Define an order $\leq_0$ on $\mathfrak{W}_0$ by extending the order $\leq_{\mathfrak{W}}$, and declaring $w <_0 x$ for all $w \in \mathfrak{W}$. Then, $(\mathfrak{W}_0, \leq_0)$ is a well-order, which moreover satisfies $(\mathfrak{W}, \leq_{\mathfrak{W}}) \prec (\mathfrak{W}_0, \leq_0)$ (Check!). This violates the maximality. Hence, $\mathfrak{W} = S$, and thus, S admits a well-ordering. $\square$

Theorem 15.6: (Ultrafilter lemma)

A filter $\mathcal{F}$ on a set X is contained in an ultrafilter on X.

Proof

Consider the collection

$$\mathfrak{F} := \{F \mid F \text{ is a filter on } X, \text{ and } \mathcal{F} \subset F.\}$$

Then, $\mathfrak{F} \neq \emptyset$ as $\mathcal{F} \in \mathfrak{F}$. Order $\mathfrak{F}$ by inclusion, i.e, $F_1 \leq F_2$ if and only if $F_1 \subset F_2$. Clearly $(\mathfrak{F}, \leq)$ is a poset. Consider a chain $\mathcal{C} = \{F_i\}_{i \in I}$ in $(\mathfrak{F}, \leq)$. Consider

$$F = \bigcup_{i \in I} F_i.$$

Clearly $\mathcal{F} \subset F$. Let us check that F is a filter on X.

i) Since $\emptyset \notin F_i$ for all $i \in I$, we have $\emptyset \notin F$.

ii) For any $A, B \in F$, by the chain condition, we have $A, B \in F_{i_0}$ for some $i_0 \in I$. But then $A \cap B \in F_{i_0} \Rightarrow A \cap B \in F$.

iii) Say $A \in F$, and $B \supset A$. Now, $A \in F_i$ for some $i \in I$, and then, $B \in F_i \Rightarrow B \in F$.

Thus, F is a filter on X, containing $\mathcal{F}$, and clearly, it is an upper bound of $\mathcal{C}$. Then, by Zorn's lemma, there exists some maximal element, say, $\mathcal{U} \in \mathfrak{F}$. We claim that $\mathcal{U}$ is an ultrafilter on X, which evidently contains $\mathcal{F}$. If not, then there exists some set $S \subset X$ such that

$$S \notin \mathcal{U}, \qquad \text{and} \qquad X \setminus S \notin \mathcal{U}.$$

Then, the collection $\mathcal{U}_0 = \mathcal{U} \cup \{S\}$ has finite intersection property (Check!). But then there is a filter, say, $\mathcal{F}_0 \supset \mathcal{U}_0 \supsetneq \mathcal{U}$, a contradiction to maximality. Hence, $\mathcal{U}$ is an ultrafilter, containing $\mathcal{F}$.$\square$

Here are some more applications, that you can try to do if you want! Or have a look at this note by Keith Conrad.

Exercise 15.7: (Existence of spanning tree)

Using Zorn's lemma, show that every connected (undirected) graph has a spanning tree.

Exercise 15.8: (Existence of maximal ideal)

Let R be a commutative ring with 1. Using Zorn's lemma, show that every ideal $I \subset R$ is contained in a maximal ideal.

Exercise 15.9: (Description of nilradical)

Let R be a commutative ring with 1. Using Zorn's lemma, show that

$$\bigcap_{\mathfrak{p} \subset R \text{ is a prime ideal}} = \{x \in R \mid x^n = 0 \text{ for some } n \geq 1\},$$

which is also known as the *nilradical* of R.

Day 16 : 26^{th} September, 2025

`locally compact space -- compactification`

16.1 Local compactness

Definition 16.1: (Neighborhood)

Given a space X, a *neighborhood* of a point $x \in X$ is any set $N \subset X$ such that $x \in \text{int}N \subset N$.

Definition 16.2: (Locally compact space)

A space X is called *locally compact at $x \in X$* if for any given open nbd $x \in U$, there exists a compact neighborhood $x \in C \subset U$. The space X is called *locally compact* if it is so at every point $x \in X$.

Proposition 16.3: (Locally compact Hausdorff)

Suppose X is a Hausdorff space. Then the following are equivalent.

a) X is locally compact.

b) For any $x \in X$ and any open nbd $x \in U \subset X$, there exists an open nbd $x \in V \subset U \subset X$, such that $\bar{V} \subset U$ and $\bar{V}$ is compact.

c) Every $x \in X$ has a cpt nbd.

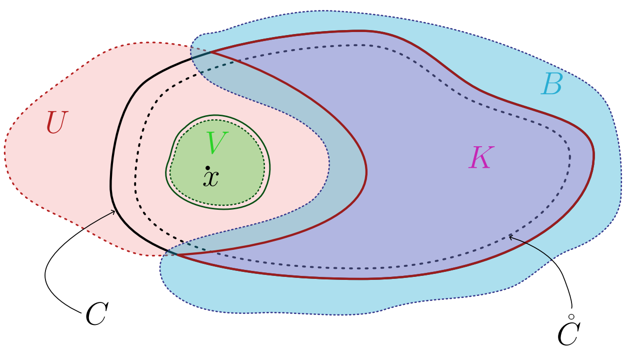
Proof

That b) implies local compactness is clear, even without the Hausdorff assumption. Now, suppose X is locally compact, T_2. For an open nbd $x \in U \subset X$, we have some compact nbd $x \in C \subset U$. By the definition of nbd, we have some open nbd $x \in V \subset C \subset U$. Now, since X is T_2, we have C is closed. Hence,

$$V \subset C \Rightarrow \bar{V} \subset \bar{C} = C \subset U.$$

Also, closed subsets of compact is always compact. Thus, $\bar{V}$ is compact. Thus, a) implies b). Again a) $\Rightarrow$ c) is clear from the definition. Suppose c) holds. Let $x \in U \subset X$ be an open nbd, and $x \in C \subset X$ be a compact nbd. Clearly $x \in W = U \cap \text{int}(C)$ is an open nbd. It follows that $K = C \setminus W$ is a closed subset of the compact set C, and hence, K is compact. Now, $x \notin K$. Since X is T_2, we have open sets $x \in A, K \subset B$, such that $A \cap B = \emptyset$ Set $V = W \cap A = U \cap \text{int}(C) \cap A$, which is an open nbd $x \in V \subset U$. We observe

$$V \subset W \subset C \Rightarrow \bar{V} \subset \bar{C} = C.$$



Consequently, $\bar{V}$ is compact, being a closed subset of a compact set. Also, $V \subset A$ and $\bar{A} \cap B = \emptyset$ (as $A \cap B = \emptyset$, and B is open). Thus,

$$\bar{V} \subset C \cap (X \setminus B) = C \setminus B = (K \sqcup W) \setminus B = W \setminus B \subset W \subset U.$$

This proves b), and hence a). ☐

Example 16.4: ($\mathbb{R}$ is locally compact)

Since $\mathbb{R}$ is Hausdorff, it is enough to check that for any $x \in \mathbb{R}$, we have $[x-1, x+1]$ is a compact nbd. Similarly, any $\mathbb{R}^n$ is also locally compact. As for $\mathbb{Q} \subset \mathbb{R}$, for any open set $U = (-\epsilon, \epsilon) \cap \mathbb{Q}$ it follows that $\bar{U} = [-\epsilon, \epsilon] \cap \mathbb{Q}$ is not compact, as it is not sequentially compact. Thus, $\mathbb{Q}$ (which is T_2) is not locally compact.

16.2 Compactification

Definition 16.5: (Compactification)

Given a space X, a *compactification* of X is a continuous injective map $\iota : X \hookrightarrow \hat{X}$, such that $\hat{X} = \overline{\iota(X)}$ is a compact space. We shall identify $X \subset \hat{X}$ as a subspace, and understand $\hat{X}$ as the compactification.

Example 16.6: (Compactification of compact space)

Suppose X is compact. Then $\mathrm{Id} : X \to X$ is trivially a compactification. In fact, if $\hat{X}$ is a Hausdorff compactification of X, then necessarily $\hat{X} = X$ (Check!).

Proposition 16.7: (Alexandroff compactification)

Given any noncompact space $(X, \mathcal{T})$, there exists a compactification $\hat{X} = X \sqcup \{\infty\}$, where ∞ is a point not in X (also denoted as $X^\star$).

Proof

Consider the space $\hat{X} = X \sqcup \{\infty\}$, along with the topology

$$\mathcal{T}_\infty := \mathcal{T} \cup \{\{\infty\} \cup (X \setminus C) \mid C \subset X \text{ is closed and compact}\}.$$

Let us verify that $\mathcal{T}_\infty$ is a topology.

i) $\emptyset \in \mathcal{T} \subset \mathcal{T}_\infty$

ii) $\hat{X} = \{\infty\} \cup (X \setminus \emptyset) \in \mathcal{T}_\infty$, since $\emptyset \subset X$ is a closed, compact subset.

iii) For any $U_\alpha = \{\infty\} \cup (X \setminus C_\alpha)$, where $C_\alpha \subset X$ is closed compact, we have $\bigcup U_\alpha = \{\infty\} \cup (X \setminus \bigcap_\alpha C_\alpha)$. Since arbitrary intersection of closed is closed, and arbitrary intersection of compact is compact, we have $\bigcap_\alpha C_\alpha \subset X$ is closed, compact. Thus, $\bigcup_\alpha U_\alpha \in \mathcal{T}_\infty$. Since finite union of closed (resp. compact) sets are closed (resp. compact), we see that $\bigcap_{i=1} U_i \in \mathcal{T}_\infty$, if $U_i = \{\infty\} \cup (X \setminus C_i)$ for some $C_i \subset X$ closed, compact.

iv) Since $\mathcal{T}$ is a topology, it is closed under arbitrary union and finite intersection.

v) Finally, let us consider some $U \subset X$ open, and some $V = \{\infty\} \cup (X \setminus C)$ for $C \subset X$ closed, compact. We have $U \cap V = U \setminus C$, which is open in X. Also,

$$U \cup V = \{\infty\} \cup (X \setminus C) \cup U = \{\infty\} \cup (X \setminus (C \setminus U)) .$$

Since $C \setminus U$ is a closed subset of a compact set, it is again closed, compact. Thus, $U \cap V \in \mathcal{T}_\infty$.

Thus, $\mathcal{T}_\infty$ is indeed a topology. It is easy to see that the inclusion $\iota : X \hookrightarrow \hat{X}$ is a homeomorphism onto the image (Check!). Also, for ∞, any open neighborhood clearly intersects X, since X itself is not compact. Thus, $\hat{X} = \overline{\iota(X)}$. Finally, let us check that $\hat{X}$ is compact. Indeed, for any open cover $\mathcal{U} = \{U_\alpha\}$, choose some $\infty \in U_{\alpha_0}$. Then, $U_{\alpha_0} = \{\infty\} \cup (X \setminus C)$, where $C \subset X$ is closed and compact. We have $\mathcal{U}$ is an open cover of X, and so, we have a finite subcover, say $C \subset \bigcup_{i=1}^k U_{\alpha_i}$. Then, $\{U_{\alpha_i},\ i = 0, \dots, k\}$ is a finite subcover of $\hat{X}$. ☐

Remark 16.8: (Alexandroff compactification of compact space)

If X is compact to begin with, then the Alexandroff compactification still produces a compact space $\hat{X} = X \sqcup \{\infty\}$, which contains X as a subspace. But here $\{\infty\}$ is an isolated point, and $\bar{X} = X \subsetneq \hat{X}$. Thus, by our definition, it is not exactly a compactification!

Exercise 16.9: (One-point compactification and Alexandroff compactification)

Consider the space

$$X = \{p, q, x_1, x_2, \dots, y_1, y_2, \dots\} .$$

Give the subspace $\{x_1, x_2, \dots, y_1, y_2, \dots\}$ the discrete topology. For p, declare the open neighborhoods as $\{p\} \cup A$, where $A \subset \{y_1, y_2, \dots\}$ is cofinite. For q, declare the open neighborhoods as $\{q\} \cup B$, where $B \subset \{x_1, x_2, \dots, y_1, y_2, \dots\}$ is cofinite. Check that X is compact with this topology. Now, consider $Y = \{p, x_1, x_2, \dots, y_1, y_2, \dots\}$, which is *noncompact* (Check!). Clearly, $\overline{Y} = X$. Thus, X is a compactification of Y. We claim that X is not the Alexandroff compactification of Y. Indeed, consider the set $K = \{p, y_1, y_2, \dots\} \subset Y$, which is compact (Check!). Also, K is closed in Y. But, $\{q\} \cup (Y \setminus K) = \{q, x_1, x_2, \dots\}$ is not open in X.

Theorem 16.10: (One-point compactification of locally compact Hausdorff space)

Let X be a noncompact space. Then, the one-point compactification $\hat{X}$ is T_2 if and only if X is locally compact, T_2.

Proof

Suppose $\hat{X}$ is T_2. Then, $X \subset \hat{X}$ is clearly T_2. Also, for any $x \in X$, we have open sets $x \in U, \infty \in V$ such that $U \cap V = \emptyset$. Then, $U \subset X$, and $V = \{\infty\} \cup (X \setminus C)$, where $C \subset X$ is a compact (and also closed, as X is T_2). Then, $x \in U \subset C$, that is, C is a compact neighborhood of x. Since X is T_2, it follows that X is locally compact.
Conversely, suppose X is locally compact, T_2. We only need to show that for any $x \in X$, there open sets $x \in U, \infty \in V$ such that $U \cap V = \emptyset$. Since X is T_2, we have an open set $x \in U \subset X$ such that $\bar{U}$ is compact (and hence closed). Then, we have $V = X \setminus \bar{U}$ is an open nbd of ∞ in $\hat{X}$. Clearly, $U \cap V = \emptyset$. Thus, $\hat{X}$ is T_2. ☐

Day 17 : 16^{th} October, 2025

`properties of Lindelöf spaces -- separable spaces`

17.1 Properties of Lindelöf spaces

Proposition 17.1: (Image of Lindelöf spaces)

A continuous image of a Lindelöf space is again Lindelöf

Proof

Suppose $f : X \to Y$ is a continuous surjection, and X is Lindelöf. Consider an open cover $Y = \bigcup_\alpha V_\alpha$. Then, we have an open cover $X = \bigcup_\alpha f^{-1}(U_\alpha)$, which admits a countable sub-cover, $X = \bigcup_{i=1}^\infty f^{-1}(U_{\alpha_i})$. Then, $Y = f(X) = \bigcup_{i=1}^\infty U_{\alpha_i}$. Thus, Y is Lindelöf. □

Lindelöf spaces are not well-behaved when considering product or subspaces.

Example 17.2: ($\mathbb{R}_\ell$ is Lindelöf)

Let us show that the lower limit topology $\mathbb{R}_\ell$ on $\mathbb{R}$ is Lindelöf. Suppose $\{U_\alpha\}$ is an open cover. For each x, we have $[x, r_x) \subset U_{\alpha_x}$, for some $r_x \in \mathbb{Q}$. Clearly, $\mathbb{R}_\ell = \bigcup_x [x, r_x)$. Let us consider the space $C = \bigcup_x (x, r_x)$. We claim that $\mathbb{R} \setminus C$ is countable. Indeed, for each $u, v \in \mathbb{R} \setminus C$, with $u < v$, we must have $r_u < r_v$, since otherwise we get $u < v < r_v \leq r_u$ and then, $v \in (u, r_u) \subset C$ a contradiction. Thus, we have an injective map

$$\begin{aligned} \mathbb{R} \setminus C &\to \mathbb{Q} \\ u &\mapsto r_u. \end{aligned}$$

But then $\mathbb{R} \setminus C$ is countable, as $\mathbb{Q}$ is countable. Say, $\mathbb{R} \setminus C = \{u_i\}_{i=1}^\infty$. On the other hand, considering $C = \bigcup_{x \in \mathbb{R}} (x, r_x)$ as a collection of open sets in the usual topology of $\mathbb{R}$, we have a countable subcover $C = \bigcup_{i=1}^\infty (x_i, r_{x_i})$. Thus, we have a countable cover,

$$\mathbb{R}_\ell = \bigcup_{i=1}^\infty [u_i, r_{u_i}) \cup \bigcup_{i=1}^\infty [x_i, r_{x_i}) \subset \bigcup U_{\alpha_{u_i}} \cup \bigcup U_{\alpha_{x_i}}.$$

Hence, $\mathbb{R}_\ell$ is Lindelöf.

Example 17.3: ($\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not Lindelöf)

Let us now show that the product $X = \mathbb{R}_\ell \times \mathbb{R}_l$ (also known as *Sorgenfrey plane*) is not Lindelöf. Consider the subset $A = \{(x, -x) \mid x \in \mathbb{R}\} \subset X$. It is easy to see that A is open. Next, for each $x \in \mathbb{R}$, consider the open set $U_x = [x, x+1) \times [-x, -x+1) \subset X$. It follows that $A \cap U_x = \{(x, -x)\}$. Now, consider the open cover

$$X = (X \setminus A) \cup \bigcup_{x \in \mathbb{R}} U_x.$$

This cannot have a countable subcover, since A is uncountable.

Definition 17.4: (Hereditarily Lindelöf)

A space X is called *hereditarily Lindelöf* if every subspace $A \subset X$ is Lindelöf.

Proposition 17.5: (Hereditarily Lindelöf if and only if open subsets are Lindelöf)

A space X is hereditarily Lindelöf if and only if every open subspace $U \subset X$ is Lindelöf.

Proof

One direction is trivial. So, suppose every open subspace of X is Lindelöf. Consider an arbitrary subset $A \subset X$, with the subspace topology. Suppose, we have an open cover $A = \bigcup_\alpha U_\alpha$, where $U_\alpha = A \cap V_\alpha$ for $V_\alpha \subset X$ open. Now, $U = \bigcup_\alpha V_\alpha$ is a open cover, which admits a countable subcover, say $U = \bigcup_{i=1}^\infty V_{\alpha_i}$. But then, $A = A \cap U = \bigcup_{i=1}^\infty A \cap V_{\alpha_i} = \bigcup_{i=1}^\infty U_{\alpha_i}$. Thus, A is Lindelöf. Since A was arbitrary, we have X is hereditarily Lindelöf. □

Example 17.6: ($\bar{S}_\Omega$ is not hereditarily Lindelöf)

Recall the space $X = \bar{S}_\Omega = S_\Omega \cup \{\Omega\}$, which was shown to be compact, and hence, Lindelöf. Now, for each $a \in S_\Omega$, consider the open sets $U_a = (a, a+2) = \{a+1\}$. Since S_Ω is uncountable, we have the uncountable discrete space $A = \bigcup_{a \in S_\Omega}(a, a+2) = \bigcup_{a \in S_\Omega} \{a+1\}$. Clearly, this is not Lindelöf. Thus, $\bar{S}_\Omega$ is not hereditarily Lindelöf.

17.2 Separable space

Definition 17.7: (Separability)

Given $A \subset X$, we say A is *dense* in X if $X = \bar{A}$. A space X is called *separable* if there exists a countable dense subset.

Exercise 17.8: (Dense set and open set)

Show that $A \subset X$ is dense if and only for any nonempty open set $U \subset X$ we have $U \cap A \neq \emptyset$.

Exercise 17.9: (Second countablity and seperability)

Show that a second countable space is separable. Check that $\mathbb{R}$ with the cofinite topology is separable, but not second countable.

Proposition 17.10: (Image of separable space)

Let $f : X \to Y$ be countinuous surjection. If X is separable, then so is Y.

Proof

Suppose $A \subset X$ is a countable dense subset. Since f is continuous, we have, $f(\bar{A}) \subset \overline{f(A)} \Rightarrow \overline{f(A)} \supset f(X) = Y \Rightarrow \overline{f(A)} = Y$. Thus, $f(A)$ is dense in Y, which is clearly countable. Hence, Y is separable. □

Proposition 17.11: (Product of separable spaces)

Suppose $\{X_\alpha\}_{\alpha \in I}$ is a countable collection of separable spaces. Then, the product $X = \Pi X_\alpha$ is separable.

Proof

Fix countable dense subsets $A_\alpha \subset X_\alpha$. Fix some $a_\alpha \in A_\alpha$. Then, consider the collection

$$A = \{(x_\alpha) \in \Pi A_\alpha \mid x_\alpha = a_\alpha \text{ for all but finitely many } \alpha \in I\}.$$

By construction, A is countable. Let us show that A is dense in X. Let $U \subset X$ be a basic open sets. Then, $U = \Pi_\alpha U_\alpha$, where $U_\alpha = X_\alpha$ for all $\alpha \in I \setminus \{\alpha_1, \ldots, \alpha_k\}$. Since $X_\alpha = \overline{A_\alpha}$, we have $b_{\alpha_i} \in U_{\alpha_i} \cap A_{\alpha_i}$ for $i = 1, \ldots, k$. Set $b_\alpha = a_\alpha$ for all $\alpha \in I \setminus \{\alpha_1, \ldots, \alpha_k\}$. Then, clearly $b \in U \cap A$. Thus, $\bar{A} = X$. Hence, X is separable. □

Example 17.12: (Subspaces of separable space)

Subspaces of a separable space need not be separable! Consider an uncountable set X, and fix a point $x_0 \in X$. Equip X with the particular point topology based at x_0 (i.e, a nonempty set is open in X if and only if it contains x_0). Then, $\{x_0\}$ is dense in X, and thus X is separable. On the other hand, the set $X \setminus \{x_0\}$ is an uncountable discrete subspace, and hence, cannot be separable.

Definition 17.13: (Nowhere dense subset)

A subset $A \subset X$ is called *nowhere dense* if $\text{int}(\bar{A}) = \emptyset$.

Example 17.14

$\mathbb{Z} \subset \mathbb{R}$ is nowhere dense, and so is the Cantor set (which is uncountable). If X has discrete topology, no subset $A \subset X$ is nowhere dense. The set $A := \mathbb{Z} \cup ((0,1) \cap \mathbb{Q}) \subset \mathbb{R}$ is not nowhere dense.

Exercise 17.15: (Nowhere dense discrete subspace of $\mathbb{R}$)

Show that any discrete subspace $A \subset \mathbb{R}$ is nowhere dense. In particular, $\left\{\frac{1}{n} \;\middle|\; n \geq 1\right\}$ is nowhere dense.

Theorem 17.16: (Nowhere dense equivalence)

Let $A \subset X$ is given. The following are equivalent.

a) $\text{int}(\bar{A}) = \emptyset$.

b) For any nonempty open set $\emptyset \neq U subset X$, we have $A \cap U$ is not dense in U (in the subspace topology).

c) $X \setminus \bar{A}$ is dense in X.

Proof

Suppose $\operatorname{int}(\bar{A}) = \emptyset$. Fix some $\emptyset \neq U \subset X$ open set. Then, $U \not\subset \bar{A}$. Pick some $y \in U \setminus \bar{A}$. Since $\bar{A}$ is closed, we have $V := U \setminus \bar{A}$ is open in X, and hence, open in U as well. Now, clearly $V \cap (U \cap A) = \emptyset$, and hence, $y \notin \overline{U \cap A}^U$. Thus, $U \cap A$ is not dense in U.

Conversely, suppose $A \cap U$ is not dense in U for any nonempty open set $U \subset X$. If possible, suppose $\operatorname{int}(\bar{A}) \neq \emptyset$. Then, there exists some nonempty open set $U \subset \bar{A}$. Pick $y \in U$ and some arbitrary open neighborhood $y \in V \subset U$. Since U is open in X, we have V is open in X as well. Now, $V \subset U \subset \bar{A} \Rightarrow V \cap A \neq \emptyset$ (since $V \cap A = \emptyset \Rightarrow V \cap \bar{A} = \emptyset$ for V open). Thus, we have $\emptyset \neq V \cap A = (V \cap U) \cap A = V \cap (U \cap A)$. Since V was an arbitrary open neighborhood of y in U, we have y is an adherent point of $U \cap A$ (in the subspace topology). Thus, we have $\overline{A \cap U}^U = U$, a contradiction. Hence, $\operatorname{int}(\bar{A}) = \emptyset$.

Let us now assume that $X \setminus \bar{A}$ is dense in X. Then, for any nonempty open set $U \subset X$, we must have $U \cap (X \setminus \bar{A}) \neq \emptyset \Rightarrow U \not\subset \bar{A}$. But then, $\operatorname{int}(\bar{A}) = \emptyset$. Conversely, suppose $\operatorname{int}(\bar{A}) = \emptyset$. Then, for any nonempty open set $U \subset X$, we have $U \not\subset \bar{A} \Rightarrow U \cap (X \setminus \bar{A})$. But this means $X \setminus \bar{A}$ is dense in X. □

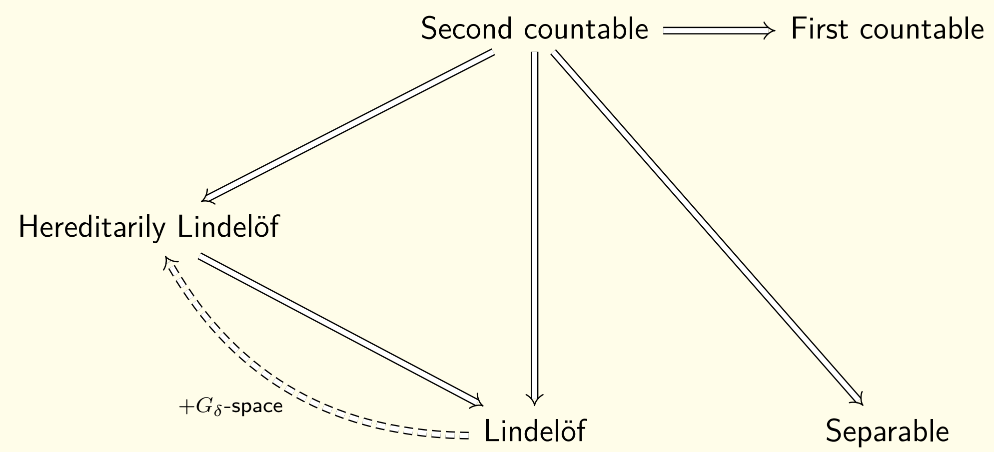
Day 18 : 17th October, 2025

`countability axioms in metric space -- Lebesgue number lemma`

18.1 Countability axioms in metric spaces

Remark 18.1

We have the implications



Recall, a space is called a G_δ-space if every closed set can be written as the intersection of countably many open sets.

Example 18.2: (Lindelöf is not separable)

Consider an uncountable space X, and fix a point $x_0 \in X$. Let $\mathcal{T}$ be the excluded point topology on X : a proper subset $U \subsetneq X$ is open if and only if $x_0 \notin U$. Then, the only open set containing x_0 is X itself, and hence, X is Lindelöf (in fact, compact). On the other hand, it cannot be separable : for any set $A \subset X$, one can see that $\bar{A} = A \cup \{p\}$. Thus, there cannot be a countable dense subset.

Example 18.3: (Separable is not Lindelöf)

Consider an uncountable space X, and fix a point $x_0 \in X$. Let $\mathcal{T}$ be the particular point topology on X based at x_0 : a nonempty set is open if and only if it contains x_0. Then, $(X, \mathcal{T})$ is separable, as the singleton $\{x_0\}$ is dense in X. But $(X, \mathcal{T})$ is not Lindelöf, as the open cover $\{\{x_0, x\} \mid x \in X\}$ does not have any countable sub-cover.

Theorem 18.4: (Metric space and countability axioms)

Suppose (X, d) is a metric space. Then, X is first countable. Moreover, the following are equivalent.

a) X is second countable.

b) X is separable.

c) X is Lindelöf.

Proof

Given any $x \in X$, consider the open balls $B_n := B_d\left(x, \frac{1}{n}\right)$. It is easy to see that $\{B_n\}$ is a countable basis at x. Thus, X is first countable.
Since any second countable space is separable and Lindelöf, clearly a) $\Rightarrow$ b) and a) $\Rightarrow$ c) holds.
Let us assume X is separable. Then, we have a countable subset $A \subset X$ which is dense in X. Consider the collection

$$\mathcal{B} := \left\{ B_d\left(a, \frac{1}{n}\right) \,\middle|\, a \in A,\ n \geq 1 \right\},$$

which is clearly a countable collection. Let us show that $\mathcal{B}$ is a basis for the topology on (X, d). Suppose $x \in X$, and pick some arbitrary open neighborhood $x \in U \subset X$. Then, for some $n \geq 1$, we have

$$x \in B_d\left(x, \frac{1}{2n}\right) \subset B_d\left(x, \frac{1}{n}\right) \subset U.$$

Since A is dense, we have some $a \in A \cap B_d\left(x, \frac{1}{2n}\right)$. Then, for any $y \in B_d\left(a, \frac{1}{2n}\right)$, we have

$$d(x, y) \leq d(x, a) + d(a, y) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} \Rightarrow y \in B_d\left(x, \frac{1}{n}\right) \subset U.$$

Thus, $B_d\left(a, \frac{1}{2n}\right) \subset U$. Also, $d(x, a) \leq \frac{1}{2n}$ and so, $x \in B_d\left(a, \frac{1}{2n}\right)$. Thus, $\mathcal{B}$ is a basis, showing b) $\Rightarrow$ a).
Now, suppose X is Lindelöf. For each $n \geq 1$, consider the collection

$$\mathcal{U}_n := \left\{ B_d\left(x, \frac{1}{n}\right) \,\middle|\, x \in X \right\},$$

which is clearly an open cover of X. Hence, there is a countable subcover $\mathcal{V}_n \subset \mathcal{U}_n$. Consider the collection $\mathcal{V} = \bigcup_{n \geq 1} \mathcal{V}_n$, which is clearly a countable collection of open sets. Let us show that $\mathcal{V}$ is a basis for the topology on (X, d). Fix some $x \in X$, and some open neighborhood $x \in U \subset X$.

Then, for some $n \geq 1$ we have $x \in B_d\left(x, \frac{1}{2n}\right) \subset B_d\left(x, \frac{1}{n}\right) \subset U$. Since $\mathcal{V}_{2n}$ is a cover, there is some $a \in X$ such that $B_d\left(a, \frac{1}{2n}\right) \in \mathcal{V}_{2n}$ and $x \in B_d\left(a, \frac{1}{2n}\right)$. Now, for any $y \in B_d\left(a, \frac{1}{2n}\right)$, we have

$$d(x,y) \leq d(x,a) + d(a,y) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} \Rightarrow y \in B_d\left(x, \frac{1}{n}\right) \subset U.$$

Thus, $x \in B_d\left(x, \frac{1}{2n}\right) \subset U$. This shows that $\mathcal{V}$ is a basis, proving c) $\Rightarrow$ a). □

Proposition 18.5: (Compact in metric space)

A compact subset of a metric space is closed and bounded.

Proof

Let (X,d) be a metric space, and $C \subset X$ is a compact subset. Since metric spaces are T_2, clearly any compact subset is closed. For any $x_0 \in C$ fixed, consider the open covering $C \subset \bigcup_{n \geq 1} B_d(x_0, n)$. This admits a finite subcover, say, $C \subset \bigcup_{i=1}^{k} B_d(x_0, n_i)$. Taking $n_0 := \max_{1 \leq i \leq k} n_i$, we have $C \subset B_d(x_0, n_0)$. Thus, C is bounded. □

Example 18.6: (Closed bounded set in metric space)

In an infinite space X, consider the metric

$$d(x,y) := \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

The induced topology is discrete, and hence, X is not compact. But clearly X is closed in itself, and bounded as $X \subset B_d(x_0, 2)$.

Lemma 18.7: (Lebesgue number lemma)

Suppose (X,d) is a compact metric space, $f : X \to Y$ is a continuous map. Let $\mathcal{V} = \{V_\alpha\}$ be an open cover of $f(X)$. Then, there exists a $\delta > 0$ (called the *Lebesgue number of the covering*) such that for any set $A \subset X$, we have

$$\operatorname{Diam}(A) := \sup_{x,y \in A} d(x,y) < \delta \Rightarrow f(A) \subset V_\alpha \text{for some } \alpha.$$

Proof

For each $x \in X$, clearly, $f(x) \in V_{\alpha_x}$ for some α_x. By continuity of f, we have some $\delta_x > 0$ such that the ball $x \in B_d(x, \delta_x) \subset f^{-1}(V_{\alpha_x})$. Now, $X = \bigcup_{x \in X} B_d\left(x, \frac{\delta_x}{2}\right)$ has a finite subcover, say, $X = \bigcup_{i=1}^{n} B_d\left(x_i, \frac{\delta_{x_i}}{2}\right)$. Set

$$\delta := \min_{1 \leq i \leq n} \frac{\delta_{x_i}}{4}.$$

We claim that δ is a Lebesgue number for the covering. Let $A \subset X$ be a set with $\operatorname{Diam}(A) < \delta$. For some $a \in A$, there exists $1 \leq i_0 \leq n$, such that $a \in B_d\left(x_{i_0}, \frac{\delta_{x_{i_0}}}{2}\right)$. Now, for any $b \in A$, we have $d(a,b) \leq \operatorname{Diam}(A) < \delta$. Then,

$$d(x_{i_0}, b) \leq d(x_{i_0}, a) + d(a,b) < \frac{\delta_{x_{i_0}}}{2} + \delta \leq \frac{\delta_{x_{i_0}}}{2} + \frac{\delta_{x_{i_0}}}{4} = \frac{3\delta_{x_{i_0}}}{4} < \delta_{x_{i_0}}.$$

Thus, $A \subset B_d(x_{i_0}, \delta_{x_{i_0}}) \Rightarrow f(A) \subset f\left(B_d\left(x_{i_0}, \delta_{x_{i_0}}\right)\right) \subset V_{\alpha_{x_{i_0}}}$. $\square$

Day 19 : 21st October, 2025

$T_{2\frac{1}{2}}$-space -- completely T_2 space -- Arens square

19.1 $T_{2\frac{1}{2}}$-space and completely Hausdorff space

Definition 19.1: ($T_{2\frac{1}{2}}$-space)

A space X is called a *$T_{2\frac{1}{2}}$-space* (or a *Urysohn space*) if given any two distinct points $x, y \in X$, there exists disjoint closed neighbrohoods of them, i.e, there are closed sets $A, B \subset X$ such that $x \in \mathring{A} \subset A, y \in \mathring{B}$ and $A \cap B = \emptyset$.

Remark 19.2: $T_{2\frac{1}{2}} \Rightarrow T_2$

Alternatively, we can define $T_{2\frac{1}{2}}$-space as follows : given any two distinct $x, y \in X$, there exists open sets $U, V \subset X$, such that $x \in U, y \in V$, and $\bar{U} \cap \bar{V} = \emptyset$. Thus, it is immediate that $T_{2\frac{1}{2}} \Rightarrow T_2$.

Example 19.3: ($T_2 \not\Rightarrow T_{2\frac{1}{2}}$)

Let us consider the *double origin plane*. Let X be $\mathbb{R}^2$, with an additional point 0^*. For any $x \in X$ with $x \neq 0, 0^*$, declare the open neighborhoods of x to be the usual open sets $x \in U \subset \mathbb{R}^2 \setminus \{0\}$. For the origin 0, declare the basic open neighborhoods

$$U_n := \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 < \frac{1}{n},\ y > 0\right\} \cup \{0\}, \quad n \geq 1,$$

and similarly, for 0^*, declare the basic open neighborhoods to be

$$V_n := \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 < \frac{1}{n},\ y < 0\right\} \cup \{0^*\}, \quad n \geq 1.$$

It is easy to see that these basic open sets form a basis for a topology on X. With this topology, X is called the double origin plane. It is easy to see that X is T_2. But for any two open neighborhoods of 0 and 0^*, there is always some point of the form $(x, 0)$ with $x \neq 0$, which is a limit point of both open sets. Thus, 0 and 0^* cannot be separated by closed neighborhoods. Hence, X is not a $T_{2\frac{1}{2}}$-space.

Definition 19.4: (Completely Hausdorff space)

A space X is said to be a *completely Hausdorff space* (or a *functionally Hausdorff space*), if given any two distinct points $x, y \in X$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.

Remark 19.5

Suppose, given $x \neq y \in X$, we have a continuous map $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$. Without loss of generality, assume $f(x) < f(y)$. Consider the function

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} f(x), & t \leq f(x), \\ t, & f(x) \leq t \leq f(y), \\ f(y), & f(y) \leq t. \end{cases} \end{aligned}$$

By the pasting lemma, g is continuous. Then, $h = g \circ f : X \to [f(x), f(y)]$ is a continuous map. By composing with a suitable homeomorphism $[f(x), f(y)] \to [0,1]$, we can then get a continuous map $F : X \to [0,1]$ such that $F(x) = 0$ and $F(y) = 1$.

Exercise 19.6

Suppose Y is a completely T_2 space. Given a space X, suppose for any $x \neq y \in X$, there is a continuous map $f : X \to Y$ such that $f(x) \neq f(y)$. Verify that X is completely T_2. In particular, subspaces and products of completely T_2 spaces are again completely T_2.

Proposition 19.7: (Metric space is completely T_2)

A metrizable space X is completely T_2. Consequently, given a space Y and a continuous injective map $\iota : Y \hookrightarrow X$, we have X is completely T_2. A space which admits a continuous injective map into a metrizable space is called a *submetrizable space*.

Proof

Any metrizable space X is T_2. Thus, we only need to show that it is regular. Suppose d is a metric on X inducing the topology. Then, $\epsilon := d(x,y) \neq 0$. Consider the function,

$$f(z) = d(x,z) + (\epsilon - d(z,y)), \quad z \in X.$$

Since distance function is continuous, it follows that $f : X \to \mathbb{R}$ is a continuous function. Also, $f(y) = 2\epsilon \neq 0 = f(x)$. But then we can get a continuous map $h : X \to [0,1]$ such that $h(x) = 0$ and $h(y) = 1$. Thus, X is completely T_2. □

Proposition 19.8: (Completely T_2-spaces are $T_{2\frac{1}{2}}$)

A completely T_2-space is $T_{2\frac{1}{2}}$.

Proof

Let X be completely T_2. For any distinct $x, y \in X$, get a continuous function $f : X \to [0,1]$ such that $f(x) = 0, f(y) = 1$. Then, consider the closed sets $A := f^{-1}([0, \frac{1}{4}]), B := f^{-1}([\frac{3}{4}, 1])$, which are clearly disjoint. Also, $x \in \underbrace{f^{-1}\left(\left[0, \frac{1}{4}\right)\right)}_{\text{open in } X} \subset A$, and so, $x \in \mathring{A}$. Similarly, $y \in \mathring{B}$. Thus, X is a $T_{2\frac{1}{2}}$-space. □

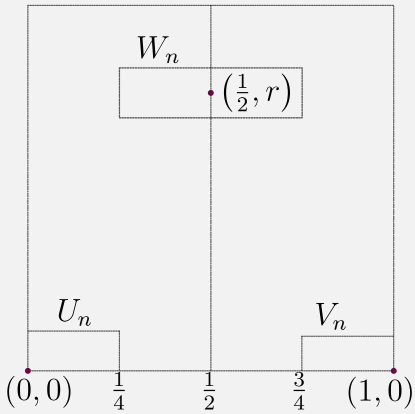
Example 19.9: (Arens square)

Consider $Q := (0,1) \cap \mathbb{Q}$, and let $Q = \sqcup_{q\in Q} Q_q$ be a disjoint union of dense subsets $Q_q \subset Q$, indexed by $q \in Q$. As an explicit example, index each prime number as $\{p_q \mid q \in Q\}$, and then consider

$$Q_q = \left\{ \frac{a}{p_q^i} \,\middle|\, 1 \le a \le p_q^i,\ \gcd(a, p_q) = 1,\ i \ge 1 \right\}.$$

Clearly, Q_q is dense in Q, and they are disjoint. Now, consider $A = Q \setminus \bigcup_{q\in Q} Q_q$. Just modify, say, $Q'_{\frac{1}{2}} = Q_{\frac{1}{2}} \cup A$. We still have disjoint dense sets.
Let us now consider the set

$$X = \{(0,0),(1,0)\} \cup \bigcup_{q\in Q} \{q\} \times Q_q \subset \mathbb{R}^2$$



Let us topologize X by declaring basic open neighborhoods for each point.

- For $(0,0)$, declare basic open neighborhoods as the collection

$$U_n := \{(0,0)\} \cup \left\{ (x,y) \in X \,\middle|\, 0 < x < \frac{1}{4},\ 0 < y < \frac{1}{n} \right\}, \quad n \ge 1$$

- For $(1,0)$, declare basic open neighborhoods as the collection

$$V_n := \{(1,0)\} \cup \left\{ (x,y) \in X \,\middle|\, \frac{3}{4} < x < 1,\ 0 < y < \frac{1}{n} \right\}, \quad n \ge 1$$

- For any $\left(\frac{1}{2}, r\right) \in \frac{1}{2} \times Q_{\frac{1}{2}}$, , declare basic open neighborhoods as the collection

$$W_n(r) := \left\{ (x,y) \,\middle|\, \frac{1}{4} < x < \frac{3}{4},\ |y - r| < \frac{1}{n} \right\}, \quad n \ge 1.$$

- Let $X \setminus \{(0,0),(1,0)\} \cup \left\{\frac{1}{2}\right\} \times Q_{\frac{1}{2}}$ inherit the usual subspace topology from $\mathbb{R}^2$.

These neighborhoods form a basis for a topology on X. This space is called the *Arens square*.

Proposition 19.10: ($T_{2\frac{1}{2}} \not\Rightarrow$ Completly T_2 : Arens square space)

Arens square is $T_{2\frac{1}{2}}$-space, but not completely T_2.

Proof

Let us consider the points $a = (0,0)$ and some $b = \left(\frac{1}{2}, r\right)$. Fix some $m, n \geq 1$ such $0 < \frac{2}{m} < r - \frac{1}{n} < r + \frac{1}{n} < 1$. Then, it is easy to see that $\overline{U_m} \cap \overline{W_n} = \emptyset$. Similar argument can be applied to b and $a' = (1,0)$. For any point $c = (q,s)$ with $q \neq \frac{1}{2}$, observe that the y-coordinate s cannot be repeated as $\left(\frac{1}{2}, s\right)$, since we started with a disjoint partition. Thus, using the denseness, we can again get some closed neighborhoods. Hence, the Arens square is a $T_{2\frac{1}{2}}$-space.
Let us show that it is not completely T_2. If possible, suppose $f : X \to [0,1]$ is a continuous map, where X is the Arens square, such that $f(0,0) = 0$ and $f(1,0) = 1$. Since f is continuous, we must have some $m, n \geq 1$ such that

$$(0,0) \in U_n \subset f^{-1}\left[0, \frac{1}{4}\right), \quad (1,0) \in V_m \subset f^{-1}\left(\frac{3}{4}, 1\right].$$

Let us fix some $r \in Q_{\frac{1}{2}}$, with $r < \min\left\{\frac{1}{n}, \frac{1}{m}\right\}$. This is possible since $Q_{\frac{1}{2}}$ is dense in Q. Now, $f\left(\frac{1}{2}, r\right)$ cannot be in both $\left[0, \frac{1}{4}\right)$ and $\left(\frac{3}{4}, 1\right]$. Without loss of generality, we can assume that exists some open interval $U \subset [0,1]$ such that

$$f\left(\frac{1}{2}, r\right) \in U, \quad \left[0, \frac{1}{4}\right] \cap \bar{U} = \emptyset.$$

Then, the pre-images $f^{-1}\left[0, \frac{1}{4}\right]$ and $f^{-1}\bar{U}$ are disjoint closed neighborhoods of $(0,0)$ and $\left(\frac{1}{2}, r\right)$ respectively. Now, $U_n \subset f^{-1}\left[0, \frac{1}{4}\right) \subset f^{-1}\left[0, \frac{1}{4}\right]$. Since $r < \frac{1}{n}$, it follows (Check!) that $\overline{U_n} \cap \overline{W_k} \neq \emptyset$ for any $k \geq 1$. This contradicts $f^{-1}\left[0, \frac{1}{4}\right] \cap \bar{U} = \emptyset$. Hence, the Arens quare is not completely T_2.
□

Remark 19.11: (Totally disconnected spaces may not be completely T_2)

It is easy to see that $\mathbb{Q}$, which is a totally disconnected set, is completetly T_2. Indeed, for any $r, s \in \mathbb{Q}$, with $r < s$, get some irrational $r < x < s$. Then,

$$f(t) = \begin{cases} 0, & t < x \\ 1, & x < t, \end{cases}$$

is a continuous function, with $f(r) = 0, f(s) = 1$. But in general, a totally disconnected space need not be completely T_2.
Indeed, we have seen that the Arens square X is not completely T_2. Let us show that it is totally disconnected. Firstly, observe that the second component projection $\pi : X \to [0,1] \cap \mathbb{Q}$ is a continuous map (but the first component projection is not continuous). Now, any two points of X cannot share the same second component, and thus π is injective. Hence, if a connected set $A \subset X$ contains more than one point, $\pi(A)$ will be a connected set of $[0,1] \cap \mathbb{Q}$, with more than one point, a contradiction. Thus, X is totally disconnected.

Day 20 : 23rd October, 2025

`regular space -- ` T_3 ` space -- half-disc topology -- Tychonoff plank -- Tychonoff corkscrew`

20.1 Regular space and T_3-space

Definition 20.1: (Regular space)

A space X is called *regular* if given any closed set $A \subset X$ and any point $x \in X \setminus A$, there exists open sets $U, V \subset X$ such that

$$x \in U, \quad A \subset V, \quad U \cap V = \emptyset.$$

Proposition 20.2: (Regularity via closed neighborhood base)

Given a space X, the following are equivalent.

a) X is regular.

b) Given any $x \in X$ and open neighborhood $x \in U \subset X$, there exists a closed neighborhood $x \in \mathring{C} \subset C \subset U$.

c) Given any $x \in X$ and open neighborhood $x \in U \subset X$, there exists an open neighborhood $x \in V \subset \bar{V} \subset U$.

In other words, regularity is equivalent to the fact that closed neighborhoods of any point forms a local base at that point.

Proof

Suppose X is regular. Let $x \in U \subset X$ be an open neighborhood. Then $A = X \setminus U$ is a closed set, and $x \notin A$. By regularity, there are open sets $P, Q \subset X$ such that

$$x \in P, \quad A \subset Q, \quad P \cap Q = \emptyset.$$

Note that

$$P \cap Q = \emptyset \Rightarrow P \subset X \setminus Q \Rightarrow \bar{P} \subset \overline{X \setminus Q} = X \setminus Q \subset X \setminus A = U.$$

Thus, we have a closed neighborhood $x \in P \subset \bar{P} \subset U$. This proves a) $\Rightarrow$ b).
Let us show b) $\Rightarrow$ c). Suppose $x \in U \subset X$ is given. Then, by b), we have some closed neighborhood $x \in \mathring{C} \subset C \subset U$. But then taking $V = \mathring{C}$, we have $x \in V \subset \bar{V} \subset \bar{C} = C \subset U$. This proves b) $\Rightarrow$ c).
Finally, suppose c) holds. Let $A \subset X$ be closed, and $x \notin A$ be a point. Then, $x \in U := X \setminus A$. By c), there is an open neighborhood such that $x \in V \subset \bar{V} \subset U$. Consider $P = V$ and $Q = X \setminus \bar{V}$. Then, $x \in V = P$, and $A = X \setminus U \subset X \setminus \bar{V} = Q$. Clearly, $P \cap Q = \emptyset$. Thus, X is regular, proving a). ☐

Definition 20.3: (T_3-space)

A space X is called a *T_3-space* if X is regular and T_0.

Example 20.4: (Regularity does not imply T_3)

Consider $X = \{0, 1\}$ with the indiscrete topology. Then, X is a regular space (in fact any indiscrete space is regular). But X is not T_0. Thus, X is not T_3.

Proposition 20.5: (T_3 is equivalent to regular, T_2)

A space X is T_3 if and only if it is regular, T_2.

Proof

Suppose X is regular, T_2. Since $T_2 \Rightarrow T_0$, we have X is T_3. Conversely, suppose X is T_3. Let us show that X is T_2. Let $x \neq y \in X$. Since X is T_0, there is an open set $U \subset X$, such that, without loss of generality, $x \in U$ and $y \notin U$. Then, there is an open neighborhood such that $x \in V \subset \bar{V} \subset U$. Take $W := X \setminus \bar{V}$. Then, $y \in X \setminus U \subset X \setminus \bar{V} = W$. Clearly, $V \cap W = \emptyset$. Thus, X is T_2. $\square$

Proposition 20.6: ($T_3 \Rightarrow T_{2\frac{1}{2}}$)

A T_3-space is $T_{2\frac{1}{2}}$.

Proof

Let $x \neq y \in X$. Since X is T_2, we have open sets $U, V \subset X$ such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

But then there are open sets $A, B \subset X$ such that $x \in A \subset \bar{A} \subset U$ and $y \in B \subset \bar{B} \subset V$. Clearly, $\bar{A} \cap \bar{B} = \emptyset$. Thus, X is $T_{2\frac{1}{2}}$. $\square$

Example 20.7: ($T_{2\frac{1}{2}} \not\Rightarrow T_3$: Arens square is $T_{2\frac{1}{2}}$, but not regular)

Recall that the Arens square X is a $T_{2\frac{1}{2}}$-space. Let us show that X is not regular. For the point $(0,0)$, consider an open neighborhood U_n. But then for any basic open neighborhood $(0,0) \in U_m \subset U_n$, we must have that $\overline{U_m}$ contains points with y-coordinate value $\frac{1}{4}$. Thus, $\overline{U_m} \not\subset U_n$. This means that the closed neighborhoods at $(0,0)$ does not form a local base. Hence, X is not regular.

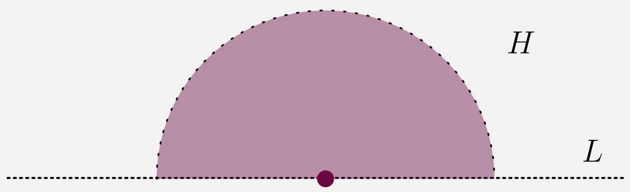
Exercise 20.8

Check that the double origin plane is not T_3.

Example 20.9: (Half-disc topology)

Consider the upper half plane $H = \{(x,y) \mid y > 0\}$ and the x-axis $L = \{(x,0) \mid x \in \mathbb{R}\}$. On the set $X := H \cup L$, consider the following topology.

- For any $(x,y) \in H$, consider the usual neighborhoods from $\mathbb{R}^2$ as the neighborhood basis.
- For $(x,0) \in L$, consider the open neighborhoods as $\{x\} \cup (H \cap U)$, where $U \subset \mathbb{R}^2$ is a usual open neighborhood of $(x,0)$.



This space X is called the *half-disc topology*.

Proposition 20.10: (Completely $T_2 \not\Rightarrow$ Regular : Half-disc topology)

The half-disc topology X is completely T_2, but not regular.

Proof

Observe that the inclusion map $\iota : X \hookrightarrow \mathbb{R}^2$ is continuous. Since $\mathbb{R}^2$ is a metric space, it is completely T_2. Consequently, it follows that X is again completely T_2. Indeed, for any $x \neq y \in X$, we have $g : \mathbb{R}^2 \to [0, 1]$ continuous such that $f(x) = 0$ and $f(y) = 1$. Then, $f := g \circ \iota : X \to [0, 1]$ gives a functional separation.

Let us now show that X is not regular (and hence not T_3 either). For any point $(x, 0) \in L$, consider the half disc $D = H \cap B((x, 0), \epsilon)$ of radius $\epsilon > 0$ and center $(x, 0)$. Then, $U = \{(x, 0)\} \cup D$ is an open set. These open sets clearly form a neighborhood basis at $(x, 0)$. Observe that $\int \bar{U}$ contains all the points on the diameter of the half disc. Hence, we cannot find neighborhood basis of regular open sets at $(x, 0)$ (recall : an open set O is regular if $\operatorname{int}(\bar{O}) = O$). Thus, the half-disc topology is not regular. □

Example 20.11: (Tychonoff Plank)

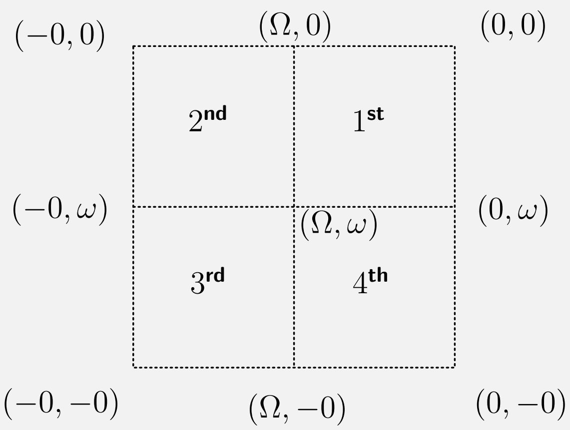
Recall the first infinite ordinal ω and the first uncountable ordinal S_Ω. We get the well-ordered "intervals" $[0, \omega]$ (which you can think of as $\{0, 1, 2, \ldots, \omega\}$), and $[0, \Omega]$ (which you can think of as $\overline{S_\Omega} = S_\Omega \cup \{\Omega\}$). These are topological spaces equipped with the order topology, and in particular, they are compact. The *Tychonoff plank* is the product $[0, \Omega] \times [0, \omega]$. You can imagine this as the first quadrant of a coordinate grid : the x-axis corresponds to the first uncountable ordinal, whereas the y-axis corresponds to the first infinite ordinal. The *deleted Tychonoff plank* is the space $[0, \Omega] \times [0, \omega] \setminus \{(\Omega, \omega)\}$

Example 20.12: (Corkscrew construction)

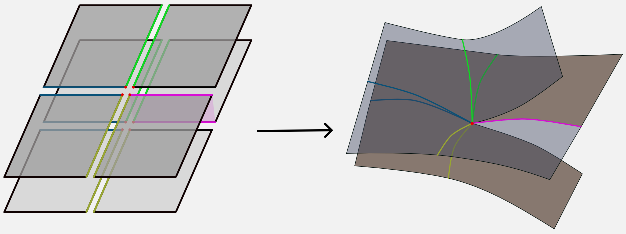
For the ordinal ω or Ω, we have the totally ordered sets

$$A_\omega := [-0, -1, \ldots, \omega, \ldots, 1, 0], \quad A_\Omega := [-0, -1, \ldots, -\omega, \ldots, \Omega, \ldots, \omega, \ldots, 1, 0],$$

equipped with the order topology. Here, the negative of an element is a new element (so, -0 and 0 different!). Taking product, we get a "coordinate plane", with all four quadrants a copy of Tychonoff plank.



Delete the "origin" (Ω,ω). Now, take countable infinitely many copies of these planes (indexed by $\mathbb{Z}$), and stack them vertically. Next, cut all the planes along the positive x-axis. Then, along the cut, identify the north edge of the fourth quadrant of one plane to the south edge of the first quadrant of the *plane just below*. This is an identification space; since the origin was removed from all the planes, there is no issue about well-definedness.



This construction can be formalized as follows. For each $k \in \mathbb{Z}$, consider the following spaces

$$T_k^1 = ([\Omega,0] \times [\omega,0] \setminus \{(\Omega,\omega)\}) \times \{k\}, \qquad T_k^2 = ([-0,\Omega] \times [\omega,0] \setminus \{(\Omega,\omega)\}) \times \{k\},$$
$$T_k^3 = ([-0,\Omega] \times [-0,\omega] \setminus \{(\Omega,\omega)\}) \times \{k\}, \qquad T_k^4 = ([\Omega,0] \times [-0,\omega] \setminus \{(\Omega,\omega)\}) \times \{k\}.$$

These are copies of the deleted Tychonoff planks, representing the four quadrants at the k^{th}-stage. Let us identify the edges to make the corkscrew (see the picture above). We consider the set $X = \bigcup_{k\in\mathbb{Z}} (T_k^1 \cup T_k^2 \cup T_k^3 \cup T_k^4)$, and on it define an equivalence relation as follows. For any $x \in X$, set $x \sim x$. Then, for each $k \in \mathbb{Z}$, consider the following collection of relations (and their reverse, to make it symmetric).

i) $x \sim y$ for $x = (\Omega, n, k) \in T_k^1$ and $y = (\Omega, n, k) \in T_k^2$ (identify the west-side of the first quadrant T_k^1 with the east-side of the second quadrant T_k^2, along the positive y-axis).

ii) $x \sim y$ for $x = (-\alpha, \omega, k) \in T_k^2$ and $y = (-\alpha, \omega, k) \in T_k^3$ (identify the south-side of the second quadrant T_k^2 with the north-side of the third quadrant T_k^3, along the negative x-axis).

iii) $x \sim y$ for $x = (\Omega, -n, k) \in T_k^3$ and $y = (\Omega, -n, k) \in T_k^4$ (identify the east-side of the third quadrant T_k^3 with the west-side of the fourth quadrant T_k^4, along the negative y-axis).

iv) $x \sim y$ for $x = (\alpha, \omega, k) \in T_k^4$ and $y = (\alpha, \omega, k-1) \in T_{k-1}^1$ (identify the north-side of the fourth quadrant T_k^4 with the south-side first quadrant T_{k-1}^1 **of the plane below**, along the positive x-axis).

The quotient space $X/_{\sim}$ looks like a corkscrew. This construction can be performed with other 'coordinate plane' whenever it makes sense!

Example 20.13: (Tychonoff Corkscrew)

Before performing the corkscrew construction as above with the Tychonoff planks, let us now add two extra points $\{\alpha_\pm\}$, and consider the space

$$Z = \{\alpha_+, \alpha_-\} \cup \bigcup_{k\in\mathbb{Z}} \left(T_k^1 \cup T_k^2 \cup T_k^3 \cup T_k^4\right).$$

The topology on Z is defined as follows. For any point $(\pm\alpha, \pm n, k)$, an open neighborhood basis is obtained from the induced topology of the deleted Tychonoff plank. Thus, basic open neighborhoods are products of intervals. For the point α_+, a basic open neighborhood consist of all of $\bigcup_{k>i} (T_k^1 \cup T_k^2 \cup T_k^3 \cup T_k^4)$ for some $i \in \mathbb{Z}$, i.e, everything above i^{th}-stage. Similarly, for α_-, open neighborhoods consist of all of $\bigcup_{k<i} (T_k^1 \cup T_k^2 \cup T_k^3 \cup T_k^4)$ for some $i \in \mathbb{Z}$, i.e, everything below i^{th}-stage. It is easy to see that these collections of neighborhood bases forms a basis for a topology on Z. Let us now perform the identification as above, the points $\{\alpha_\pm\}$ are identified only to themselves,i.e, $\alpha_+ \sim \alpha_+$, $\alpha_- \sim \alpha_-$, and no other point. The quotient space $Z/_\sim$ is called the *Tychonoff corkscrew*.

Day 21 : 24th October, 2025

`Tychonoff corkscrew property -- completely regular space`

21.1 Regular space and T_3 space (cont.)

Proposition 21.1: (Continuous map from S_Ω is eventually constant)

Given any continuous map $f : S_\Omega \to \mathbb{R}$, there exists some $\alpha \in S_\Omega$ such that $f(x) = c$ for all $x \geq \alpha$. Consequently, f can only have countably many distinct values.

Proof

If possible, suppose there exists some $\epsilon > 0$ such that for any $\alpha \in S_\Omega$ there exists some $\beta(\alpha) > \alpha$ with $|f(\alpha) - f(\beta)| \geq \epsilon$. Otherwise, for each $n \geq 1$, there exists some α_n such that for all $\beta > \alpha_n$, we have $|f(\beta) - f(\alpha_n)| < \frac{1}{n}$. If the sequence $\{\alpha_n\}$ is finite (i.e, there are finitely many points), then just take $\theta = \max \alpha_n$. It follows that for any $\beta > \theta$, we have $|f(\beta) - f(\theta)| < \frac{1}{n}$ for all n. In particular, $f(\beta) = f(\theta)$ for all $b > \theta$, proving the claim. If the sequence is not finite, without loss of generality, assume $\alpha_1 < \alpha_2 < \dots$. Now, recall that $[0, \Omega)$ is sequentially convergent. Hence, without loss of generality, the sequence $\{\alpha_n\}$ converges to some $\theta \in [0, \Omega)$, and $\theta \geq \alpha_i$ for all i. Then, by continuity of f we have $f(\theta) = \lim_n f(\alpha_n)$. Now, for any $\beta > \theta$, we have

$$|f(\beta) - f(\theta)| \leq |f(\beta) - f(\alpha_n)| + |f(\alpha_n) - f(\theta)| \to 0, \quad n \to \infty.$$

Thus, $f(\beta) = f(\theta)$ for any $\beta > \theta$, again proving the claim.

Thus, let us now assume that there exists some $\epsilon > 0$ such that for any $\alpha \in S_\Omega$ there exists some $\beta(\alpha) > \alpha$ with $|f(\alpha) - f(\beta)| \geq \epsilon$. Starting with $\alpha_0 = 0$, we can construct an increasing sequence $\alpha_0 < \alpha_1 < \dots$, where each α_j is inductively obtained as some $\beta(\alpha_{j-1})$. Now, $\{\alpha_j\}$ is a countable set, and hence, upper bounded. Suppose $\theta \in S_\Omega$ is the least upper bound of $\{\alpha_j\}$. Now, by continuity, we have some $\delta < \theta$ such that

$$f\left((\delta, \theta]\right) \subset \left(f(\theta) - \frac{\epsilon}{2}, f(\theta) + \frac{\epsilon}{2}\right).$$

Since θ is the least upper bound of the strictly increasing sequence α_j, there exists some $\delta < \alpha_{j_0} \leq \theta$. Now, for $\alpha_j < \alpha_{j+1} \leq \theta$. But then, $|f(\alpha_{j+1}) - f(\alpha_j)| < \epsilon$, a contradiction.
Hence, we have that there is some $\alpha \in S_\Omega$ such that $f(x)$ is constant for all $x \geq \alpha$. □

Proposition 21.2: ($T_3 \not\Rightarrow$ Completely T_2 : Tychonoff Corkscrew)

The Tychonoff corkscrew is T_3, but not completely T_2.

Proof

For any point other than $\alpha_\pm$, one can easily construct a basis of open sets which are regular (i.e, $\operatorname{int}(\bar{O}) = O$). Indeed, if the point is not on any of the "slits", we can take product of intervals. For a point on the slit, we might need to take the intervals in two different planks, but we can still get a basis of regular open sets. For α_+, the image of the basic open neighborhoods are open (Check!), and they are clearly regular open sets. Similar argument works for α_-. Thus, the Tychonoff corkscrew is a regular space. In fact, it is T_0 as every point is closed, and hence, T_3.

Let us now show that the space is not completely T_2. Suppose f is a real-valued continuous function. Observe that for $n \neq 0$, on each of the horizontal lines $A_\Omega \times \{n\} \times \{k\}$, the function f is constant on an interval of the form $[-\alpha, \alpha]$ about Ω. Same argument works for the x-axis as well, and we get a deleted neighborhood about $\{(\Omega, \omega, k)\}$ where f is constant. Now, there are countable infinitely many such intervals, on each of which f is constant. Indeed, on each stage, there are countable infinitely many horizontal lines (counting two lines for the x-axis), and there are countable infinitely many stages (the positive x-axes are getting counted twice, which is not an issue). Again, using the well-ordering, we can get a common α such that f is constant on each of the $[-\alpha, \alpha] \times \{\pm n\} \times \{k\}$ and on $([-\alpha, \alpha] \times \{\omega\} \setminus \{(\Omega, \omega)\}) \times \{k\}$, for all $k \in \mathbb{Z}$.

Fix some $-\beta \in [-\alpha, \Omega)$, and the corresponding $\beta \in (\Omega, \alpha]$. Then, denote the same points (i.e, their equivalence classes) in different stages as

$$-\beta^k = (-\beta, \omega, k), \quad \beta^k = (\beta, \omega, k).$$

Next, get the sequences

$$-\beta^k_{\pm n} = (-\beta, \pm n, k), \quad \beta^k_{\pm n} = (\beta, \pm n, k).$$

Clearly, as $\pm n \to \omega$, we have

$$-\beta^k_{\pm n} \to -\beta^k, \quad \beta^k_n \to \beta^k, \quad \beta^k_{-n} \to \beta^{k-1},$$

where the last convergence follows since the north edge of the fourth quadrant is identified with the south edge of the first quadrant of the stage just below! Now, $f\left(-\beta^k_{\pm n}\right) = f\left(\beta^k_{\pm n}\right)$. Hence, by continuity,

$$f(-\beta^k) = \lim f\left(-\beta^k_n\right) = \lim f\left(\beta^k_n\right) = f(\beta^k),$$

and also,

$$f(-\beta^k) = \lim f\left(-\beta^k_{-n}\right) = \lim f\left(\beta^k_{-n}\right) = f\left(\beta^{k-1}\right).$$

But then, inductively we see that $f(\pm\beta^k)$ are all constant. This implies that f is constant on the union of deleted intervals

$$\mathcal{I} = \bigcup_{k\in\mathbb{Z}} ([-\alpha, \alpha] \times \{\omega\} \setminus \{(\Omega, \omega)\}) \times \{k\}.$$

We can now get a sequence $\{a_i\}_{i=-\infty}^{\infty} \in \mathcal{I}$ (in fact, taking $a_{\pm i} = \pm\beta^i$ will do) such that $\lim_{i\to\infty} a_i = \alpha_+$ and $\lim_{i\to-\infty} a_i = \alpha_-$. This follows since the basic open neighborhoods of $\{\alpha_{pm}\}$ contains all the stages after (resp. below) a certain 'height'. By continuity of f, we have $f(\alpha_+) = f(\alpha_-)$. Thus, Tychonoff corkscrew is not functionally T_2, as no continuous function is able to distinguish the points $\alpha_\pm$. □

21.2 Completely regular space

Definition 21.3: (Completely regular space)

A space X is called a *completely regular space* if given any closed set $A \subset X$ and a point $x \in X \setminus A$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$.

Remark 21.4

It is immediate that a completely regular space is regular.

Definition 21.5: ($T_{3\frac{1}{2}}$-space)

A space X is called a *$T_{3\frac{1}{2}}$-space* (or a *Tychonoff space*) if it is completely regular, and T_0.

Remark 21.6

It is immediate that a $T_{3\frac{1}{2}}$-space is completely T_2, and hence, $T_{2\frac{1}{2}}$. Also, $T_{3\frac{1}{2}} \Rightarrow T_3$ is clear as well. Moreover, one can check that a completely regular space is $T_{3\frac{1}{2}}$ if and only if it is T_2. Thus, one can define $T_{3\frac{1}{2}}$-space as a completely regular, Hausdorff space.

Proposition 21.7: (Metrizable $\Rightarrow$ Tychonoff)

Metrizable spaces are Tychonoff.

Proof

Say (X, d) is a metric space. Let $A \subset X$ be closed, and $p \in X \setminus A$ be a point. Consider the map

$$f(x) := \frac{d(p, x)}{d(p, x) + d(A, x)}, \qquad x \in X.$$

It is easy to see that $f : X \to \mathbb{R}$ is continuous, and $f(p) = 0, f(A) = 1$. Thus, X is completely regular, and hence, Tychonoff. □

Proposition 21.8: ($T_3 \not\Rightarrow T_{3\frac{1}{2}}$: Tychonoff corkscrew)

The Tychonoff corkscrew X is T_3 but not $T_{3\frac{1}{2}}$.

Proof

We have seen that X is T_3 but not completely T_2. Since $T_{3\frac{1}{2}}$ implies completely T_2, it follows that X is not $T_{3\frac{1}{2}}$. □

Proposition 21.9: (Completely $T_2 \not\Rightarrow T_{3\frac{1}{2}}$: Half-disc topology)

The half-disc topology X is a completely T_2 space, which is not $T_{3\frac{1}{2}}$.

Proof

We have seen X is completely T_2 (as it was submetrizable), but not regular (in fact not even semiregular). Hence, X cannot be $T_{3\frac{1}{2}}$. □

Day 22 : 29th October, 2025

`normal space -- Urysohn's lemma`

22.1 Normal space

Definition 22.1: (Normal space)

A space X is called a *normal space* if given any two disjoint closed sets $A, B \subset X$, there exists disjoing open sets separting them, i.e, there are open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$

Remark 22.2: (Normal $\not\Rightarrow$ Regular)

Consider the space $X = \{-1, 0, 1\}$, with the topology $\mathcal{T} = \{\emptyset, X, \{-1\}, \{1\}, \{-1, 1\}\}$. This space is the *excluded point topology on the three point set*. It is easy to see that X is normal, since there are no disjoint nonempty closed sets! Indeed, the closed sets are $\{\emptyset, X, \{0\}, \{0, 1\}, \{0, -1\}\}$. Now, consider $A = \{0, 1\}$ and the point $x = -1 \in X \setminus A$. If possible, suppose $f : X \to [0, 1]$ is a continuous map, with $f(x) = 0$ and $f(A) = 1$. But then, $A = f^{-1}\left(\frac{1}{2}, 1\right]$ must be open, a contradiction. Thus, X is not completely regular. In fact, X is not regular either : $1 \in \{-1, 1\}$, but $\overline{\{1\}} = X$, which implies that we can not find any open U such that $1 \in U \subset \bar{U} \subset \{-1, 1\}$.

Proposition 22.3: (Normality by closed neighborhood)

X is normal if and only if given any closed set A and an open set $U \subset X$, with $A \subset U$, there exists an open set $V \subset X$ such that $A \subset V \subset \bar{V} \subset U$.

Proof

Suppose X is normal. Let $A \subset X$ be closed and $U \subset X$ be open, with $A \subset U$. Then, $B = X \setminus U$ is a closed set, disjoint from A. We have open sets $P, Q \subset X$ such that $A \subset P, B \subset Q$ and $P \cap Q = \emptyset$. Note that

$$P \subset X \setminus Q \Rightarrow \bar{P} \subset \overline{X \setminus Q} = X \setminus Q \subset X \setminus B = U.$$

That is, we have $A \subset P \subset \bar{P} \subset U$.

Conversely, suppose for any closed A and open U, with $A \subset U$, we have some open V such that $A \subset V \subset \bar{V} \subset U$. Let A, B be disjoint closed sets. Then, $A \subset X \setminus B$, which is open. Get open set U such that $A \subset U \subset \bar{U} \subset X \setminus B$. Let us take $V := X \setminus \overline{U}$, which is open. Then, $\bar{U} \subset X \setminus B \Rightarrow B \subset X \setminus \bar{U} = V$. Clearly, $U \cap V \subset \bar{U} \cap V = \emptyset \Rightarrow U \cap V = \emptyset$. Thus, X is a normal space. □

Exercise 22.4: (Normality is equivalent to separation by closed neighborhoods)

Check that a space X is normal if and only if for any closed sets $A, B \subset X$ with $A \cap B$, there are closed sets $P, Q \subset X$ such that $A \subset \mathring{P} \subset P, B \subset \mathring{Q} \subset Q$ and $P \cap Q = \emptyset$.

Theorem 22.5: (Urysohn's Lemma)

A space X is normal if and only if given disjoint closed sets $A, B \subset X$, there exists a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.

Proof

Let X be a normal space. Fix two closed sets $A, B \subset X$ with $A \cap B = \emptyset$.

Step 1: Let us consider the dyadic rationals $D = \left\{ \frac{m}{2^n} \,\middle|\, m, n \geq 0,\ m \text{ odd} \right\} \cap (0,1)$ in $[0,1]$. For each $r \in D$, using the normality, we shall inductively construct an open set $U_r \subset X$ and a closed $V_r \subset X$, satisfying the following.

i) $A \subset U_r$ and $V_r \subset X \setminus B$ for all $r \in D$.

ii) $U_r \subset V_r$ for all $r \in D$.

iii) $V_r \subset U_s$ whenever $r < s$ in D.

Here are the first few steps of the induction.

$$\begin{array}{ccccccccccccccc} A & & & & & & \subset & & & & & & B^c \\ A & & \subset & & U_{\frac{1}{2}} & \subset & V_{\frac{1}{2}} & & \subset & & & & B^c \\ A & \subset & U_{\frac{1}{4}} & \subset & V_{\frac{1}{4}} & \subset & U_{\frac{1}{2}} & \subset & V_{\frac{1}{2}} & \subset & U_{\frac{3}{4}} & \subset & V_{\frac{3}{4}} & \subset & B^c \end{array}$$

Let us describe this formally. We induct over $n \geq 1$ where n appears as the exponent of 2 in $\frac{m}{2^n} \in D$, where $1 \leq m < 2^{k+1}$ are odd numbers. For notational convenience, let us denote $U_1 = B^c$ and $V_0 = A$.

Base case $n = 1$**:** We just have one value $\frac{1}{2}$ in this case. Since $A \subset B^c$, by normality, we have an open set $U_{\frac{1}{2}}$ and a closed set $V_{\frac{1}{2}} = \overline{U_{\frac{1}{2}}}$ such that $A \subset U_{\frac{1}{2}} \subset V_{\frac{1}{2}}$.

Inductive assumption $n = k$**:** Suppose, we for some $k \geq 1$, we have constructed the open and closed sets for all $\frac{m}{2^l} \in D$ with $l \leq k$.

Induction step $n = k + 1$**:** We need to get the sets labeled by $\left\{ \frac{1}{2^{k+1}}, \frac{3}{2^{k+1}}, \ldots, \frac{2^{k+1}-1}{2^{k+1}} \right\}$. But these appear in the middle of two sets already defined. As an example, for any $1 \leq m =$

$2l+1 < 2^{k+1}$, we already have defined $V_{\frac{m-1}{2^{k+1}}} = V_{\frac{l}{2^k}} \subset U_{\frac{l+1}{2^k}} = U_{\frac{m+1}{2^{k+1}}}$ (after reducing the fractions $\frac{l}{2^k}$ and $\frac{l}{2^k}$ as needed, and noting, $V_0 = A, U_1 = B$ are the edge cases). Using normality, we get open and closed sets satisfying $V_{\frac{m-1}{2^{k+1}}} \subset U_{\frac{m}{2^{k+1}}} \subset V_{\frac{m}{2^{k+1}}} \subset U_{\frac{m}{2^{k+1}}}$.

Since every dyadic rational appears like this, we can construct the collection $\{U_r, V_r\}_{r\in D}$ with the desired properties.

Step 2: Let us now define a function $f : X \to [0,1]$ as follows.

a) Set $f(x) = 1$ if $x \notin U_r$ for all $r \in D$.

b) For any other x, define
$$f(x) = \inf \{r \in D \mid x \in U_r\}.$$

In particular, since $A \subset U_r$ for all r, we see that $f(x) = 0$ for $x \in A$. Similarly, as $U_r \subset V_r \subset X \setminus B \Rightarrow B \subset X \setminus U_r$ for all r, we see that $f(x) = 1$ for $x \in B$. Thus, f satisfies the desired properties. We need to show that f is continuous.

Step 3: Let us prove the continuity of the function defined in the previous step. We consider three cases.

a) Suppose $f(x) = 0$. If possible, suppose $x \notin U_{r_0}$ for some $r_0 \in D$. Then, for any $r \in D$ with $0 < r < r_0$, we must have $x \notin U_r$, as we have $U_r \subset V_r \subset U_{r_0}$. But this means $f(x) = \inf \{r \in D \mid x \in U_r\} \geq r_0 > 0$, a contradiction. Thus, $f(x) = 0 \Rightarrow x \in U_r$ for all $r \in D$. Now, for any open set $[0, \epsilon) \subset [0,1]$, we have some $r \in D \cap (0, \epsilon)$. Then, for any $y \in U_r$, we have $f(x) \leq r < \epsilon$. In other words, $x \in U_r \subset f^{-1}[0, \epsilon)$. Thus, f is continuous at x whenever $f(x) = 0$.

b) Suppose $f(x) = 1$. If possible, suppose $x \in V_{r_0}$ for some $r_0 \in D$. But then, $x \in U_r$ for any $r \in D$ with $r_0 < r$, and hence, $f(x) \leq r_0 < 1$, a contradiction. Thus, we have $f(x) = 1 \Rightarrow x \notin V_r$ for all $r \in D$. Now, for any open set $(1-\epsilon, 1] \subset [0,1]$, we have some $s \in D$ with $1 - \epsilon < s < 1$. Consider the open set $W = X \setminus V_s$. Clearly, $x \in W$. Then, for any $r < s$ in D, we have $U_r \subset V_r \subset U_s \subset V_s$. Thus, it follows that for any $y \notin V_s \Rightarrow y \not\in U_s$ we have $f(y) \geq r > 1 - \epsilon$. In other words, $x \in W \subset f^{-1}(1-\epsilon, 1]$. Thus, f is continuous at x whenever $f(x) = 1$.

c) Finally, suppose $0 < f(x) < 1$. Set $\delta := f(x)$, and get an open set $(\delta - \epsilon, \delta + \epsilon) \subset (0,1) \subset [0,1]$. Next, get $r_1, r_2 \in D$ satisfying $\delta - \epsilon < r_1 < \delta < r_2 < \delta + \epsilon$. Since D is dense in $(0,1)$, this is always possible. Consider the open set $W = U_{r_2} \setminus V_{r_1}$. Note that $f(x) = \delta < r_2 \Rightarrow x \in U_{r_2}$. Also, for any $r \in D$ with $r_1 < r < \delta$, we have $V_{r_1} \subset U_r$. Thus, $x \in V_{r_1} \Rightarrow y \in U_r \Rightarrow f(y) \leq r < \delta$, a contradiction. Thus, $x \in W$. Now, for any $r < r_1$, we have $U_r \subset V_{r_1}$, and thus, $y \in W \Rightarrow f(y) \geq r_1$. Also, $y \in W \subset U_{r_2} \Rightarrow f(y) \leq r_2$. Thus, for any $y \in W$ we have $f(y) \in [r_1, r_2] \subset (\delta - \epsilon, \delta + \epsilon)$. In other words, $x \in W \subset f^{-1}(\delta - \epsilon, \delta + \epsilon)$. Thus, f is continuous at x whenever $0 < f(x) < 1$.

Hence, we have proved that $f : X \to [0,1]$ is a continuous map. This concludes the theorem. □

Remark 22.6: (Onion Lemma!)

The construction in Uryshon's lemma has a resemblance of peeling an onion layer by layer: the space X is the onion, and any $U_r \setminus V_s$ for $s < r$ behaves like a layer. The function constructed in the lemma is called the *Urysohn's function* (for the sets A, B).

Day 23 : 30th October, 2025

T_4-space -- completely normal space -- T_5-space -- perfectly normal space -- T_6-space

23.1 T_4-space

Definition 23.1: (T_4-space)

A space X is called a *T_4-space* if it is normal and T_1.

Remark 23.2: (Normal + T_0 is not T_4)

As normal spaces are regular, $T_4 \Rightarrow T_3$. The excluded point topology on the three point set is normal, but not even T_1 (and hence, not T_2, T_3, T_4 either).

Proposition 23.3: ($T_4 \Rightarrow T_{3\frac{1}{2}}$)

Any T_4 space X is also a $T_{3\frac{1}{2}}$.

Proof

Let $A \subset X$ be a closed set, and $x \in X \setminus A$. Since X is T_1, we have $\{x\}$ is closed as well. Since X is normal, by Urysohn's lemma, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$. But this means that X is completley regular. As X is T_0, we have X is $T_{3\frac{1}{2}}$. $\square$

Proposition 23.4: (Compact $+ T_2 \Rightarrow T_4$)

A compact T_2 space X is T_4.

Proof

Let $A, B \subset X$ be disjoint closed sets. Fix some $a \in A$. Then, for each $b \in B$, there are open sets $U_{a,b}, V_{a,b}$ such that $a \in U_{a,b}, b \in V_{a,b}$ and $U_{a,b} \cap V_{a,b}$. Since B is closed in a compact space, B is compact. Thus, the cover $B \subset \bigcup_{b\in B} V_{a,b}$ has finite subcover $B \subset V_a := \bigcup_{i=1}^{k} V_{a,b_i}$. Then, $U_a := \bigcap_{i=1}^{k} U_{a,b_i}$ is an open set, with $a \in U_a$. Clearly, $U_a \cap V_a = \emptyset$. Now, we have a cover $A \subset \bigcup_{a\in A} U_a$, which again admits a finite subcover $A \subset U := \bigcup_{i=1}^{l} U_{a_i}$. We have an open set $V := \bigcap_{i=1}^{l} V_{a_i}$. Clearly, $B \subset V$ and $U \cap V = \emptyset$. Thus, we have that X is normal. Since X is T_2, we get X is T_4. $\square$

Proposition 23.5: (Metrizable $\Rightarrow T_4$)

Metrizable spaces are T_4.

Proof

Fix a metric space (X, d). Let $A, B \subset X$ be disjoint closed sets. For each $a \in A$, fix $r_a := \frac{1}{3}d(a, B) > 0$, and for each $b \in B$, fix $s_b := \frac{1}{3}d(b, A)$. Consider the open sets

$$U := \bigcup_{a\in A} B_d(a, r_a), \qquad V := \bigcup_{b\in B} B_d(b, s_b).$$

Clearly, $A \subset U$ and $B \subset V$. If possible, suppose $z \in U \cap V$. Then, for some $a \in A$ and $b \in B$, we have

$$d(a, z) < r_a, \qquad d(b, z) < s_b.$$

Without loss of generality, assume $s_b \leq r_a$. Then,

$$3r_a = d(a, B) \leq d(a, b) \leq d(a, z) + d(z, b) < r_a + s_b \leq r_a + r_a = 2r_a,$$

a contradiction. Thus, $U \cap V = \emptyset$. Hence, X is normal. As X is T_2, we have X is T_4. □

Proposition 23.6: ($T_{3\frac{1}{2}} \not\Rightarrow T_4$: Deleted Tychonoff plank)

The deleted Tychonoff plank $X := [0, \Omega] \times [0, \omega] \setminus \{(\Omega, \omega)\}$ is a $T_{3\frac{1}{2}}$ space, which is not T_4.

Proof

Recall that the ordinal spaces $[0, \Omega]$ and $[0, \omega]$ are compact, T_2, and hence, so is their product $T = [0, \Omega] \times [0, \omega]$. Thus, the Tychonoff plane T is T_4 and in particular, $T_{3\frac{1}{2}}$. Since being completely regular is hereditary (Check!), the subspace $X \subset T$ is $T_{3\frac{1}{2}}$.

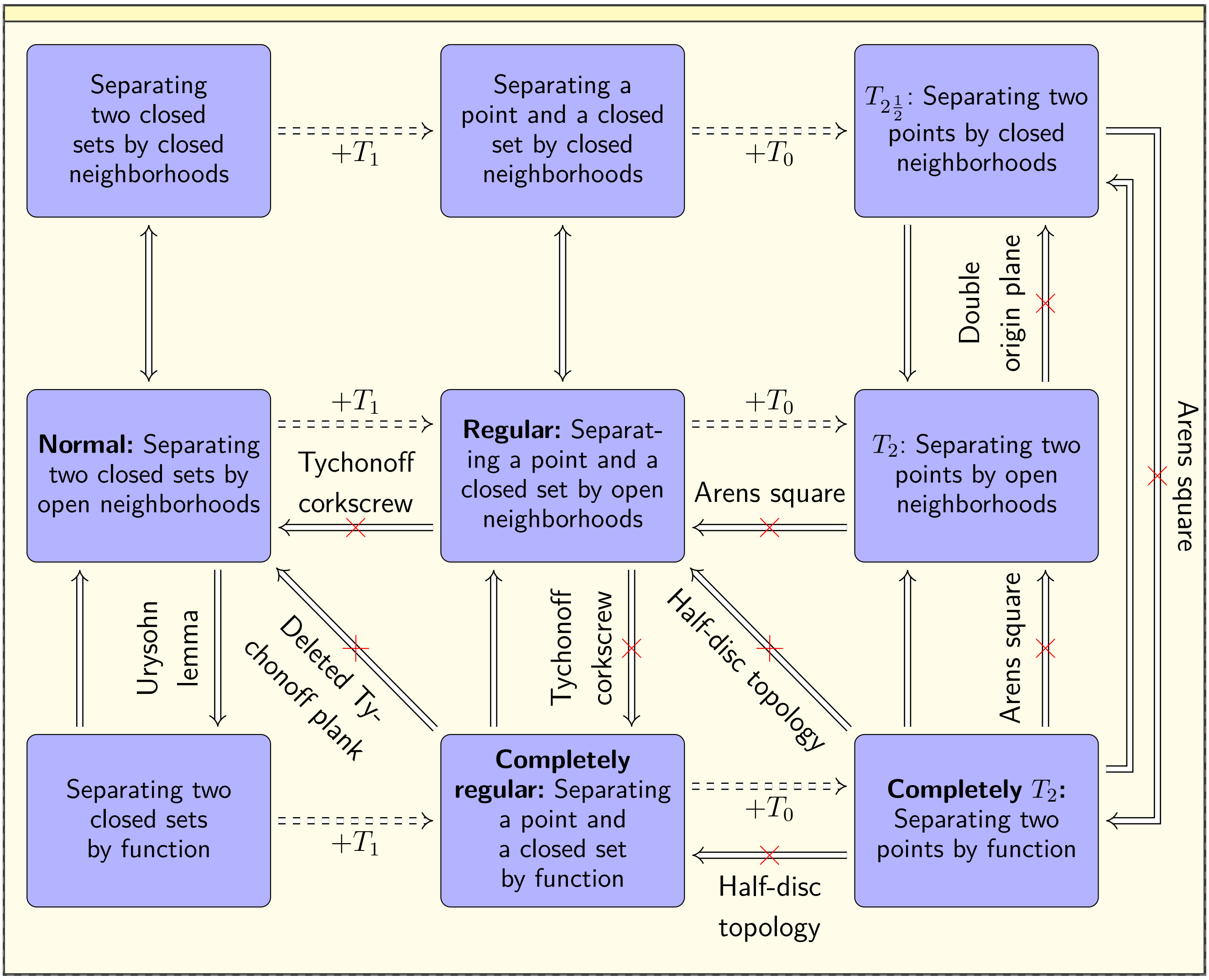
Let us show that X is not normal. Consider the sets $A = [0, \Omega) \times \{\omega\}$ and $B = \{\Omega\} \times [0, \omega)$, which are closed in the subspace topology of X. If possible, suppose there are open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$. Then, for each $0 \leq n < \omega$, there is some $0 \leq \alpha_n < \Omega$ such that $(\alpha_n, \Omega] \times \{n\} \subset B$. Now $\{\alpha_n\}_n \subset [0, \Omega)$ is a countable set, and hence, there is an upper bound $\beta \in [0, \Omega)$. Then, we have the (open) set

$$(\beta, \Omega] \times [0, \omega) = \bigcup_{0\leq n<\omega} (\beta, \Omega] \times \{n\} \subset \bigcup_{0\leq n<\omega} (\alpha_n, \Omega] \times \{n\} \subset V.$$

Now, the basic open sets of $(\beta + 1, \omega) \in A$ are of the form $(\gamma, \delta) \times (n, \omega)$, where $\beta + 1 \in (\gamma, \delta) \subset [0, \Omega)$ is an open interval. In particular, any open neighborhood of $(\beta + 1, \omega)$ will contain the set $\{\beta + 1\} \times [n, \omega)$ for some n large. Consequently, any open set containing $(b + 1, \omega)$ (and in particular, the open set U) will intersect the set V. This is a contradiction to $U \cap V = \emptyset$. Thus, X is not normal, and hence, not T_4. □

Remark 23.7: (Separation axioms implications)

Let us summarize all the observations about separation axioms so far.



23.2 Completely normal and T_5-spaces

Definition 23.8: (Completely normal space)

A normal space is called a *completely normal space* (or *hereditarily normal space*) if every subspace is again a normal space.

Proposition 23.9

Given a space X, the following are equivalent.

a) X is completely normal.

b) Every open subset of X is normal.

c) Given any two subsets $A, B \subset X$, with $\bar{A} \cap B = \emptyset = A \cap \bar{B}$, there are open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

Proof

Suppose X is completely normal. Then, clearly any open set of X is again normal. Conversely, suppose every open set of X is normal. Let $Y \subset X$ be arbitrary subspace, and let $A, B \subset Y$ be closed sets with $A \cap B = \emptyset$. Note that $A = \overline{A}^Y = Y \cap \bar{A}$ and $B = \overline{B}^Y = Y \cap \bar{B}$. Consider the open set $W = X \setminus \bar{A} \cap \bar{B}$, which is normal. Now, $Y \cap (\bar{A} \cap \bar{B}) = (Y \cap \bar{A}) \cap (Y \cap \bar{B}) = A \cap B = \emptyset$. Thus, $Y \subset W$. Now, we have the closed sets $C = \bar{A} \cap W$ and $D = \bar{B} \cap W$ in the subspace W. Then, there are open sets $U, V \subset W$ (which are also open in X as W is open), such that $C \subset U, D \subset V$ and $U \cap V = \emptyset$. Then, we have

$$A = \bar{A} \cap Y \subset \bar{A} \cap W \subset U, B = \bar{B} \cap Y \subset \bar{B} \cap W \subset V.$$

Set $U' = U \cap Y, V' = V \cap Y$, which are open in Y, and clearly disjoint. Also, $A \subset U', B \subset V'$. Thus, Y is normal. Since Y was arbitrary, we have X is completely normal.

Next, let us assume X is completely normal. Let $A, B \subset X$ be arbitrary, with $\bar{A} \cap B = \emptyset = A \cap \bar{B}$. Consider $W = X \setminus \bar{A} \cap \bar{B}$. Then, W is normal. Also, $A \cap \bar{B} = \emptyset \Rightarrow A \subset X \setminus \bar{B} \subset W$, and similarly, $B \subset W$. Consider $C = W \cap \bar{A}$ and $D = W \cap \bar{B}$, which are closed in W. Note that $C \cap D = W \cap \bar{A} \cap \bar{B} = \emptyset$. Then, there are open sets $U, V \subset W$ (which are open in X, as W is open), such that $C \subset U, D \subset V$ and $U \cap V = \emptyset$. Clearly, $A \subset C \subset U, B \subset D \subset V$. Conversely, suppose given any two sets $A, B \subset X$ with $\bar{A} \cap B = \emptyset = A \cap \bar{B}$, we have open sets $U, V \subset X$ such that $A \subset U, B \subset V, U \cap V = \emptyset$. Let us show that X is completely normal. Fix some subspace $Y \subset X$, and closed sets $A, B \subset Y$ with $A \cap B = \emptyset$. Then, $A = Y \cap \bar{A}, B = Y \cap \bar{B}$. Now, $\bar{A} \cap B = \bar{A} \cap (B \cap Y) = (\bar{A} \cap Y) \cap B = A \cap B = \emptyset$, and similarly, $A \cap \bar{B} = \emptyset$. Then, there are open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$. But then, consider $U' = Y \cap U, V' = Y \cap V$, which are open in Y. Clearly, $A \subset Y', B \subset V'$ and $U' \cap V' = \emptyset$. Thus, Y is normal. Since Y was arbitrary, we have X is completely normal. $\square$

Definition 23.10: (T_5-space)

A completely normal, T_1 space is called a *T_5-space*.

Remark 23.11: ($T_4 \not\Rightarrow T_5$: Tychonoff plank)

Clearly $T_5 \Rightarrow T_4$. But the Tychonoff plank is a T_4-space, which is not T_5, since the (open) subspace deleted Tychonoff plank is not normal.

Theorem 23.12: (Order topology $\Rightarrow T_5$)

Any order topology is T_5.

Proof

Let $(X, \leq)$ be a totally ordered space, equipped with the order topology. Clearly X is T_2. Without loss of generality, assume that $|X| \geq 2$, so that even if X has end-points, they are distinct.

Let $A, B \subset X$ be arbitrary sets, with $\bar{A} \cap B = \emptyset = A \cap \bar{B}$.

Step 1: Consider the set $Y = X \setminus (A \cup B)$. On Y, let us define an equivalence relation : $x \sim y$ if and only if the closed interval

$$[\min\{x, y\}, \max\{x, y\}] := \{z \in X \mid \min\{x, y\} \leq z \leq \max\{x, y\}\}$$

is contained in Y. Then, the equivalence classes represent the largest connected intervals in Y. By *axiom of choice*, let us choose a representative, say, $f(C)$ from each of the class C.

Step 2: For each $a \in A$, which is not the right end-point of X (if it exists at all), let us define $a < q_a$ as follows.

a) If a has an immediate successor in X, choose that to be q_a.

b) If a has no immediate successor, then for any $a < x$, we have $[a, x)$ contains a point of X. That is, a is then a *right* accumulation point. We consider two possibilities.

   i) Suppose a is a right accumulation point of A. Choose any $q_a \in A$ such that $a < q_a$ and $(a, q_a) \cap B = \emptyset$. This is possible since $A \cap \bar{B} = \emptyset$.

   ii) Suppose a is a right accumulation point of X, but not of A. In this case, consider $Z \coloneqq \{z \in A \cup B \mid z > x\}$. Since $A \cap \bar{B} = \emptyset$, we have some interval $[x, a) \cap Z = \emptyset$. Consequently, it follows that x is least upper bound of a unique component, say, C of Y. Let us take q_a to be the chosen point $f(C)$.

Observe that $[a, q_a)$ is always disjoint from B. Similarly, for each $a \in A$, which is not the left end-point of X, we choose $p_a < a$ as follows.

a) If a has an immediate predecessor in X, choose that to be p_a.

b) If a has no immediate predecessor in X, then a is a *left* accumulation point. We consider two possibilities.

   i) If a is an accumulation point of A, choose $p_a < a$ such that $(p_a, a) \cap B = \emptyset$.

   ii) If a is not an accumulation point of A, then as argued earlier, a is greatest lower bound of a unique component, say, C of Y. Take p_a to be the chosen point $f(C)$.

Note that a point $a \in A$ cannot be simultaneous both the end-points, since $|X| \geq 2$. Reversing the role of A and B, for each $b \in B$, we choose $p_b < b < q_b$ accordingly as well. Finally, for any $x \in A \cup B$, let us define the interval

$$I_x = (p_x, q_x) \quad \text{or,} \quad (p_x, x], \quad \text{or,} \quad [x, q_x),$$

as necessary. Clearly, for $a \in A$, we have I_a is an open neighborhood of a, disjoint from B. Similarly, for $b \in B$, we have I_b is an open neighborhood of b, disjoint from A.

Step 3: Say, $a \in A$ and $b \in B$ are fixed. Without loss of generality, assume $a < b$. Let us show that $I_a \cap I_b = \emptyset$. Suppose not. Then, $I_a \cap I_b = (p_b, q_a) \neq \emptyset$, and in particular, $p_b < q_a$. Clearly $b \notin I_a$, as $I_a \cap B = \emptyset$, and similarly, $a \notin I_b$. Thus, it follows that $a \leq p_b$ and $q_a \leq b$. Now, if q_a was the immediate successor of a, then, $I_a \cap I_b = (p_b, q_a) = \emptyset$. Thus, a must be defined by the other two cases (in particular, a is a right accumulation point). By the same argument, p_b is not the immediate predecessor of b, and consequently b is a left accumulation point. Now $p_b \notin B$, as otherwise $I_a \cap B \neq \emptyset$, and similarly, $q_a \notin A$. Thus, by previous step, p_b is not an accumulation point of B and q_a is not an accumulation point of A. Hence, there are components $C_1, C_2 \subset Y$ such that $(a, q_a) \subset C_1$ and $(p_b, b) \subset C_2$, where $q_a = f(C_1)$ and $p_b = f(C_2)$. Now,

$$\emptyset \neq I_a \cap I_b = (a, q_a) \cap (p_b, b) \subset C_1 \cap C_2.$$

Since C_1, C_2 are equivalence classes, the only possibility is $C_1 = C_2$, whence, $q_a = f(C_1) = f(C_2) = p_b$. But then, $I_a \cap I_b = \emptyset$, a contradiction.

Step 4: As a final step, consider the open sets

$$U := \bigcup_{a \in A} I_a, \qquad V := \bigcup_{b \in B} I_b.$$

Clearly, $A \subset U, B \subset V$. Moreover, $U \cap V = \emptyset$ by the previous step. Thus, X is perfectly normal. In particular, any linearly ordered space is T_5. □

Corollary 23.13: (Ordinal spaces are T_5)

Every ordinal space is T_5. In particular, $[0, \omega], [0, \Omega], [0, \Omega)$ are all T_5.

Remark 23.14: (LOTS)

A totally ordered space $(X, \leq)$ equipped with the order topology is also known as *LOTS*, i.e, *linearly ordered topological space*. A subspace of a LOTS is known as a *GO-space*, i.e, *generalized order space*. The above theorem essentially proves that a GO-space is T_5.
Note that the subspace topology of a GO-space may be strictly finer than the induced order topology from the restriction of the total order! As an example, consider $A = \{0\} \cup (1, 2]$ as a subspace of $\mathbb{R}$, which is a LOTS. In the subspace topology of A, $\{0\}$ is open, but it is not open in the induced order topology. On the other hand, A is homeomorphic to $(1, 2] \cup \{3\}$, which is clearly a LOTS. Thus, A is actually a LOTS, but under a different order.

23.3 Perfectly normal and T_6-spaces

Definition 23.15: (Perfectly normal space)

A space X is called a *perfectly normal space* if given closed sets $A, B \subset X$ with $A \cap B = \emptyset$, there is a continuous function $f : X \to [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$. That is, a function *precisely* separates any two disjoint closed sets.

Theorem 23.16: (Vedenissoff's theorems)

Given a space X, the following are equivalent.

a) X is perfectly normal.

b) X is normal, and every closed set of C can be written as a countable intersection of open sets (i.e, X is a G_δ-space).

c) Every closed set $A \subset X$ is the zero set of a continuous function, i.e, there is a continuous function $f : X \to \mathbb{R}$ such that $A = f^{-1}(0)$.

Proof

Suppose X is perfectly normal. Then clearly X is normal, as functional separation leads to separation by open neighborhoods. Let $C \subset X$ be an arbitrary closed set. We show that C is a

G_δ-set, i.e, countable intersection of open sets of X. We have a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = C$ and $f^{-1}(1) = \emptyset$. Then, we have open sets $U_n := f^{-1}\left[0, \frac{1}{n}\right)$. Clearly, $C = \bigcap_{n\geq 1} U_n$. Thus, X is a normal, G_δ-space.

Next, suppose X is a normal, G_δ-space. Let $A \subset X$ be a closed set. Then, $A = \bigcap_{n\geq 1} U_n$ for some open sets $U_n \subset X$. Without loss of generality, we can assume that $U_{n+1} \subset U_n$ for each $n \geq 1$. Now, for each $n \geq 1$, we have disjoint closed sets A and $B_n := X \setminus U_n$. Then, as X is normal, by Urysohn's lemma we have a continuous map $f_n : X \to [0,1]$ such that $f_n(A) = \{0\}$ and $f_n(B_n) = \{1\}$. Consider a function $f : X \to [0,1]$ defined by

$$f(x) := \sum_{n\geq 1} \frac{f_n(x)}{2^{n+1}}, \qquad x \in X.$$

It follows that f is continuous. Clearly, $f(A) = 0$. Suppose $x \notin A$. Then, $x \notin U_{n_0}$ for some n_0. So, $x \in B_{n_0} \subset B_n$ for all $n \geq n_0$, and hence, $f_n(x) = 1$ for $n \geq 1$. We have

$$f(x) \geq \sum_{n\geq n_0} \frac{f_n(x)}{2^{n+1}} = \sum_{n\geq n_0} \frac{1}{2^{n+1}} = \frac{1}{2^{n_0+1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots\right) = \frac{1}{2^{n_0}} > 0.$$

Hence, $f^{-1}(0) = A$. As A was arbitrary closed set, this proves c).

Finally, suppose every closed set is the 0-set of some continuous function. Let $A, B \subset X$ be closed set with $A \cap B = \emptyset$. We have $f, g : X \to \mathbb{R}$ such that $f^{-1}(0) = A$ and $g^{-1}(0) = B$. As $A \cap B = \emptyset$, we have $f + g$ is nonvanishing. Consider the continuous function $h = \frac{f}{f+g}$. Clearly, $h : X \to [0,1]$. Also, $h(x) = 0 \Leftrightarrow f(x) = 0 \Leftrightarrow x \in A$, and $h(x) = 1 \Leftrightarrow f(x) = f(x) + g(x) \Leftrightarrow g(x) = 0 \Leftrightarrow x \in B$. Thus, $h^{-1}(0) = A$ and $h^{-1}(1) = B$. Hence, X is perfectly normal. □

Proposition 23.17: ($T_6 \Rightarrow T_5$)

Any subspace of a perfectly normal space is again perfectly normal. Consequently, a perfectly normal space is completely normal.

Proof

Let X be a perfectly normal space. Say, $Y \subset X$ is arbitrary subset, and $A \subset Y$ be closed. Then, $A = Y \cap \bar{A}$. We have a continuous function such that $\bar{A} = f^{-1}(0)$. Then, the restriction $g := f|_Y$ is again continuous, and clearly, $g^{-1}(0) = f^{-1}(0) \cap Y = \bar{A} \cap Y = A$. Thus, Y is perfectly normal, and hence, normal. In particular, X is completely normal. □

Definition 23.18: (T_6-space)

A space is called a *T_6-space* if it is perfectly normal, and T_1.

Proposition 23.19: (Metrizable $\Rightarrow T_6$)

Any metrizable space is T_6.

Proof

Fix a metric d on X. Given any closed sets $A, B \subset X$ with $A \cap B = \emptyset$, we have the continuous map

$$f(x) := \frac{d(x, A)}{d(x, A) + d(x, B)}, \qquad x \in X.$$

Then, $f^{-1}(0) = A$ and $f^{-1}(B)$. Clearly X is T_2 (and hence, T_1). Thus, X is T_6. ☐

Day 24 : 31st October, 2025

`product of normal space`

24.1 Separation axioms : More properties and counterexamples

Proposition 24.1: ($T_5 \not\Rightarrow T_6$: The uncountable ordinal space $\overline{S_\Omega} = [0, \Omega]$)

The uncountable ordinal space $[0, \Omega]$ is a T_5-space, which is not T_6.

Proof

Since $[0, \Omega]$ is a linearly ordered space, we have $[0, \Omega]$ is T_5. Let us show that it is not G_δ. Consider $\{\Omega\}$, which is closed. If possible, suppose $\{\Omega\} = \bigcap_{n \geq 1} O_n$ for some open neighborhoods $\Omega \in O_n \subset [0, \Omega]$. Then, there is some $\alpha_n \in [0, \Omega)$ such that $\Omega \in (\alpha_n, \Omega] \subset O_n$. Since any countable collection of $[0, \Omega)$ is bounded above, we have some $\beta \in [0, \Omega)$ such that $\beta > \alpha_n$ for all $n \geq 1$. But then, $\{\Omega\} \subsetneq (\beta, \Omega] \subset \bigcap_{n \geq 1} O_n$. Thus, $\{\Omega\}$ fails to be a G_δ-set. Hence, $[0, \Omega]$ is not T_6. ☐

Remark 24.2

It is fact that the first uncountable ordinal $S_\Omega = [0, \Omega)$ is also not a G_δ-space, and hence, is not a T_6-space. Clearly, S_Ω, being a linearly ordered space, is T_5. Moreover, any ordinal space which is also a G_δ-space, is necessarily countable. Thus, all uncountable ordinal spaces are T_5 but not T_6.

Proposition 24.3: (Product of T_5 is not T_5)

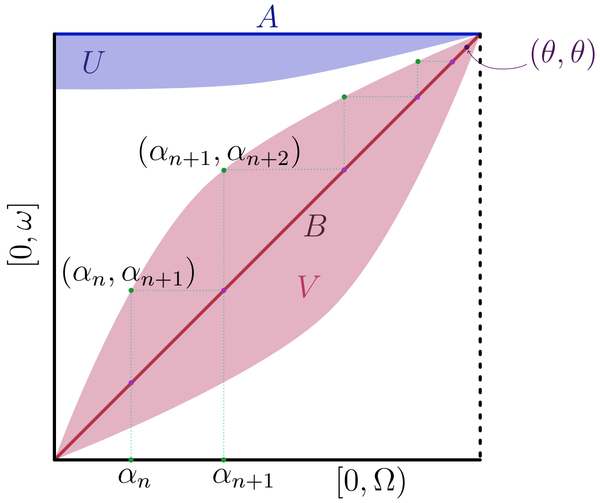
The product space $X = [0, \Omega) \times [0, \Omega]$ of two T_5 spaces is not T_5. In fact, the product is not even normal. Thus, product of T_4-spaces need not be T_4 either.

Proof

Since linearly ordered spaces are T_5, we have both $[0, \Omega)$ and $[0, \Omega]$ are T_5. Let us show that it fails to be normal. Consider

$$A := [0, \Omega) \times \{\Omega\}, \quad B := \{(\alpha, \alpha) \mid \alpha \in [0, \Omega)\}.$$

Note that A is the intersection of the closed set $[0, \Omega] \times \{\Omega\} \subset [0, \Omega] \times [0, \Omega]$ with the subspace $[0, \Omega) \times [0, \Omega]$. Similarly, B is the intersection of the diagonal $\Delta = \{(\alpha, \alpha) \mid \alpha \in [0, \Omega]\}$, which is closed in $[0, \Omega] \times [0, \Omega]$ as the space $[0, \Omega]$ is T_2, with the subspace X. Clearly, $A \cap B = \emptyset$. If possible, suppose there are open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.



For each $0 \leq \alpha < \Omega$, consider any $\alpha < \beta < \Omega$. If for all such β, we have $(\alpha, \beta) \in V$, then the limit (α, Ω) will be a limit point of V. But this contradicts $(\alpha, \Omega) \in U$ and $U \cap V = \emptyset$. Thus, there is some $\alpha < \beta < \Omega$ such that $(\alpha, \beta) \notin V$. Let $\beta(\alpha)$ be the least such element, which exists as $[0, \Omega)$ is well-ordered. Let us now construct a sequence $\{\alpha_n\} \subset [0, \Omega)$ as follows. Start with $\alpha_1 = 0$. Then, set $\alpha_{n+1} = \beta(\alpha_n)$ for all $n \geq 1$. By construction, $\alpha_1 < \alpha_2 < \dots$. Let $\theta \in [0, \Omega)$ be the least upper bound of the sequence, and we have $\theta = \lim_n \alpha_n$. Then, $\lim_n (\alpha_n, \beta(\alpha_n)) = \lim_n (\alpha_n, \alpha_{n+1}) = (\theta, \theta) \in B \subset V$. But by construction, $(\alpha_n, \beta(\alpha_n)) \notin V$ for all $n \geq 1$. This is a contradiction. Hence, A, B cannot be separated by open neighborhoods. Thus, X is not normal, and hence, not T_5. □

Proposition 24.4: (Image of $T_{3\frac{1}{2}}$ need not be $T_{3\frac{1}{2}}$)

Continuous image of a $T_{3\frac{1}{2}}$-space need not be $T_{3\frac{1}{2}}$.

Proof

Recall the deleted Tychonoff plank $X = [0, \Omega] \times [0, \omega] \setminus \{(\Omega, \omega)\}$. In X, we have seen two closed sets $A = [0, \Omega) \times \{\omega\}$ and $B = \{\Omega\} \times [0, \omega)$, which are disjoint, but cannot be separated by open sets. Consider the quotient map $q : X \to X/A$. In X/A, observe that $q(B)$ is a closed set, since $q^{-1}(q(B)) = B$ is closed. Also, the point $a_0 = q(A)$ is not in $q(B)$. If possible, suppose there are open sets $U, V \subset X/A$ such that $a_0 \in U, A \subset V$ and $U \cap V = \emptyset$. Then, $A \subset q^{-1}(U), B \subset q^{-1}(V)$ are open sets such that $q^{-1}(U) \cap q^{-1}(V) = q^{-1}(U \cap V) = \emptyset$. This is a contradiction. Hence, X/A is not even regular, and in particular, not completely regular. □

24.2 Urysohn's metrization theorem

Proposition 24.5

Let X be a completely regular space, and $\mathcal{B}$ be a fixed basis of X. Assume $\mathcal{B}$ is infinite. Then, there exists a family $\mathcal{F}$ of continuous functions $X \to [0, 1]$, with $|\mathcal{F}| \leq |\mathcal{B}|$, such that given any closed $A \subset X$ and $x \in X \setminus A$, there is a function $f \in \mathcal{F}$ such that $f(x) = 0$ and $f(A) = 1$.

Proof

Given any pair of sets $(U, V) \in \mathcal{B} \times \mathcal{B}$, call it *good* if there is a continuous map $f : X \to [0, 1]$ such that $f(U) = 0$ and $f(X \setminus V) = 1$. Denote by $\mathcal{G}$ the collection of good pairs. Clearly,

$|\mathcal{G}| \leq |\mathcal{B} \times \mathcal{B}| = |\mathcal{B}|$. For each good pair $(U, V) \in \mathcal{G}$, choose a function $f_{U,V}$, and denote the family $\mathcal{F} = \{f_{U,V} \mid (U, V) \in \mathcal{B}\}$. Again, $|\mathcal{F}| = |\mathcal{G}| \leq |\mathcal{B}|$. We claim that $\mathcal{F}$ separates any closed set and a disjoint point.

Let $A \subset X$ be a closed set, and $x \in X \setminus A$ be a point. Get a basic open set $V \in \mathcal{B}$ such that $x \in V \subset X \setminus A$. By complete regularity, there is a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(X \setminus V) = 1$. Now, $x \in f^{-1}\left[0, \frac{1}{2}\right)$ is an open neighborhood, so there is a basic open set $U \in \mathcal{B}$ such that $x \in U \subset f^{-1}\left[0, \frac{1}{2}\right)$. Construct the function $g : X \to [0, 1]$ by

$$g(y) = \begin{cases} 0, & f(y) \leq \frac{1}{2}, \\ 2\left(f(y) - \frac{1}{2}\right), & f(y) \geq \frac{1}{2}. \end{cases}$$

By pasting lemma, g is continuous. Moreover, $g(U) = 0, g(X \setminus V) = 1$. Thus, $(U, V) \in \mathcal{G}$ is a good pair. But then we have a $f_{U,V} \in \mathcal{F}$. Clearly, $f_{U,V}$ separates x and A, since $x \in U$ and $V \subset X \setminus A \Rightarrow A \subset X \setminus V$. ☐

Corollary 24.6

Let X be a second countable, completely regular space. Then there is a countable collection $\mathcal{F}$ of functions such that any closed set $A \subset X$ and any point $x \in X \setminus A$ can be separated by some function $f \in \mathcal{F}$.

Theorem 24.7: (Tychonoff embedding theorem)

Let X be a Tychonoff space (i.e, $T_{3\frac{1}{2}}$), and $\mathcal{B}$ be a fixed basis. Then, X is homeomorphic to a subspace of the cube $\mathcal{C} = [0, 1]^{|\mathcal{B}|}$

Proof

Get a family $\mathcal{F}$ of functions, with $|\mathcal{F}| \leq |\mathcal{B}|$. We prove an embedding $X \hookrightarrow [0, 1]^{|\mathcal{F}|}$, which is sufficient. Indeed, we have a map $\mathfrak{F} : X \to [0, 1]^{|\mathcal{F}|}$ defined by

$$\pi_f\left(\mathfrak{F}(x)\right) = f(x), \quad f \in \mathcal{F}, \qquad x \in X.$$

By the properties of the product topology, $\mathfrak{F}$ is continuous. As the space X is T_1, it follows that $\mathcal{F}$ separates points, and consequently, $\mathfrak{F}$ is injective. We show that $\mathfrak{F}$ is open onto its image.

Let $O \subset X$ be open, and $y \in \mathfrak{F}(O)$. Pick $x \in \mathfrak{F}^{-1}(y) \cap O$. Since $\mathcal{F}$ separates points and closed sets, there is some $f \in \mathcal{F}$ such that $f(x) = 0$ and $f(X \setminus O) = 1$. Consider $W := \pi_f^{-1}([0, 1))$, which is open in the cube. Moreover, $W \cap \mathfrak{F}(X) \subset \mathfrak{F}(O)$. Indeed, for any $z \in Z$, with $\mathfrak{F}(z) \in W$, we must have $f(z) \neq 1 \Rightarrow z \notin X \setminus O \Rightarrow z \in O$, and thus, $\mathfrak{F}(z) \in \mathfrak{F}(O)$. In particular, $f(x) = 0 \Rightarrow y = \mathfrak{F}(x) \in W \Rightarrow y \in W \cap \mathfrak{F}(X) \subset \mathfrak{F}(O)$. As y was arbitrary, we have $\mathfrak{F}(O)$ is open. But then $\mathfrak{F}$ is a homeomorphism onto its image. In particular, X can be identified as a subspace of $[0, 1]^{|\mathcal{F}|}$. If $|\mathcal{F}| < \mathcal{B}$, then one can canonically see $[0, 1]^{|\mathcal{F}|}$ as a subspace of $[0, 1]^{|\mathcal{B}|}$. This concludes the proof. ☐

Theorem 24.8: (Urysohn's metrization theorem)

Any T_3, second countable space is metrizable.

Proof

Since X is second countable, it is Lindelöf. A regular, Lindelöf space is normal. Thus, X is T_4, and hence, $T_{3\frac{1}{2}}$. But then by the Tychonoff embedding theorem, X can be identified as a subspace of $[0,1]^\omega$, where $\omega = |\mathbb{N}|$. Now, $[0,1]^\omega$ is a metric space (being the countable product of metric spaces). Hence, X is a metric space. $\square$

Day 25 : 5$^{\text{th}}$ November, 2025

`Lebesgue number property -- Tietze extension`

25.1 Lebesgue number property

Definition 25.1: (Lebesgue number property)

A metric space (X,d) is said to have the *Lebsgue number property* if given any open cover $\{U_\alpha\}$, there is a number $\delta > 0$ (known as the *Lebesgue number of the covering*) such that for any set $A \subset X$ with $\mathrm{Diam} A < \delta$, we have $A \subset U_\alpha$ for some α.

Theorem 25.2: (Lebesgue number property and uniform continuity)

Let (X,d) be a metric space. Suppose every continuous map $f : X \to \mathbb{R}$ is uniformly continuous. Then, X has the Lebesgue number property

Proof

Suppose not. Then there exists an open cover $\mathcal{U} = \{U_\alpha\}$ without a Lebesgue number. Consequently, for each $n \geq 1$, there is a point x_n such that the set $A_n = B_d\left(x_n, \frac{1}{n}\right)$ is *not contained in any of the* U_α, i.e, $U_\alpha \setminus A_n \neq \emptyset$ for all α. Choose some $y_n \in A_n$ with $y_n \neq x_n$. Note that A_n is not singleton, otherwise $A_n \subset U_\alpha$ for some α, and so, y_n can always be chosen.

Let us observe that $\{x_n\}$ and $\{y_n\}$ has no convergent subsequence. If possible, suppose $x_{n_k} \to x$. Then, $x \in U_\alpha$ for some α. Now, there is some $\epsilon > 0$ such that $x \in B_d(x, 2\epsilon) \subset U_\alpha$. Since $x_{n_k} \to x$, there exists some $N \geq 1$ such that $x_{n_k} \in B_d(x,\epsilon) \subset B_d(x, 2\epsilon) \subset U_\alpha$ for all $n_k \geq N$. But then for some n_k sufficiently large, it follows from the triangle inequality that $A_{n_k} \subset B_d(X, 2\epsilon) \subset U_\alpha$, a contradiction. On the other hand, if $y_{n_k} \to y$, then it is clear that the subsequence $x_{n_k} \to y$, which is a contradiction. Thus, none of the sequences admit a convergent subsequence.

Next, we construct two disjoint closed sets from the two sequences. Set $x_{n_1} = x_1, y_{n_1} = y_1$. Clearly $\{x_{n_1}\} \cap \{x_{n_1}\} = \emptyset$. Choose $n_2 > n_1$, such that $x_{n_2} \neq x_{n_1}, y_{n_2} \neq y_{n_1}$, and $\{x_{n_1}, x_{n_2}\} \cap \{y_{n_1}, y_{n_2}\}$. This is possible, since otherwise the sequence will have to be eventually constant. Inductively, assume that we have constructed $\{x_{n_1}, \dots, x_{n_k}\}$ and $\{y_{n_1}, \dots, y_{n_k}\}$, which are disjoint sets of distinct points, with $n_1 < n_2 < \cdots < n_k$. Now, each of the points $\{x_{n_1}, \ldots, x_{n_k}, y_{n_1}, \ldots, y_{n_k}\}$ can only repeat finitely many times in $\{x_n\}$ and in $\{y_n\}$ (since otherwise there will be a convergent subsequence). Hence, we can choose $x_{n_{k+1}}, y_{n_{k+1}}$ at the induction step, so that $\{x_{n_1}, \ldots, x_{n_{k+1}}\}, \{y_{n_1}, \ldots, y_{n_{k+1}}\}$ are disjoint set of distinct points, with $n_{k+1} > n_k$. Set $A := \{x_{n_i}\}$ and $B := \{y_{n_i}\}$. By construction, $A \cap B = \emptyset$. Also, A, B are closed, since there are no (sub)sequential limits, and thus, A, B contains all of their limit points (which are none).

Now, (X, d) is a T_4-space. Hence, there is a continuous function $f : X \to \mathbb{R}$ with $f(A) = 0$ and $f(B) = 1$. We claim that f is not uniformly continuous. Indeed, for $\epsilon = \frac{1}{2}$ fixed, consider any $\delta > 0$ small. We must have some n_k with $\frac{1}{n_k} < \delta$. Now, $d(x_{n_k}, y_{n_k}) < \delta$, but $|f(x_{n_k}) - f(y_{n_k})| = |0 - 1| = 1 > \epsilon$. This contradicts the hypothesis. Hence, X has the Lebesgue number property. □

Exercise 25.3

Show that a metric space X has the Lebesgue number property if and only for any metric space Y any continuous map $f : X \to Y$ is uniformly continuous.

25.2 Tietze extension theorem

Theorem 25.4: (Tietze Extension Theorem)

A space X is normal if and only if given any closed set $A \subset X$ and continuous map $f : A \to \mathbb{R}$, there is an extension $\tilde{f} : X \to \mathbb{R}$, i.e, there is a continuous map $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}(a) = f(a)$ for all $a \in A$.

Proof

Suppose X is normal. Firstly, let us consider a map $f : A \to [-1, 1]$. Define

$$A_1 :=:= \left\{ x \in A \,\middle|\, f(x) \geq \frac{1}{3} \right\} = f^{-1}\left[\frac{1}{3}, 1\right], \quad B_1 := \left\{ x \in A \,\middle|\, f(x) \leq -\frac{1}{3} \right\} = f^{-1}\left[-1, -\frac{1}{3}\right].$$

Clearly A_1, B_1 are disjoint closed sets of A, and hence, closed in X. As X is normal, by the Urysohn's lemma, we have continuous function $f_1 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ such that

$$f_1(A_1) = \frac{1}{3}, \quad f_1(B_1) = -\frac{1}{3}.$$

Now, for any $x \in A$ we have 3 cases.

a) $x \in A_1 \Rightarrow f_1(x) = \frac{1}{3}, f(x) \in \left[\frac{1}{3}, 1\right] \Rightarrow |f(x) - f_1(x)| \leq \frac{2}{3}$.

b) $x \in B_1 \Rightarrow f_1(x) = -\frac{1}{3}, f(x) \in \left[-1, -\frac{1}{3}\right] \Rightarrow |f(x) - f_1(x)| \leq \frac{2}{3}$.

c) $x \in A \setminus A_1 \cup B_1 \Rightarrow f_1(x), f(x) \in \left[-\frac{1}{3}, \frac{1}{3}\right] \Rightarrow |f(x) - f_1(x)| \leq \frac{2}{3}$.

In other words, we have a continuous map $g_1 := f - f_1 : A \to \left[-\frac{2}{3}, \frac{2}{3}\right]$. We repeat the process for g_1 instead of f. That is, we define $A_2 := g_1^{-1}\left[-\frac{2}{3}, -\frac{2}{9}\right]$ and $B_2 := g_1^{-1}\left[\frac{2}{9}, \frac{2}{3}\right]$. We get a function $f_2 : X \to \left[-\frac{2}{9}, \frac{2}{9}\right]$, such that $f_2(A_2) = \frac{2}{9}, f_2(B_2) = -\frac{2}{9}$. Clearly, $|g_1 - f_2| \leq \left(\frac{2}{3}\right)^2$ on points of A. Define, $g_2 := g_1 - f_2 = f - f_1 - f_2$, clearly, $g_2 : A \to \left[-\frac{2}{9}, \frac{2}{9}\right]$. Inductively, we define

$f_n : X \to \left[-\frac{2}{3^n}, \frac{2}{3^n}\right]$, such that

$$\left| f - \sum_{i=1}^{n} f_i \right| \leq \left(\frac{2}{3}\right)^n, \qquad \text{on points of } A.$$

Let us define $F(x) = \sum_{i=1}^{\infty} f_i(x)$. Observe that for any fixed $x \in X$, the series sum converges, since the partial sums

$$\left| \sum_{i=1}^{n} f_i(x) \right| \leq \sum_{i=1}^{n} \frac{2}{3^i}$$

are dominated by the geometric series. Moreover, for $a \in A$ we have,

$$\left| f(a) - \sum_{i=1}^{n} f_i(a) \right| \leq \left(\frac{2}{3}\right)^n \to 0 \Rightarrow F(a) = f(a).$$

In other words, F extends f. Let us show that F is continuous.

Fix some $x \in X$ and $\epsilon > 0$. Then, pick $N \geq 1$ such that $\sum_{n>N} \left(\frac{2}{3}\right)^n < \frac{\epsilon}{4}$. For $i = 1, \ldots, N$, using the continuity of f_i, pick open neighborhoods $x \in U_i \subset X$ such that

$$y \in U_i \Rightarrow |f_i(y) - f_i(x)| < \frac{\epsilon}{2N}.$$

Set $U = \bigcap_{i=1}^{N} U_i$, which is an open neighborhood of x. Then, for any $y \in U$ we have

$$\begin{aligned} |F(y) - F(x)| &= \left| \sum_{i=1}^{\infty} f_i(y) - f_i(x) \right| \\ &\leq \sum_{i=1}^{N} |f_i(y) - f_i(x)| + \sum_{i>N} |f_i(y) - f_i(x)| \\ &< N \cdot \frac{\epsilon}{2N} + 2 \sum_{i>N} \left(\frac{2}{3}\right)^i \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Consequently, F is continuous at $x \in X$. Since x was arbitrary, we have the continuous extension $F : X \to [-1, 1]$ of $f : A \to [-1, 1]$.

Now, let us consider the general case. If $f : A \to [a, b]$ was given, we can use any homeomorphism $[a, b] \to [-1, 1]$ and its inverse, to get an extension $X \to [a, b]$. In case $f : A \to \mathbb{R}$ is given, we can use a homeomorphism $\mathbb{R} \to (-1, 1)$ to assume that the map is $f : A \to (-1, 1)$. Then, we end up with an extension $F_0 : X \to [-1, 1]$. Consider the set $A_0 = \{x \in X \mid F_0(x) \in \{\pm 1\}\} = F_0^{-1}(\{\pm 1\})$, which is clearly a closed set, disjoint from A. Then, by Urysohn's lemma, we have continuous map $\phi : X \to [0, 1]$ such that $\phi(A_0) = 0$ and $\phi(A) = 1$. Consider the function $F = \phi F_0$. Then, F is continuous, and clearly, $F(a) = F_0(a) = f(a)$ for any $a \in A$. Observe that $F : X \to (-1, 1)$. This concludes one direction of the proof.

Conversely, assume that given any closed $A \subset X$, and any continuous function $f : A \to \mathbb{R}$, there is a continuous extension $\tilde{f} : X \to \mathbb{R}$. Let $A, B \subset X$ be closed sets with $A \cap B = \emptyset$. Then, on the closed set $C = A \cup B$ consider the function $f_0 : C \to [0, 1]$ given by $f_0(a) = 0$ for all $a \in A$, and $f_0(b) = 1$ for all $b \in B$. Clearly it is continuous. Then, we have an extension $f : X \to \mathbb{R}$ such that $f(A) = 0$ and $f(B) = 1$. By modifying the range of f, we can get the function $X \to [0, 1]$ as well. Thus, X is a normal space. □

Exercise 25.5

Assuming Tietze extension theorem, prove the Urysohn's lemma!

Day 26 : 6^{th} November, 2025

`completely metrizable space -- completion --` G_δ`-subspace of completely metrizable space`

26.1 Completely metrizable space

Definition 26.1: (Cauchy sequence)

A sequence x_n in a metric space (X, d) is called a *Cauchy sequence* if given $\epsilon > 0$, there exists some $N = N_\epsilon \geq 1$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 26.2: (Complete metric space)

A metric space (X, d) is called *complete* if every Cauchy sequence in (X, d) converges.

Exercise 26.3

Given a metric space (X, d), we have a new metric $\bar{d}(x, y) = \min\{d(x, y), 1\}$, which is clearly bounded. Show that (X, d) is complete if and only if $(X, \bar{d})$ is complete.

Example 26.4

$\mathbb{R}$ with the usual metric is complete, but $X = (0, \infty)$ is not complete. Indeed, $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence (with the usual distance metric), which does not converge. On the other hand, consider

$$d(x, y) = |x - y| + \left|\frac{1}{x} - \frac{1}{y}\right|, \quad x, y \in X.$$

Check that d is a complete metric on X, inducing the same topology. Indeed, if $\{x_n\}$ is a Cauchy sequence in this metric, then both $\{x_n\}$ and $\left\{\frac{1}{x_n}\right\}$ are Cauchy in $\mathbb{R}$ with the usual metric, which implies $x_n \to c \neq 0$ (as we must have $\frac{1}{x_n} \to \frac{1}{c}$). Thus, $(0, \infty)$ is completely metrizable.

Example 26.5: ($\mathbb{Q}$ is not complete)

In $\mathbb{Q}$, consider the following sequence

$$x_1 = 1, \quad x_{n+1} = \frac{x_n + \frac{2}{x_n}}{2}, \; n \geq 1.$$

This sequence converges to $\sqrt{2}$ in $\mathbb{R}$, and hence, is a Cauchy sequence. Clearly, $\{x_n\} \subset \mathbb{Q}$ does not converge. Thus, $\mathbb{Q}$ is not complete with the usual metric.

Definition 26.6: (Completely metrizable space)

A space X is called a *completely metrizable space* if there exists a complete metric d on X inducing the topology.

Exercise 26.7

Check that complete metrizability is a topological property. That is, check that if X is homeomorphic to Y, and if Y is completely metrizable, then so is X.

Theorem 26.8: ($\mathbb{Q}$ is not completely metrizable)

A completely metrizable space, without any isolated point, is uncountable. Consequently, $\mathbb{Q}$ is not a completely metrizable space.

Proof

Suppose (X, d) is a complete metric space, without isolated points. Choose two distinct point $x_0, x_1 \in X$. This is possible, as X has no isolated point. Get open balls U_0, U_1 of radius ≤ 1 such that

$$x_0 \in U_0,\ x_1 \in U_1,\ \overline{U_0} \cap \overline{U_1} = \emptyset.$$

This is possible as X is T_3. Next, get more distinct points $x_{00}, x_{01} \in U_0 \setminus \{x_0\}$ and $x_{10}, x_{11} \in U_1 \setminus \{x_1\}$. Again, this is possible since there are no isolated points. Get open neighborhoods of radius $\leq \frac{1}{2}$

$$x_{00} \in U_{00} \subset \overline{U_{00}} \subset U_0,\ x_{01} \in U_{01} \subset \overline{U_{01}} \subset U_0, \quad x_{10} \in U_{10} \in \overline{U_{10}} \subset U_1,\ x_{11} \in U_{11} \subset \overline{U_{11}} \subset U_1,$$

with

$$\overline{U_{00}} \cap \overline{U_{01}} = \emptyset = \overline{U_{10}} \cap \overline{U_{11}}.$$

Inductively contitue getting points and open sets with disjoint closures. Thus, for any finite length word s formed by $\{0,1\}$ we have a unique point x_s contained in an open set U_s of radius $\leq \frac{1}{|s|}$, where $|s|$ is the length of the word. Note that this is a countable infinite collection of points (and open sets), since the collection of all finite words formed by $\{0,1\}$ is countable infinite. Moreover, for two distinct words s, t, if they are not sub-word of the other, then $\overline{U_s} \cap \overline{U_t} = \emptyset$. If $s \subset t$, then $\overline{U_t} \subset U_s$.

Let us now consider s to be an infinite word formed by $\{0,1\}$. Denote s_n to be the initial word of s of length n, and set $x_n := x_{s_n}$. Let us check that $\{x_{s_n}\}$ is Cauchy. Let $\epsilon > 0$ be given, and fix $N \geq 1$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Observe that for any $n, m \geq N$, we have $x_n, x_m \in U_{s_N}$, and by construction, U_{s_N} is a ball with radius $\leq \frac{1}{|s_N|} = \frac{1}{N} < \frac{\epsilon}{2}$. Hence, $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$. Thus, $\{x_n\}$ is Cacuhy, which converges to a point, which we denote by x_s (where s is the infinite word).

Now, suppose s, t are two distinct infinite words of $\{0,1\}$. Then, they differ at, say, the n^{th} position. But then $\overline{U_{s_{n+1}}} \cap \overline{U_{t_{n+1}}} = \emptyset$. This implies that $x_s \neq x_s$. Consequently, for each infinite word, we have unique point in X. Since the number of infinite words are uncountable (in fact, the cardinality is same as $\mathbb{R}$), it follows that X must be uncountable.

Since $\mathbb{Q}$ is a (metrizable) space without any isolated point, it cannot be completely metrizable. □

26.2 Completion of a metric space

Definition 26.9: (Isometry)

A function $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is said to be an *isometry* if

$$d_Y(f(x), f(y)) = d_X(x, y), \quad \forall x, y \in X.$$

Definition 26.10: (Completion of a metric space)

Given a metric space (X, d_X), a complete metric space (Y, d_Y) is said to be a *completion* of X, if there exists an isometry $\iota : X \hookrightarrow Y$ such that the image $\iota(X)$ is dense in Y.

Theorem 26.11: (Completion : Existence and uniqueness)

Every metric space admits a completion, which is unique up to an isometry.

Proof

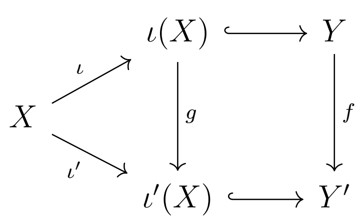
Let us first prove the uniqueness. Suppose, we have two completions $\iota : X \hookrightarrow Y$ and $\iota' : X \hookrightarrow Y'$. We have a well-defined continuous map

$$g := i' \circ \iota^{-1} : \iota(X) \to \iota'(X),$$

from a dense subset of Y to a dense subset of Y'. Note tha g is an isometry. Now, for any $y \in Y$, get a sequence $y_n \in \iota(X)$ such that $y_n \to y$. Then, $\{y_n\}$ is a Cacuchy sequence, and hence, so is $\{y'_n := g(y_n)\}$. Since Y' is complete, there is a point $y' \in Y'$ such that $y'_n \to y'$. Let us define $f(y) = y'$. We need to check that f is well-defined. Suppose $\{z_n\}$ is another sequence converging to y. Denote, $z'_n = g(z_n)$, and suppose $z'_n \to z' \in Y'$. Now,

$$d_{Y'}(y', z') = \lim d_{Y'}(y'_n, z'_n) = \lim d_{Y'}(g(y_n), g(z_n)) = \lim d_Y(y_n, z_n) = d_Y(y, y) = 0.$$

Thus, $y' = z'$, proving that f is well-defined.



Clearly f is surjective. Let us show that f is an isometry. Let $y, z \in Y$ be given. Suppose $y_n \to y, z_n \to z$, with $\{y_n\}, \{z_n\} \subset \iota(X)$. Denote, $y'_n = g(y_n), z'_n = g(z_n)$, and then, $y'_n \to y' = f(y), z'_n \to z' = f(z)$. We have,

$$d_{Y'}(f(y), f(z)) = d_{Y'}(y', z') = \lim d_{Y'}(y'_n, z'_n) = \lim d_Y(y_n, z_n) = d_Y(y, z).$$

Now, let us consider $h : Y' \to Y$ to be the isometry defined in the same way by using $\iota \circ (\iota')^{-1} : \iota'(X) \to \iota(X)$. Let us check that $h = f^{-1}$. It is clear that on points of $\iota(X)$, we have

$h\circ f=(i'\circ\iota^{-1})\circ\left(\iota\circ(\iota')^{-1}\right)=\mathrm{Id}$. Now, for any $y\in Y$ we have $y=\lim y_n$ for some $y_n\in\iota(X)$. Then,

$$(h\circ f)(y)=h\left(f\left(\lim y_n\right)\right)=\lim h\left(f\left(y_n\right)\right)=\lim y_n=y.$$

Thus, $h\circ f=\mathrm{Id}_Y$. Similarly, $f\circ h=\mathrm{Id}_{Y'}$. Thus, we have $Y=Y'$ up to an isometry.

Let us now actually prove that a completion exists! The construction is similar to how one constructs $\mathbb{R}$ from $\mathbb{Q}$. Denote $\mathcal{C}(X)$ to be the collection of all Cauchy sequences in X. Note that given two Cauchy sequences $\{x_n\},\{y_n\}$, we have $\{d(x_n,y_n)\}$ is a Cauchy sequence in $\mathbb{R}$, and hence, converges. Indeed, for any $\epsilon>0$, we have $N_1,N_2\geq 1$ such that $d(x_n,x_m)<\frac{\epsilon}{2}$ for $n,m\geq N$, and $d(y_n,y_m)<\frac{\epsilon}{2}$ for $n,m\geq N_2$. Set $N=\max\{N_1,N_2\}$. Then, for any $n,m\geq N$ we have

$$|d(x_n,y_n)-d(x_m,y_m)|\leq d(x_n,x_m)+d(y_n,y_m)<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

The first inequality follows from the triangle inequality and the symmetry! Now, define an equivalence relation $\sim$ on $\mathcal{C}(X)$ by

$$\{x_n\}\sim\{y_n\}\Leftrightarrow\lim d(x_n,y_n)=0$$

Denote $X^\star=\mathcal{C}(X)/_\sim$ to be the collection of equivalence classes. Define $d^\star:X^\star\times X^\star\to X^\star$ by

$$d^\star\left([x_n],[y_n]\right)=\lim d(x_n,y_n).$$

Let us check that $d^\star$ is well-defined. Let $\{x_n'\}$ and $\{y_n'\}$ be some other representative. Then, we have

$$|d(x_n,y_n)-d(x_n',y_n')|\leq d(x_n,x_n')+d(y_n,y_n')\to 0,\quad \text{as } n->\infty.$$

Hence, in the limit, we have $\lim d(x_n,y_n)=\lim d(x_n',y_n')$. It is easy to see that $d^\star$ is a metric on $X^\star$ (Check!). For any $x\in X$, define $\iota(x)$ to be the equivalence class of the constant sequence $\{x_n=x\}$. It follows that $\iota:X\hookrightarrow X^\star$ is an isometry (Check!).

Let us verify that $\iota(X)$ is dense in $X^\star$. Let $x^\star\in X^\star$ is represented by some Cauchy sequence $\{x_n\}\subset X$. Then, for any $\epsilon>0$, there is some $N\geq 1$ such that $d(x_n,x_m)<\frac{\epsilon}{2}$ for all $n,m\geq N$. Let $z=x_N$, and consider the point $\iota(z)$ formed by the constant sequence. Then,

$$d^\star(x^\star,\iota(z))=\lim_n d(x_n,x_N)\leq\frac{\epsilon}{2}<\epsilon.$$

Since $\epsilon>0$ and $x^\star$ is arbitrary, it follows that $\iota(X)$ is dense in $X^\star$.

Finally, we check that $d^\star$ is a complete metric. Let $\{z_n\}$ be a Cauchy sequence in $X^\star$. For $k\geq 1$, there is an $N_k\geq 1$ such that $d(z_n,z_m)<\frac{1}{k}$ for all $n,m\geq N$. For each N_k, we have some $w_k\in\iota(X)$ such that $d(w_k,z_{N_k})<\frac{1}{k}$. Now, for any $\epsilon>0$, choose some N such that $\frac{1}{N}<\frac{\epsilon}{3}$. Then, for $k,l\geq N$ we have

$$d^\star\left(w_k,w_l\right)\leq d^\star\left(w_k,z_{N_k}\right)+\underbrace{d^\star\left(z_{N_k},z_{N_l}\right)}_{<\max\{\frac{1}{k},\frac{1}{l}\}}+d^\star\left(z_{N_l},w_l\right)<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.$$

In other words, $\{w_k\}$ is a Cauchy sequence in $\iota(X)$. Without loss of generality, assume that each w_k represented as a constant sequence $w_k\in X$. Since ι is an isometry, it follows that $\{w_k\}$ is a Cauchy sequence in X, and hence, represents a point $w^\star \in X^\star$. We claim that the subsequence $\{z_{N_k}\}$ converges to $w^\star$. It is easy to see that $w_k \to w^\star$ (Check!). But then by construction, $z_{N_k} \to w^\star$. Since a subsequence of the Cacuchy sequence $\{z_n\}$ converges to $w^\star$, the Cauchy sequence $\{z_n\}$ also converges to $w^\star$. Thus, $X^\star$ is a completee metric space. In particular, completion of a metric space exists, unique up to isometry. □

Exercise 26.12

Fill in the details of the proof of the previous theorem.

Exercise 26.13

If X is a completely metrizable space, show that the completion $X^\star$ is homeomorphic to X.

26.3 Subspace of a completely metrizable space

Theorem 26.14: (G_δ-subspace of a completely metrizable space)

A G_δ subspace of a completely metrizable space is again completely metrizable.

Proof

Let (X, d) be a complete metric space. Fix an open set $U \subset X$. Then, we have a continuous function

$$\begin{aligned} f : U &\longrightarrow (0, \infty) \\ x &\longmapsto \frac{1}{d(x, X \setminus U)}. \end{aligned}$$

Since $X \setminus U$ is closed, the distance never vanishes, and thus f is indeed continuous. Let us now define $\rho : U \times U \to (0, \infty)$ by

$$\rho(x, y) = d(x, y) + |f(x) - f(y)|, \quad x, y \in U$$

It is easy to see that ρ is a metric on U. Moreover, ρ induces the subspace topology on U.

Let us show that (U, ρ) is complete. Say, $\{x_n\}$ is a Cauchy sequence in (U, ρ). Then, $\{x_n\}$ is Cauchy in (X, d) as well. Also, for any $\epsilon > 0$, there is some $N \geq 1$ such that for $n \geq N$ we have

$$|f(x_N) - f(x_n)| = \left| \frac{1}{d(x_N, X \setminus U)} - \frac{1}{d(x_n, X \setminus U)} \right| < \epsilon$$

Then, it follows that $d(x_n, X \setminus U)$ is bounded away from 0. In other words, there is some $\delta > 0$ such that

$$\{x_n\} \subset X_\delta \coloneqq \{x \in X \mid d(x, X \setminus U) \geq \delta\} \subset U.$$

Now, X_δ is a closed subset, and hence, complete. Thus, we have $x_n \to x \in X_\delta \subset U$. Thus, (U, ρ) is a complete metric space.

Next, consider a G_δ-set $G = \bigcap_{n=1}^\infty U_n$, where $U_n \subset X$ is open. Now, U_n is completely metrizable, and hence, so is the product $\mathcal{U} = \prod_{n=1}^\infty U_n$. Insider $\mathcal{U}$ we have the diagonal,

$$\Delta_{\mathcal{U}} = \{x \in \mathcal{U} \mid x_i = x_j \; \forall i, j\}.$$

Note that $\Delta_{\mathcal{U}} = \Delta \cap \mathcal{U}$, where Δ is the diagonal in $\mathcal{X} = \prod_{n\geq 1} X$. Since Δ is closed in $\mathcal{X}$, it follows that Δ_U is closed in $\mathcal{U}$, and hence, completely metrizable. Now, the map

$$\begin{aligned} f : G &\longrightarrow \Delta_{\mathcal{U}} \\ x &\longmapsto (x, x, x, \dots) \end{aligned}$$

is clearly a continuous, bijection from G to Δ, with continuous inverse (given by any projection map). Indeed, it is the restriction of the usual diagonal map $X \hookrightarrow \mathcal{X}$. Thus, G is homeomorphic to Δ, and hence, G is completely metrizable. □

Example 26.15: (Irrationals are completely metrizable)

Since $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q\in\mathbb{Q}} (\mathbb{R} \setminus \{q\})$ is a G_δ-set in the complete metric space $\mathbb{R}$, it follows that the set of irrationals is a completely metrizable space.

Day 27 : 7th November, 2025

`product of complete metric space -- Lavrentieff's theorem -- completely metrizable and` G_δ

27.1 Product of metric spaces

Proposition 27.1: (Metric on Product Topology)

Suppose (X_i, d_i) is a countable collection of metric spaces. Let $X = \prod_{i=1}^{\infty} X_i$ be the product. Define

$$\rho_n(a, b) := \min\{d_n(a, b), 1\}, \ a, b \in X_n, \qquad \rho(x, y) := \sum_{i=1}^{\infty} \frac{\rho_i(x_i, y_i)}{2^i}, \ x, y \in X.$$

Then, ρ is a metric on X, inducing the product topology.

Proof

Since each ρ_n is a bounded metric, it follows that ρ is well-defined. The metric properties can be checked easily. Let us show that the induced metric is the product topology. For some open $U \subset X_i$, consider the sub-basic open set $\mathcal{U} = \pi_i^{-1}(U)$. Without loss of generality, assume $U = B_{\rho_i}(x_i, r_i)$. Fix some $y \in U$. Set $\epsilon := \frac{r_i - \rho_i(x_i, y_i)}{2^i}$. Consider the metric ball $B_\rho(y, \epsilon)$. Then, for any $z \in B_\rho(y, \epsilon)$, we have

$$\begin{aligned} \rho_i(x_i, z_i) &\leq \rho_i(x_i, y_i) + \rho_i(y_i, z_i) \\ &\leq \rho_i(x_i, y_i) + 2^i \rho(y, z) \\ &< \rho_i(x_i, y_i) + (r_i - \rho_i(x_i, y_i)) = r_i \\ &\Rightarrow z_i \in U \Rightarrow z \in \mathcal{U}. \end{aligned}$$

Thus, $B_\rho(y, \epsilon) \subset \mathcal{U}$. This proves that the metric topology is finer than the product topology.

Conversely, consider a metric ball $B := B_\rho(x, \epsilon)$. Get some $N \geq 1$ with $\sum_{i>N} \frac{1}{2^i} < \frac{\epsilon}{2}$. Consider the set

$$V = \prod_{i=1}^{N} B_{\rho_i}\left(x_i, \frac{2^i \epsilon}{2N}\right) \times \prod_{i>N} X_i,$$

which is open in the product topology. Now, for any $y \in V$ we have

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{\rho_i(x_i, y_i)}{2^i} \leq \sum_{i=1}^{N} \frac{\frac{2^i \epsilon}{2N}}{2^i} + \sum_{i>N} \frac{1}{2^i} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $V \subset B$. This proves that the product topology is finer than the metric topology. Hence, the two topologies coincide. □

Remark 27.2: (Arbitrary product of metric spaces)

Any uncountable product of (nonempty) metric space fails to be metrizable. In fact, the product topology fails to be first countable. There is a notion of *uniform metric* on an uncountable product, but the induced topology is strictly finer than the product topology, and strictly coarser than the box topology.

Theorem 27.3: (Countable product of completely metrizable spaces)

Let $\{X_n\}$ be a countable collection of nonempty spaces, and denote $X = \prod_{n=1}^{\infty} X_n$ be the product space. Then the following are equivalent.

a) X is completely metrizable.

b) X_n is completely metrizable for each $n \geq 1$.

Proof

Suppose X is completely metrizable. Fix some $a_i \in X_i$. Then, for each n, we have the subspace

$$X_n^\star = \{x \mid x_i = a_i \text{ if } i \neq n\} = \bigcap_{i \neq n} \pi_i^{-1}(a_i),$$

which is closed being the intersection of closed sets, and hence, completely metrizable. But X_n is homeomorphic to $X_n^\star$, and thus, X_n is completely metrizable as well.

Conversely, suppose each X_n is completely metrizable. Fix some complete metric d_n on X_n, and set

$$\rho_n(x, y) = \min\{d_n(x, y), 1\}, \quad x, y \in X_n.$$

Then, ρ_n is a bounded, complete metric, inducing the same topology. On $X = \prod X_n$, define

$$\rho(x, y) := \sum_{i=1}^{\infty} \frac{\rho_i(x_i, y_i)}{2^i}, \quad x, y \in X.$$

Then, ρ induces the product topology on X. Let us check that ρ is complete. Say, $\{x^n\} \subset X$ is a Cauchy sequence. Then, for a fixed i, consider the sequence $\{x_i^n\}_{n \geq 1} \subset X_n$. For $\epsilon > 0$, get $N \geq 1$ such that $\rho(x^n, x^m) < \frac{\epsilon}{2^i}$ for all $n, m \geq N$. Then, for $n, m \geq N$ we have

$$\rho_n(x_i^n, x_i^m) = 2^i \frac{\rho_n(x_i^n, x_i^m)}{2^i} \leq 2^i \rho(x^n, x^m) < \epsilon.$$

Thus, $\{x_i^n\} \subset X_i$ is a Cauchy sequence, and hence, converges to some $y_i \in X_i$. Consider the point $y = (y_i) \in X$. Fix some $\epsilon > 0$. Then, get some $K \geq 1$ such that $\sum_{n>N} \frac{1}{2^n} < \frac{\epsilon}{2}$. Also, for each $1 \leq i \leq K$, get some N_i such that

$$\rho_i(x_i^n, y_i) < \frac{2^i \cdot \epsilon}{2N}, \quad n \geq N_i.$$

Set $N = \max\{K, N_1, \dots, N_k\}$. Then, for $n \geq N$ we have

$$\rho(x^n, y) = \sum_{i=1}^{\infty} \frac{\rho_i(x_i^n, y_i)}{2^i} =\leq \sum_{i=1}^{N} \frac{\rho_i(x_i^n, y_i)}{2^i} + \sum_{i>N} \frac{1}{2^i} < N \cdot \frac{\epsilon}{2N} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $x^n \to y$. Hence, (X, ρ) is a completely metric space. □

27.2 Lavrenthieff's Theorem

Proposition 27.4

Let X be a metrizable space, and Y be a completely metrizable space. Suppose, for some $A \subset X$, we have a continuous map $f : A \to Y$. Then, there exists a G_δ-set, say, $A^\star \subset X$ with $A \subset A^\star \subset \bar{A}$, and a continuous map $f^\star : A^\star \to Y$, which extends f.

Proof

Fix a complete metric d_Y on Y. For any $x \in \bar{A}$, denote the *oscillation*

$$\operatorname{osc}(f, x) := \inf\{\operatorname{Diam} f(U \cap A) \mid U \subset X \text{ is open, } x \in U\}.$$

As $x \in \bar{A}$, for any open neighborhood $x \in U$, we have $A \cap U \neq \emptyset$. Let us consider

$$A_n := \left\{x \in \bar{A} \,\middle|\, \operatorname{osc}(f, x) < \frac{1}{n}\right\}, \qquad A^* := \{x \in \bar{A} \mid \operatorname{osc}(f, x) = 0\}$$

Clearly $A^\star = \bigcap_{n \geq 1} A_n$. Moreover, for any $a \in A$, by continuity of f, we have some open $U \subset X$ such that $x \in U$ and $\operatorname{Diam} f(U \cap A) < \frac{1}{n}$. Thus, $a \in A_n$ for any $n \geq 1$. In particular, $A \subset A^\star \subset \bar{A}$ is clear.

Let us check that A_n is open in $\bar{A}$. For any $x \in A_n$, we have some open $U \subset X$ such that $x \in U$, and $\operatorname{Diam} f(U \cap A) < \frac{1}{n}$. But then for any $w \in U \cap \bar{A}$, it follows that $\operatorname{osc}(f, w) < \frac{1}{n}$. Thus, $x \in U \cap \bar{A} \subset A_n$. Since $x \in A_n$ is arbitrary, we have A_n is open in $\bar{A}$. Then, $A_n = \bar{A} \cap B_n$ for some open $B_n \subset X$. We have,

$$A^\star = \bigcap_{n \geq 1} A_n = \bigcap_{n \geq 1} \bar{A} \cap B_n = \bar{A} \cap \bigcap_{n \geq 1} B_n.$$

Since $\bar{A}$ is a closed set in a metric space, it is itself G_δ. Hence, we have $A^\star$ is a G_δ set in X.

Let us get a function $f^\star : A^\star \to Y$. For $x \in A^\star$, let $x_n \in A$ be a sequence with $\lim x_n = x$. Fix $\epsilon > 0$. Since $\operatorname{osc}(f, x) = 0$, we have some open set $U \subset X$ such that $x \in U$ and $\operatorname{Diam} f(U \cap A) < \epsilon$. As $x_n \to x$, we have some $N \geq 1$, such that for all $n, m \geq N$ we have $x_n, x_m \in U$. Then, it follows that $d_Y(f(x_n), f(x_m)) < \epsilon$ for all $n, m \geq N$. In other words,

$\{f(x_n)\}$ is a Cauchy sequence in (Y, d_Y). Since d_Y is complete, we have $f(x_n) \to y \in Y$. Set, $f^\star(x) = y$.

Let us check that $f^\star$ is well-defined. Suppose $z_n \in A$ is another sequence, with $z_n \to x \in A^\star$. Then, $\{f(z_n)\}$ is again Cauchy, and converges to some $w \in Y$. Fix some $\epsilon > 0$. Then, there is some $U \subset X$ open such that $x \in U$, and $\operatorname{Diam} f(U \cap A) < \frac{\epsilon}{3}$. As $\lim y_n = x = \lim z_n$, we have some $N \geq 1$, such that $y_n, z_n \in U$ for all $n \geq N$. Taking N larger, we may assume $d(f(y_n), y) < \frac{\epsilon}{3}$ and $d(f(z_n), w) < \frac{\epsilon}{3}$ for all $n \geq N$. Then, we have

$$d_Y(y, w) \leq d_Y(y, f(y_N)) + d_Y(f(y_N), f(z_N)) + d_Y(f(z_N), w) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Since ϵ is arbitrary, it follows that $d_Y(y, w) = 0 \Rightarrow y = w$. Thus, $f^\star$ is well-defined.

Finally, let us check that $f^\star$ is a continuous extension. For any $a \in A$, we can consider the constant sequence $\{a_n = a\}$ that converges to a. Then, $f^\star(a) = \lim f(a_n) = \lim f(a) = f(a)$. Thus, $f^\star$ extends f. Let us check continuity. Let $x \in A^\star$, and fix $\epsilon > 0$. Then, there is some open set $U \subset X$ such that $\operatorname{Diam} f(U \cap A) < \frac{\epsilon}{3}$. Fix a sequence $y_n \in U \cap A$ such that $y_n \to y$. Now, for any $z \in U \cap A^\star$, consider a sequence $z_n \in U \cap A$ such that $z_n \to z$. There exists some $N \geq 1$ such that $d_Y\left(f(y_n), f^\star(y)\right) < \frac{\epsilon}{3}$ and $d_Y\left(f(z_n), f^\star(z)\right) < \frac{\epsilon}{3}$ for all $n \geq N$. We have,

$$d_Y(f^\star(y), f^\star(z)) \leq d_Y(f^\star(y), f(y_N)) + d_Y\left(f(y_N), f(z_N)\right) + d_Y\left(f(z_N), f^\star(z)\right) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This proves $f^\star$ is continuous at y. Since $y \in A^\star$ is arbitrary, we have $f^\star : A^\star \to Y$ is a continuous extension. □

Theorem 27.5: (Lavrentieff's Theorem)

Suppose X, Y are completely metrizable spaces, and $f : A \to B$ is a homeomorphism, where $A \subset X, B \subset Y$. Then, f extends to a homeomorphism $f^\star : A^\star \to B^\star$, where $A^\star \subset X, B^\star \subset Y$ are G_δ-sets, with $A \subset A^\star \subset \bar{A}$ and $B \subset B^\star \subset \bar{B}$.

Proof

Let us denote $g = f^{-1}$. Since f, g are both continuous, we have G_δ-sets $A_1 \subset X, A_2 \subset Y$, with $A \subset A_1 \subset \bar{A}, B \subset B_1 \subset \bar{B}$, and extensions $f_1 : A_1 \to Y, g_1 : B_1 \to X$ of f and g respectively. Let us consider

$$A^\star := \{x \in A_1 \mid f_1(x) \in B_1\} = (f_1)^{-1}(B_1), \quad B^\star := \{x \in B_1 \mid g_1(x) \in A_1\} = (g_1)^{-1}(A_1).$$

Since these are inverse images of G_δ-sets, they are again G_δ. Clearly, $A \subset A^\star \subset \bar{A}$ and $B \subset B^\star \subset \bar{B}$. Let us denote $f^\star = f_1|_{A^\star}$ and $g^\star = g_1|_{B^\star}$. Clearly, $f^\star$ and $g^\star$ are continuous maps, extending f and g respectively. For any $x \in A^\star$, we have $f_1(x) \in B_1$, and so, $g_1 f_1(x) \in A_1$ is defined. Thus, $g_1 \circ f^\star : A^\star \to A_1$ is continuous. Say, $x_n \in A$ is a sequence, such that $x_n \to x \in A^\star$. Then,

$$g_1 f^\star(x) = \lim g_1 f^\star(x_n) = \lim g_1 f(x_n) = \lim g f(x_n) = \lim x_n = x.$$

Thus, $g_1 \circ f^\star : A^\star \to A^\star$ is the identity map. In particular, we have $g^\star \circ f^\star = \mathrm{Id}_{A^\star}$. Similarly, we have $f^\star \circ g^\star = \mathrm{Id}_{B^\star}$. Thus, $f^\star : A^\star \to B^\star$ is a homeomorphism, with inverse $g^\star : B^\star \to A^\star$. □

Theorem 27.6

Suppose X is a metrizable space, and $A \subset X$ is a completely metrizable space. Then, A is a G_δ-set in X.

Proof

Fix metric d on X. Consider $\iota : (X, d) \hookrightarrow (X^\star, d^\star)$ be the completion. Then, the restriction $f = \iota|_A : A \hookrightarrow X^\star$ is also an embedding, i.e, homeomorphism onto the image. Thus, we have a homeomorphism $A \supset A \to f(A) \subset X^\star$, where $A, X^\star$ are completely metrizable. By Lavrenteiff's theorem, f has an extension to a homeomorphism of G_δ sets of A and $X^\star$, containing A and $\iota(A)$ respectively. But then the extension must be ι itself, as on the left-hand side, the extended domain can only possibly be A. Thus, $f^\star(A^\star) = f(A) = \iota(A)$ is the extended set on the right-hand side. But then $\iota(A)$ is a G_δ set in $X^\star$. Taking inverse, it follows that A is then a G_δ set of X. □

Corollary 27.7: (Characterization of Completely Metrizable Space)

Given a metric space (X, d), the following are equivalent.

a) X is completely metrizable.

b) X is G_δ in the completion $X^\star$.

Corollary 27.8: ($\mathbb{Q}$ is not G_δ in $\mathbb{R}$)

$\mathbb{Q}$ is not G_δ in $\mathbb{R}$.

Day 28 : 14th November, 2025

`game of Choquet -- strongly Choquet space -- Baire space`

28.1 A digression: Game of Choquet

Given a space X, let us assume that two players are playing a game.

Round 0: Player I goes first by choosing an open set $U_0 \subset X$ and a point $x_0 \in U_0$. Then, player II chooses another open set V_0 satisfying $x_0 \in V_0 \subset U_0$.

Round 1: Player I now chooses an open set $U_1 \subset V_0$, and a point $x_1 \in U_1$. Then, player II chooses another open set V_1 satisfying $x_1 \in V_1 \subset U_1$.

Round n: At this stage, player I chooses an open set $U_n \subset V_{n-1}$ and a point $x_n \in U_n$. Player II then chooses an open set V_n satisfying $x_n \in V_n \subset U_n$.

Thus, we have an infinite game that goes like this:

Player I :	(U_0, x_0)		(U_1, x_1)		$\dots$	(U_n, x_n)	$\dots$	
Player II :		V_0		V_1		$\dots$	V_n	$\dots$

This game is known as the *strong game of Choquet*. The usual *game of Choquet* is played the same way, but player I does not choose any points $x_n \in U_n$ at any stage, and thus, player II does not care about the points either. Observe that

$$\bigcap_{n\geq 0} U_n = \bigcap_{n\geq 0} V_n.$$

We say *player II wins the game* if $\bigcap V_n \neq \emptyset$ at the end of the game. Conversely, player I wins the game if $\bigcap U_n = \emptyset$ at the end of the game.

Remark 28.1: (Winning strategy)

To formalize the concept of winning strategy (for player II), let us consider the following. Given a space, $(X, \mathcal{T})$, let us consider the sets

$$\mathcal{T}_* := \{U \in \mathcal{T} \mid U \neq \emptyset\}, \qquad \mathcal{S} := \{(U, x) \mid U \in \mathcal{T}_*,\ x \in U\}.$$

Then, a *winning strategy for player II* is a map

$$f : \mathcal{S} \to \mathcal{T}_*$$

such that the following holds.

i) For any $(U, x) \in \mathcal{S}$, we have
$$x \in f(U, x) \subset U.$$

ii) For any sequence $(U_n, x_n) \in \mathcal{S}$ defined inductively, such that,
$$U_0 \supset V_0 := f(U_0, x_0) \supset U_1 \supset V_1 := f(U_1, x_1) \supset \cdots \supset U_n \supset V_n := f(U_n, x_n) \supset \ldots,$$
we always have $\bigcap V_n \neq \emptyset$.

Definition 28.2: (Strong) Choquet space

A space X is called a *Choquet space* (resp. *strongly Choquet space*) if in a game of Choquet (resp. *strong game of Choquet*), player II always has a winning strategy.

Remark 28.3

Winning a strong game of Choquet is more difficult for player II, as at the n^{th}-stage they have to choose an open set $V_n \subset U_n$ satisfying the extra condition $x_n \in V_n$. Thus, a strongly Choquet space is always a Choquet space. Also, since player I's goal is to make the intersection empty, player I is also denoted as player E (Empty). In this convention, player II is denoted as player N (Nonempty).

Proposition 28.4: (Completely metrizable space is strongly Choquet)

Let X be a completely metrizable space. Then X is strongly Choquet.

Proof

Let us fix a complete metric d on X, inducing the underlying topology. At the n^{th}-stage, after player E has chosen $x_n \in U_n \subset V_{n-1}$, player N chooses $x_n \in V_n \subset \overline{V_n} \subset U_n$, such that $\text{Diam}\overline{V_n} < \frac{1}{2^n}$. This is always possible in the metric space (X, d). Now, observe that

$$\bigcap U_n = \bigcap V_n = \bigcap \overline{V_n}.$$

But $\{\overline{V_n}\}$ is a decreasing sequence of closed sets in a complete metric space with diameter going to zero. Hence, $\bigcap \overline{V_n} \neq \emptyset$ (Check!). Thus, player N always wins. Hence, X is a strongly Choquet space. ☐

Theorem 28.5: (Strongly Choquet implies Complete Metrizability)

Suppose X is a metrizable space. If X is strongly Choquet, then X is completely metrizable.

Proof

Fix an arbitrary metric d on $(X, \mathcal{T})$, and consider the completion $(X, d) \hookrightarrow (X^\star, d^\star)$. We shall show that X is G_δ in $X^\star$.

Let us fix a winning strategy player N, and denote it by

$$\mathsf{f} : \{(U, x) \mid x \in U \in \mathcal{T}\} \longrightarrow \{U \in \mathcal{T} \mid U \neq \emptyset\}.$$

For each $n \geq 1$, let us consider $\mathcal{W}_n$ to be the collection of open sets $W \subset X^\star$, such that for some $x \in U \subset X$ we have

i) $U = X \cap \tilde{U}$, for some $\tilde{U} \subset X^\star$ open, with $\tilde{U} \subset B_{d^\star}\left(x, \frac{1}{n}\right)$,

ii) $W \cap X = \mathsf{f}(U, x)$, and

iii) $W \subset B_{d^\star}\left(x, \frac{1}{n}\right)$.

Denote,

$$G_n = \bigcup \{W \mid W \in \mathcal{W}_n\}.$$

Clearly $G_n \subset X^\star$ is open (possibly empty). Let us check that $X \subset G_n$. For any $x \in X$, let player E choose $U_0 = B_{d^\star}\left(x, \frac{1}{n}\right) \cap X$ and $x_0 = x$. At the n^{th}-stage, say player E chooses an open set $U_n = X \cap \tilde{U}_n$, where $x_n = x \in U_n$, and $\tilde{U}_n \subset B_{d^\star}\left(x, \frac{1}{n}\right)$. Then, player N chooses $V_n = \mathsf{f}(U_n, x)$, such that $x \in V_n \subset U_n$. But then, $V_n = X \cap W'$ for some $W' \subset X^\star$ open. Consider $W = W' \cap \tilde{U}_n$. Clearly,

$$X \cap W = X \cap \left(W' \cap \tilde{U}_n\right) = (X \cap W') \cap \left(X \cap \tilde{U}_n\right) = V_n \cap U_n = V_n = \mathsf{f}(U_n, x).$$

Also, $x \in W \subset B_{d^\star}\left(x, \frac{1}{n}\right)$. Thus, $x \in G_n$. Note that this argument requires *strong* game of Choquet. Consequently, we have $X \subset \bigcap G_n$.

Let us show that $\bigcap G_n \subset X$. Let $x \in \bigcap U_n$. For $n_1 = 1$, as $x \in G_{n_1}$, we have some $\tilde{V}_1 \in \mathcal{W}_{n_1}$ such that $x \in \tilde{V}_1$. Then, for some $y_1 \in U_1 \subset X$, we have $V_1 := \tilde{V}_1 \cap X = \mathsf{f}(U_1, y_1)$, and moreover, $\tilde{V}_1 \subset B_{d^\star}\left(y_1, \frac{1}{n_1}\right)$. As $x \in \tilde{V}_1$, we have $\epsilon_1 := d^\star(x, X^\star \setminus \tilde{V}_1) > 0$. Choose some $n_2 > n_1$ such that

$\frac{1}{n_2} < \frac{\epsilon_1}{2}$. As $x \in G_{n_2}$, we have some $\tilde{V}_2 \in \mathcal{W}_{n_2}$ such that $x \in \tilde{V}_2$. Then, for some $y_2 \in U_2 \subset X$, we have $V_2 := \tilde{V}_2 \cap X = \mathsf{f}(U_2, y_2)$, and moreover, $\tilde{V}_2 \subset B_{d^\star}\left(y_2, \frac{1}{n_2}\right)$. Note that $U_2 \subset \tilde{V}_1$. Indeed, $U_2 = X \cap \tilde{U}_2$ for some $\tilde{U} \subset X^\star$ open with $\tilde{U}_2 \subset B_{d^\star}\left(y_2, \frac{1}{n_2}\right)$. Then, for any $z \in U_2$ we have $d^\star(y_2, z) < \frac{1}{n_2}$. Also, we have $x \in \tilde{V}_2 \subset B_{d^\star}\left(y_2, \frac{1}{n_2}\right)$. Thus,

$$d(x,z) \le d(x,y_2) + d(y_2,z) < \frac{1}{n_2} + \frac{1}{n_2} < \epsilon_1 = d^\star\left(x, X^\star \setminus \tilde{V}_1\right) \Rightarrow z \in \tilde{V}_1.$$

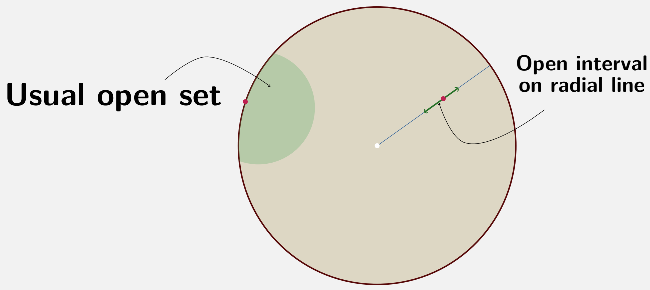
Thus, $U_2 \subset \tilde{V}_1$ holds, which implies $U_2 \subset V_1 = X \cap V_1$. Inductively, we continue this (strong) game of Choquet in a similar way. Since player N is playing by a winning strategy, it follows that $\bigcap U_n = \bigcap V_n \neq \emptyset$. Now, by construction, $x \in \bigcap \tilde{V}_n$. Since $(X^\star, d^\star)$ is a complete metric space, and since the diameters of $\tilde{V}_n$ are going to 0, it follows that $\bigcap \tilde{V}_n = \{x\}$, a singleton. But then,

$$\emptyset \neq \bigcap V_n = X \cap \bigcap \tilde{V}_n = X \cap \{x\} \Rightarrow x \in X.$$

Thus, we have $X = \bigcap G_n$, i.e, X is a G_δ-set in $X^\star$. Hence, X is completely metrizable. □

Example 28.6: (Wheel with its Hub)

Let $X = \{(x,y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \le 1\}$ be the closed unit disc with the center removed. Consider the collection of usual open sets in D as a subspace of $\mathbb{R}^2$, and additionally, consider every open intervals (in the usual sense) on every open radial line. It is easy to see, this collection is a basis for a topology on X. The space X is called the *wheel without its hub*.



Observe that X is not second countable, since the set $A = \left\{(x,y) \;\middle|\; x^2 + y^2 = \frac{1}{2}\right\}$ is a closed discrete subspace of X. Nevertheless, X is metrizable. Let us explicitly define a metric.

Consider the function $h : X \to [0,\infty)$ defined by $h(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|} - 1$, and the function $r : X \to [0,\infty)$ defined by $r(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}$. Here, for any $\mathbf{x} = (x,y)$, we have $\|\mathbf{x}\| = \sqrt{x^2 + y^2}$. It is easy to see that h, r are continuous maps. Define $d : X \times X \to [0,\infty)$ via

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} |h(\mathbf{x}) - h(\mathbf{y})|, & \text{if } r(\mathbf{x}) = r(\mathbf{y}), \\ h(\mathbf{x}) + h(\mathbf{y}) + \|r(\mathbf{x}) - r(\mathbf{y})\| & \text{otherwise.} \end{cases}$$

One can easily check that d is a metric on X, inducing the same topology (Check!). Moreover, one can show that d is a complete metric as well.

Let us instead play a strong game of Choquet on X! If at any stage, player E plays an open set U, and a point $\mathbf{x} \in U$ on some open radial line ℓ, then player N plays an open set V which is an open interval containing x on the radial line ℓ, such that the closed interval has length half that of the component of $\ell \cap U$ containing $\mathbf{x}$ (which is going to be interval), and is contained in said component. Then, we get a decreasing sequence closed intervals of ℓ with diameters going to 0. The intersection is nonempty by the completeness of $\mathbb{R}$, and so, player N wins. Suppose player E plays an open set U and a point $x \in U$ with x on the boundary circle. Then, player N plays a usual open neighborhood $V \subset U$ of x, such that $\bar{V} \subset U$. If player E never plays a point on any radial line (so the points are always on the circle), then we get a decreasing sequence closed sets in the boundary circle, which is a compact set. Thus, player N again wins. This proves X is strongly Choquet, and hence, completely metrizable.

Example 28.7: (Discrete rational extension of $\mathbb{R}$)

Consider X to be the discrete rational extension of $\mathbb{R}$, i.e, $X = \mathbb{R}$, with the topology $\mathcal{T}$ generated by the basis

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{Q},\ a < b\} \cup \{\{q\} \mid q \in \mathbb{Q}\}.$$

It is easy to see that $\mathcal{B}$ is a basis of clopen sets, and hence, X is a completely regular, second countable space, which is clearly T_1. By Urysohn's metrization theorem, X is then metrizable. Let us show that X is strongly Choquet.

If at any stage player E plays an open set U and a rational $q \in U$, player N can play $V = \{q\}$, and thereby winning the game. Suppose player E plays an open set $U \in \mathcal{T}$ and an irrational $x \in U$. Then, there is a finite length interval $x \in (a, b) \subset [a, b] \subset U$, such that $b - a < \frac{1}{2}\mathrm{Diam} U$. Player N chooses (a, b). Then, in a game, where player E never plays a rational point, we have $V_n = (a_n, b_n)$ for finite intervals, which are nested, with strictly decreasing diameter. In particular, $\bigcap V_n \neq \emptyset$, as $\mathbb{R}$ is complete. Thus, X is strongly Choquet. Consequently, the discrete rational extension of $\mathbb{R}$ is a completely metrizable space.

28.2 Baire Space

Definition 28.8: (Baire space)

A space X is called a *Baire space* if a countable intersection of dense, open sets of X is again dense.

Definition 28.9: (First and second category)

A subset $A \subset X$ is called of *first category* (or *meager*) if $A = \bigcup_{n \geq 1} A_n$ for some nowhere dense set $A_n \subset X$ (i.e., $\mathrm{int}\overline{A_n} = \emptyset$). If A cannot be written as the countable union of nowhere dense sets, then A is called of *second category* (or *non-meager*).

Exercise 28.10: (Subset of meager set)

Verify that a subset of a meager set is again meager.

Proposition 28.11

X is Baire if and only if countable union of closed nowhere dense sets have empty interior. In particular, a (nonempty) Baire space is non-meager (in itself).

Proof

Suppose X is a Baire space. Let A_n be a collection of closed nowhere dense sets. Then, $U_n = X \setminus A_n$ is a collection of open dense sets. We have $\bigcap U_n$ is dense. Now, for any nonempty open set $O \subset X$, we have $O \cap \bigcap U_n \neq \emptyset \Rightarrow O \not\subset X \setminus \bigcap U_n = \bigcup A_n$. Thus, $\bigcup A_n$ has empty interior.

Now, suppose countable union of closed nowhere dense sets in X has empty interior. Let U_n be a collection of open dense sets. Then, $A_n = X \setminus U_n$ is closed, nowhere dense. We have $\bigcup A_n$ has empty interior. So, for any nonempty open set $O \subset X$, we have $O \not\subset \bigcup A_n \Rightarrow O \cap (X \setminus \bigcup A_n) \neq \emptyset \Rightarrow O \cap \bigcap U_n \neq \emptyset$. Thus, $\bigcap U_n$ is dense. Hence, X is a Baire space.

Now, for a Baire space X, suppose $X = \bigcup A_n$ for some nowhere dense sets. Then, $X = \bigcup \overline{A_n}$, where $\overline{A_n}$ is closed, nowhere dense. But this contradicts that $\bigcup \overline{A_n}$ has empty interior. □

Remark 28.12: (Non-meager spaces need not be Baire)

There are non-meager spaces, which fail to be Baire. Consider $X = \mathbb{R} \times \{0\} \cup \mathbb{Q} \times \{1\} \subset \mathbb{R}^2$. Then, for each $q \in U$, we have $U_q := X \setminus \{(q, 1)\}$, an open dense set. Clearly, $\bigcup U_n = \mathbb{R} \times \{0\}$ is not dense in X. Thus, X is not Baire. On the other hand, if possible, let us write $X = \bigcup A_n$ for some nowhere dense sets A_n. Then, $\mathbb{R} \times \{0\} = \bigcup (A_n \cap \mathbb{R} \times \{0\})$. Note that $B_n := A_\times \mathbb{R} \times \{0\}$ is nowhere dense in $\mathbb{R} \times \{0\}$. But this implies $\mathbb{R} \cong \mathbb{R} \times \{0\}$ is a meager (in itself) space. This is a contradiction, as $\mathbb{R}$, being a completely metrizable space, is Baire, and hence, non-meager.

Day 29 : 18th November, 2025

`Baire category theorem -- paracompactness`

29.1 Baire Category Theorems

Theorem 29.1: (Baire Category Theorem)

A G_δ-set in a compact T_2 space is a Baire space.

Proof

Let X be compact, T_2-space. Note that X is a T_4-space. Let us first show that X itself is Baire. Let $G_n \subset X$ be a countable collection of open dense sets, and $U \subset X$ be a fixed nonempty open set. Denote $V_0 = U$. Now, $U \cap G_1 \neq \emptyset$. Then, by regularity, there is a nonempty open set V_1, with $\overline{V_1} \subset U \cap G_1$. Inductively, assume that there is a nonempty open set V_n such that $\overline{V_n} \subset V_{n-1} \cap G_n$. Since $V_n \cap G_{n+1} \neq \emptyset$, again by regularity, we have a nonempty open set V_{n+1} with $\overline{V_{n+1}} \subset V_n \cap G_{n+1}$. Now, by construction, $\{\overline{V_n}\}_{n \geq 1}$ are closed sets, with $\overline{V_1} \supset \overline{V_2} \supset \dots$. Consequently, $\{\overline{V_n}\}$ is a collection of (nonempty) closed sets with finite intersection property.

Hence, $\bigcap \overline{V_n} \neq \emptyset$. But, $\bigcap \overline{V_n} \subset U \cap \bigcap G_n$ by construction. Thus, $U \cap \bigcap G_n \neq \emptyset$. As U is arbitrary nonempty open set, we have $\bigcap G_n$ is dense in X. Thus, X is a Baire space.

Now, let us consider a G_δ-set $K = \bigcap U_n$, where $U_n \subset X$ is open. Consider $\bar{K}$, which is closed, hence compact, and also T_2. Now, $V_n = U_n \cap \bar{K}$ is an open set in $\bar{K}$. Note that $\bigcap V_n = \bigcap U_n \cap \bar{K} = K \cap \bar{K} = K$. Also, $K \subset V_n \subset \bar{K} \Rightarrow \bar{K} = \overline{V_n}$. Thus, V_n is an open dense set in the compact, T_2 space $\bar{K}$. Now, suppose $W_i \subset K$ are open, dense subsets. Then, $W_i = K \cap G_i$ for some $G_i \subset \bar{K}$ open. Clearly, G_i is also dense in $\bar{K}$, since for any nonempty open set $V \subset \bar{K}$ we have,

$$V \cap K \neq \emptyset \underset{\text{as } W_i \text{ is dense in } K}{\Rightarrow} (V \cap K) \cap W_i \neq \emptyset \Rightarrow V \cap G_i \neq \emptyset.$$

Thus, we have a countable collection $\{G_i\} \cup \{V_n\}$ of open dense subsets in $\bar{K}$. Hence, the intersection

$$\bigcap_i G_i \cap V_i = \left(\bigcap_i G_i\right) \cap \left(\bigcap_i V_i\right) = \left(\bigcap G_i\right) \cap K = \bigcap G_i \cap K = \bigcap W_i$$

is dense in $\bar{K}$. But then $\bigcap W_i$ is dense in K as well. Hence, K is a Baire space. □

Corollary 29.2: (BCT 1)

A locally compact T_2 space is a Baire space.

Proof

Suppose X is locally compact, T_2. A locally compact, T_2 noncompact space embeds as an open subset in its one point compactification $\hat{X}$, which is compact, T_2. Thus, X is a G_δ-set in $\hat{X}$, and hence, a Baire space. □

Theorem 29.3: (BCT 2)

A completely metrizable space is a Baire space

Proof

Let (X, d) be a complete metric space. Suppose $G_i \subset X$ is a countable collection of open dense set, and $U \subset X$ is a fixed nonempty open set. Proceeding as in the proof of Baire category theorem, consider $V_0 = U$, and get open balls $V_n = B_d(x_n, r_n)$ of radius $r_n < \frac{1}{n}$, such that $\overline{V_{n+1}} \subset V_n \cap G_{n+1}$ holds. In particular, we have a decreasing sequence of closed balls $V_0 \supset \overline{V_1} \supset \overline{V_2} \supset \ldots$, and moreover, $\bigcap \overline{V_n} \subset U \cap \bigcap G_n$ holds.

We claim that the sequence $\{x_n\}$ is Cauchy. Indeed, for any $\epsilon > 0$, get $N \geq 1$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Then, for any $n, m \geq N$ we have $x_n, x_m \in V_N$. Hence,

$$d(x_n, x_m) \leq d(x_n, x_N) + d(x_N, x_m) < r_N + r_N < \frac{2}{N} < \epsilon.$$

As X is complete, we have $x_n \to x$. Clearly, $x \in \overline{V_n}$ for all n. Hence, $x \in U \cap G_n$ for all $n \geq 1$. Thus, $U \cap \bigcap_n U_n \neq \emptyset$. As U is arbitrary nonempty open set, we have $\bigcap G_n$ is dense. Thus, X is a Baire space. □

Corollary 29.4: ($\mathbb{Q}$ is not G_δ)

The set of rationals $\mathbb{Q} \subset \mathbb{R}$ is not a G_δ-set.

Proof

If possible, suppose $\mathbb{Q}$ is G_δ. Then, $\mathbb{Q} = \bigcap_n U_n$ for some open sets $U_n \subset \mathbb{R}$. Clearly, U_n is dense in $\mathbb{R}$, since $\mathbb{Q} \subset U_n$ is already dense. Now, foreach $q \in \mathbb{Q}$, consider $V_q = \mathbb{R} \setminus \{q\}$, which are also open and dense. Note that $\bigcap_{q\in\mathbb{Q}} V_q = \mathbb{R} \setminus \mathbb{Q}$. Now, $\{U_n\}_{n\geq 1} \cup \{V_q\}_{q\in\mathbb{Q}}$ is a countable collection of open dense sets. Since $\mathbb{R}$ is a Baire space, there intersection must be dense. But, $\bigcap_{n\geq 1} U_n \cap \bigcap_{q\in\mathbb{Q}} V_q = \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$, a contradiction. Hence, $\mathbb{Q}$ is not a G_δ-set. □

Remark 29.5

Since for a function $f : \mathbb{R} \to \mathbb{R}$ the set of continuities must be a G_δ-set, it follows that there does not exist a function which is continuous only at the rationals.

Theorem 29.6: (Choquet spaces are Baire space)

Let X be a nonempty space. Then, X is a Choquet space if and only if X is a Baire space.

Proof

Let X be a Choquet space. Suppose G_n is a countable collection of open dense sets. Fix some nonempty open set $O \subset X$. Let player E choose the open set $U_0 := G_1 \cap O$, which is nonempty as G_1 is dense. Suppose at the n^{th}-stage, player N chooses $V_n \subset U_n$ according to their winning strategy. Then, player E chooses $U_{n+1} := V_n \cap G_{n+1}$, which is again nonempty as G_{n+1} is dense. At the end of the game, since N must win, we have

$$\emptyset \neq \bigcap_{n\geq 0} U_n = (O \cap G_1) \cap \bigcap_{n\geq 1} V_n \cap G_{n+1} \subset O \cap \bigcap_{n\geq 1} G_n.$$

As O is an arbitrary nonempty open set, we have $\bigcap G_n$ is dense in X.

Conversely, let X be a Baire space. If possible, suppose player E has a winning strategy,

$$\mathsf{f} : \mathcal{T}_\star \to \mathcal{T}_*,$$

where $\mathcal{T}_\star$ denotes the set of nonempty open sets of X. Say, according to this strategy, player E chooses the open set $U_0 \subset X$. We shall show that U_0 is not a Baire space.

Fix some open $U \subset U_0$. Given any collection $\mathcal{O}$ of nonempty open subsets of U, call $\mathcal{O}$ is *good* if

$$\mathcal{O}^\star = \{\mathsf{f}(O) \mid O \in \mathcal{O}\}$$

is a pairwise disjoint collection of (necessarily nonempty) open subsets of U. Let $\mathfrak{O}_U$ be the collection of all good sub-collections of U, partially ordered by inclusion. For a chain $\{\mathcal{O}_\alpha\}$ in $\mathfrak{O}_U$, consider the union $\mathcal{O} = \bigcup \mathcal{O}_\alpha$. If possible, suppose there are $O_\alpha \in \mathcal{O}_\alpha$ and $O \in \mathcal{O}_\beta$ such that $\mathsf{f}(O_\alpha) \cap \mathsf{f}(O_\beta)$. Without loss of generality, $\mathcal{O}_\alpha \subset \mathcal{O}_\beta$. But as $\mathcal{O}_\beta$ is good, we have a contradiction. Thus, $\mathcal{O}$ is a good sub-collection of nonempty open sets of U. Hence, by Zorn's lemma, we can then get a *maximal* good collection, say, $\mathcal{O}_U^{\max}$. Let us denote

$$U^\star := \bigcup_{O\in\mathcal{O}_U} \mathsf{f}(O).$$

Clearly, $U^\star$ is a nonempty open set of $U^\star$. We claim that $U^\star$ is dense in U. If not, then there is some nonempty open set $O \subset U$ such that $O \cap U^\star = \emptyset$. Then, $\mathsf{f}(O) \subset O$ is a nonempty open set, and clearly, $\mathsf{f}(O) \cap U^\star = \emptyset$. But then, $\mathcal{O}_U^{\max} \cup \{O\}$ is also good, violating the maximality of $\mathcal{O}_U$. Hence, for any $U \subset U_0$, we have constructed $U^\star$, which is open and dense in U_0, and given as the union of pairwise disjoint open sets of the form $\mathsf{F}(O)$ for open subsets $O \subset U$.

Let us now inductively construct the following open dense sets. Set $G_1 = U_0^\star$. Assuming G_n is defined, set $G_{n+1} = \bigcup_{W \in G_n} W^\star$. Observe that each G_n is a *disjoint* union of open sets of the form $\mathsf{f}(U)$ for some open $U \subset U_0$. Moreover, G_{n+1} is dense in G_n, and hence, by a simple induction, each G_n is dense in U_0 as well. If possible, let $x \in \bigcap G_n$. Since $x \in G_1$, we have a unique open $V_0 \subset U_0$, such that $x \in \mathsf{f}(V_0)$ (as G_1 is a disjoint union). Set $U_1 = \mathsf{f}(V_0)$. Inductively, suppose we have constructed $(U_0, V_0, U_1, V_1, \ldots, U_n)$. Now, $x \in G_{n+1}$. Hence, there is a unique open set $V_n \subset U_n$, such that $x \in \mathsf{f}(V_n)$ (as G_{n+1} is a disjoint union). Set $U_n = \mathsf{f}(V_n)$. This is a game of Choquet! Now, by construction, $x \in \bigcap U_n = \bigcap V_n$. Thus, player N wins in this game. This is a contradiction, since player E is playing by a winning strategy by assumption. Hence, we must have $\bigcap G_n = \emptyset$. But then, U_0 is an open set of X, which is not Baire. Consequently, X itself cannot be a Baire space. □

Corollary 29.7: (BCT 1 and 2 by game of Choquet)

X is a Choquet space (and hence, a Baire space) if either a) X is completely metrizable, or b) X locally compact T_2.

Proof

Suppose X completely metrizable. At the n^{th}-stage of any Choquet game, let player N choose $V_n \subset U_n$ satisfying $V_n \subset \overline{V_n} \subset U_n$, and $\mathrm{Diam}\overline{V_n} < \frac{1}{2}\mathrm{Diam}U_n$. Then, a usual argument using Cauchy sequence shows that $\bigcap V_n = \bigcap \overline{V_n} \neq \emptyset$. Thus, X is a Choquet space.

Next, suppose X is a locally compact T_2 space. This time, at the n^{th}-stage, let player N choose $V_n \subset U_n$ satisfying $V_n \subset \overline{V_n} \subset U_n$, and $\overline{V_n}$ compact (this is possible, as the space is locally compact, T_2). It follows that $\bigcap V_i = \bigcap \overline{V_i} \neq \emptyset$, as the intersection of decreasing nonempty closed sets in a compact space (here, the compact space is $\overline{V_1}$) is always nonempty. □

29.2 Paracompactness

Definition 29.8: (Refinement)

Given an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X, a *refinement* of $\mathcal{U}$ is an open cover $\mathcal{V} = \{V_j\}_{j \in J}$, such that there exists a function $\phi : I \to J$ for which

$$V_j \subset U_{\phi(i)}, \quad j \in J$$

holds. In words, each $V_j \in \mathcal{V}$ is contained in some $U_i \in \mathcal{U}$.

Definition 29.9: (Paracompact space)

A space X is called *paracompact* if any open cover of X admits a locally finite refinement.

Example 29.10: ($\mathbb{R}^n$ is Paracompact)

Suppose $\mathcal{U} = \{U_i\}_{i\in I}$ be an arbitrary open cover. Denote, $B_n = B_d(0, n)$ be the open ball of radius n, centered at orgin, and $\bar{B}_n$ be the closed ball. Note that each $\bar{B}_n$ is compact. Hence, for each n, there is a finte subset $I_n \subset I$ such that $\bar{B}_n \subset \bigcup_{i\in I_n} U_i$. Denote,

$$\mathcal{V}_1 := \{U_i \mid i \in I_1\}, \qquad \mathcal{V}_n := \left\{U_i \setminus \bar{B}_{n-1} \;\middle|\; i \in I_n\right\}, \quad n \geq 2.$$

Set, $\mathcal{V} = \bigcup \mathcal{V}_n$. By construction, each element of $\mathcal{V}$ is a subset of some $U_i \in \mathcal{U}$. For any $x \in \mathbb{R}^n$, consider $n \geq 1$ to be the least integer such that $x \in \bar{B}_n$. Then, $x \notin \bar{B}_{n-1}$. Clearly, we have $x \in U_i \setminus \bar{B}_{n-1}$ for some $i \in I_n$. Thus, $\mathcal{V}$ is a refinement of $\mathcal{U}$. More over, for any $x \in \mathbb{R}^n$, we have some $n \geq 1$ such that $x \in B_n$. It is clear that B_n can intersect only the open sets from $\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_n$, which is a finite collection. Thus, $\mathcal{V}$ is a locally finite refinement. Consequently, $\mathbb{R}^n$ is paracompact.

Exercise 29.11: (Exhaustion by Compacts)

A space X is said to be *exhaustible by compacts* if there are compact sets $K_n \subset X$ such that $X = \bigcup_{n\geq 1} K_n$, and $K_n \subset \mathring{K}_{n+1}$. Show that a T_2-space, which is exhaustible by compacts, is paracompact.

Remark 29.12: (Metric space is Paracompact)

Note that $\mathbb{R}$ with discrete topology is a metrizable space, which is not exhaustible by compacts, and hence, we cannot use the previous exercise! It is a deep theorem that any metric space is paracompact. The original proof was by Stone, which was simplified significantly by Mary Ellen Rudin.

Theorem 29.13: (M.E. Rudin's Proof : Metric Spaces are Paracompact)

A metrizable space is paracompact.

Proof

Let (X, d) be a metric space. Suppose $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover. By the well-ordering principle, we assume that the indexing set Λ is well-ordered! Note that for any $x \in X$, there exists a least $\alpha \in \Lambda$ such that $x \in U_\alpha$, since Λ is a well-order and $\mathcal{U}$ is a cover.

By induction over n, we construct a locally finite refinement as follows. Firstly, for each $\alpha \in \Lambda$, define $A_{\alpha,n}$ to be the set of points $x \in X$, satisfying the following.

i) $\alpha \in \Lambda$ is the least index such that $x \in U_\alpha$.

ii) For any $j < n$, we have $d(x, y) \geq \frac{1}{2^j}$ whenever $y \in \bigcup_{\beta\in\Lambda} A_{\beta,j}$

iii) $B_d\left(x, \frac{3}{2^n}\right) \subset U_\alpha$.

Note that for $n = 1$, the second condition is vacuous, and thus $A_{\alpha,1}$ consists of $x \in X$ satisfying only the first and third condition. Moreover, at the n^{th}-step, the second condition does not involve any $A_{\alpha,n}$. Thus, one can inductively construct all $A_{\alpha,n}$. We allow the possibility that $A_{\alpha,n} = \emptyset$ for some $\alpha \in \Lambda$ and $n \geq 1$. Once these sets are constructed, whenever $A_{\alpha,n} \neq \emptyset$, denote

$$D_{\alpha,n} := \bigcup \left\{ B_d\left(x, \frac{1}{2^n}\right) \;\middle|\; x \in A_{\alpha,n} \right\}, \qquad \alpha \in \Lambda,\ n \geq 1.$$

If $A_{\alpha,n} = \emptyset$, set $D_{\alpha,n} = \emptyset$ as well. We claim that $\mathcal{D}$, the collection of all $D_{\alpha,n}$ as defined, is a locally finite refinement of $\mathcal{U}$.

Let us check $\mathcal{D}$ covers X. For any $x \in X$, there is a least $\alpha \in \Lambda$ such that $x \in U_\alpha$, and $x \notin U_\beta$ for all $\beta < \alpha$. Now, U_α is open, and hence, there is some $n \geq 1$ such that $B_d\left(x, \frac{3}{2^n}\right) \subset U_\alpha$. We claim that $x \in D_{\beta,j}$ for some $\beta \in \Lambda$ and some $j \leq n$. We have two possibilities. Suppose $x \in A_{\alpha,n}$. Then, clearly $x \in D_{\alpha,n}$ and we are done. Suppose $x \notin A_{\alpha,n}$. Since the first and third condition is satisfied, we must have that the second condition is violated. Thus, for some $j < n$, we have some $y \in A_{\beta,j}$ such that $d(x,y) < \frac{1}{2^j}$. But then, $x \in B_d\left(y, \frac{1}{2^j}\right) \subset D_{\beta,j}$. Thus, we see that $\mathcal{D}$ covers X.

By construction, each $D_{\alpha,n} \subset U_\alpha$, and hence, $\mathcal{D}$ is indeed a refinement of $\mathcal{U}$.

Finally, let us show that $\mathcal{D}$ is locally finite. Let $x \in X$. Get the least $\alpha \in \Lambda$ such that $x \in D_{\alpha,n}$ for some $n \geq 1$. Then, choose some $j \geq 1$ such that $B_d\left(x, \frac{1}{2^j}\right) \subset D_{\alpha,n}$. Fix the ball $U := B_d\left(x, \frac{1}{2^{n+j}}\right)$. We show the following.

a) For any $i \geq n+j$, we have $U \cap D_{\beta,i} = \emptyset$ for all $b \in \Lambda$.

b) For any $i < n+j$, we have $U \cap D_{\beta,i} \neq \emptyset$ for at most a single $\beta \in \Lambda$.

Let $i \geq n+j$. In particular, $i > n$. Fix some $y \in A_{\beta,i}$. We then have $d(y,z) \geq \frac{1}{2^n}$ whenever $z \in A_{\alpha,n}$, and hence, $y \notin D_{\alpha,n}$. As $B_d\left(x, \frac{1}{2^j}\right) \subset D_{\alpha,n}$, we then get $d(x,y) \geq \frac{1}{2^j}$ as well. Now, $i \geq j+1$ and $n+j \geq j+1$. Hence, it follows from triangle inequality that

$$B_d\left(x, \frac{1}{2^{n+j}}\right) \cap B_d\left(y, \frac{1}{2^i}\right) = \emptyset.$$

Indeed, if $z \in B_d\left(x, \frac{1}{2^{n+j}}\right) \cap B_d\left(y, \frac{1}{2^i}\right)$, then we have

$$d(x,y) \leq d(x,z) + d(z,y) < \frac{1}{2^{n+j}} + \frac{1}{2^j} \leq \frac{1}{2^{j+1}} + \frac{1}{2^{j+1}} = \frac{1}{2^j},$$

a contradiction. Thus, for any $y \in A_{\beta,i}$, we have $U \cap B_d\left(y, \frac{1}{2^i}\right) = \emptyset$. But then clearly, $U \cap D_{\beta,i} = \emptyset$ holds for any $i \geq n+j$ and any $\beta \in \Lambda$.

Now, let $i < n+j$. Suppose $\beta \neq \gamma \in \Lambda$, without loss of generality, assume $\beta < \gamma$. Fix some $p \in D_{\beta,i}$ and $q \in D_{\gamma,i}$. Then, there are $y \in A_{\beta,i}, z \in A_{\gamma,i}$ such that $d(y,p) < \frac{1}{2^i}$ and $d(z,q) < \frac{1}{2^i}$. By construction, $B_d\left(y, \frac{3}{2^i}\right) \subset U_\beta$, and also, $z \notin U_\beta$ (as γ is the least one so that $z \in U_\gamma$). So, we must have $d(y,z) \geq \frac{3}{2^i}$. But then,

$$\frac{3}{2^i} \leq d(y,z) \leq d(y,p) + d(p,q) + d(q,z) < \frac{1}{2^i} + d(p,q) + \frac{1}{2^i} \Rightarrow d(p,q) > \frac{1}{2^i} \geq \frac{1}{2^{n+j-1}}.$$

Now, if U intersects both $D_{\beta,i}$ and $D_{\gamma,i}$ (with $\beta < \gamma$), then we can choose $p \in U \cap D_{\beta,i}$ and $q \in D_{\gamma,i}$. As argued above, we have $d(p,q) > \frac{1}{2^{n+j-1}}$. But, $p, q \in U = B_d\left(x, \frac{1}{2^{n+j}}\right)$. We have,

$$d(p,q) \leq d(p,z) + d(z,q) < \frac{1}{2^{n+j}} + \frac{1}{2^{n+j}} = \frac{1}{2^{n+j-1}},$$

a contradiction. Thus, U can intersect at most one $D_{\beta,i}$ whenever $i < n + j$.

But then it is clear U can intersect at most finitely many elements of $\mathcal{D}$, proving that $\mathcal{D}$ is a locally finite collection.

Thus, starting with the open cover $\mathcal{D}$, we have obtained a locally finite refinement $\mathcal{D}$ of $\mathcal{U}$. Consequently, any metric space is a paracompact space. □

Day 30 : 20^{th} November, 2025

paracompactness -- partition of unity

30.1 Paracompactness (Cont.)

Proposition 30.1

uppose $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of X, which admits a locally finite refinement $\mathcal{V} = \{V_j\}_{j\in J}$. Then, there exists a locally finite refinement $\mathcal{W} = \{W_i\}_{i\in I}$ such that $W_i \subset U_i$ for all $i \in I$.

Proof

Suppose $\phi : J \to I$ is the function such that $V_j \subset U_{\phi(j)}$ for each $j \in J$. For each $i \in I$, consider the set

$$W_i := \bigcup \{V_j \mid \phi(j) = i\} = \bigcup_{j\in\phi^{-1}(i)} V_j.$$

Clearly, $W_i \subset V_i$ for all $i \in I$, and $\mathcal{W} = \{W_i\}_{i\in I}$ still covers X. Thus, $\mathcal{W}$ is a refinement of $\mathcal{U}$ (but now with same indexing). We need to show that $\mathcal{W}$ is locally finite. Let $x \in X$ be fixed. Then, there is an open neighborhood $x \in N \subset X$, such that $N \cap V_j = \emptyset$ for all $j \in J \setminus A$, where $A \subset J$ is a finite set. Then, $B = \phi(A) \subset I$ is also a finite set. If possible, for some $i \in I \setminus B$, suppose $N \cap W_i \neq \emptyset$. Then, $N \cap \left(\bigcup_{\phi(j)=i} V_j\right) \neq \emptyset$. So, for some $j \in J$ with $\phi(j) = i$, we must have $N \cap V_j \neq \emptyset$. But then we must have $j \in B \Rightarrow i = \phi(j) \in \phi(B) = A$, a contradiction. Hence, $N \cap W_i = \emptyset$ for all $i \in I \setminus B$. Thus, $\mathcal{W}$ is a locally finite refinement of $\mathcal{U}$. □

Example 30.2: (Compact and Lindelöf space)

Since for a compact space, you can get a finite sub-cover of any open cover, it will clearly be a locally finite refinement. Thus, any compact space is paracompact. A Lindelöf space may not be paracompact! As an example, consider the double-origin plane. We have seen that it is $T_{2\frac{1}{2}}$ but not T_3. Also, it is second countable, and hence, Lindelöf. On the other hand, it cannot be paracompact, as a paracompact T_2 space is T_4.

Proposition 30.3: (Closed subset of Paracompact)

Let X be a paracompact space, and $C \subset X$ be closed. Then, C is paracompact.

Proof

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of C. Suppose $U_i = C \cap \tilde{U}_i$, where $\tilde{U}_i \subset X$ is open. Then, $\widetilde{\mathcal{U}} = \{X \setminus C\} \cup \left\{\tilde{U}_i\right\}_{i\in I}$ is an open cover of X. By paracompactness, we have a locally finite refinement, say, $\mathcal{V} = \{V_0\} \cup \{V_i\}_{i\in I}$, so that $V_0 \subset X \setminus C$ and $V_i \subset \tilde{U}_i$ for all $i \in I$. Now, for any $x \in C$, there is some open set $x \in N \subset X$ such that $N \cap V_i = \emptyset$ for all $i \in I_0 \setminus F$, where $F \subset I_0 := I \sqcup \{0\}$ is a finite subset. Then, clearly $N \cap X \cap V_i = \emptyset$ for any $i \in I \setminus F$. Thus, $\{V_i \cap C\}_{i\in I}$ is a locally finite refinement of $\mathcal{U}$. Consequently, C is paracompact. □

Theorem 30.4: (Paracompact T_2 is T_4)

A paracompact T_2 space is T_4.

Proof

Let X be a paracompact T_2 space. Let us first proof regularity of X. Say, $A \subset X$ is closed, and $x \in X\setminus A$ is a point. As X is T_2, for each $a \in A$ there are open sets U_a, V_a such that $x \in U_a, a \in V_a$ and $U_a \cap V_a = \emptyset$. Now, $\mathcal{V} = \{X \setminus A\} \cup \{V_a\}_{a\in A}$ is an open cover of X, and hence, there is a locally finite refinement, say, $\mathcal{W}$. Define

$$V := \bigcup \{W \in \mathcal{W} \mid W \cap A \neq \emptyset\}.$$

Note that $A \subset V$. Since $\mathcal{W}$ is a locally finite collection (and hence, so is any subcollection of $\mathcal{W}$), we also have

$$\bar{V} = \bigcup \left\{\bar{W} \;\middle|\; W \in \mathcal{W},\ W \cap A \neq \emptyset\right\}.$$

Now, any $W \in \mathcal{W}$ with $W \cap A \neq \emptyset$ is contained in some V_a for some $a \in A$, and hence, $\bar{W} \subset \bar{V}_a$. Thus,

$$\bar{V} \subset \bigcup_{a\in A} \bar{V}_a.$$

As $a \in U_a$ and $U_a \cap V_a = \emptyset$, we have $a \notin \bar{V}_a$, and hence, $a \not\in \bar{V}$. Then, consider $U = X \setminus \bar{V}$. Clearly, $x \in U, A \subset V$ and $U \cap V = \emptyset$. Thus, X is a regular space.

Now, consider $A, B \subset X$ be closed sets, with $A \cap B = \emptyset$. For each $a \in A$, there are open sets $U_a, V_a \subset X$ such that $a \in U_a, B \subset V_a$ and $U_a \cap V_a = \emptyset$. In particular, $B \cap \bar{U}_a = \emptyset$. Again, consider the open cover $\{X \setminus A\} \cup \{U_a\}_{a\in A}$ of X, and get a locally finite refinement, say, $\mathcal{G}$. Define $U = \bigcup \{G \in \mathcal{G} \mid G \cap A \neq \emptyset\}$. Then, $\bar{U} = \bigcup \left\{\bar{G} \;\middle|\; G \in \mathcal{G},\ G \cap A \neq \emptyset\right\}$ follows from local finiteness. Observe that $B \cap \bar{U} = \emptyset$. Then, set $V = X \setminus \bar{U}$. Clearly, $A \subset U, B \subset V$ and $U \cap V = \emptyset$. Thus, X is normal. As X is T_2, we have X is T_4. □

Example 30.5: ($T_4 \not\Rightarrow$ Paracompact)

Consider $[0, \Omega)$, the first uncountable ordinal with the order topology. We have seen that X is T_4 (in fact, T_5). Now, the product space $[0, \Omega) \times [0, \Omega]$ was shown to be not T_4. If $[0, \Omega)$ was paracompact, then the product with the compact space must be paracompact again. But then the product being paracompact and T_2 has to be normal, a contradiction. Hence, $[0, \Omega)$ is not paracompact. On the other hand, a Lindelöf, regular space is paracompact.

Exercise 30.6: (Product of Compact and Paracompact)

Show that the product of a compact and a paracompact space is again paracompact.

Hint

Use the tube lemma.

30.2 Partition of Unity

Definition 30.7: (Support)

Given a continuous map $f : X \to \mathbb{R}$, the *support of f* is defined as

$$\operatorname{supp}(f) := \overline{f^{-1}(\mathbb{R} \setminus \{0\})}.$$

In words, support is the smalles closed set containing the non-zero set of f.

Definition 30.8: (Partition of Unity)

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. A *partition of unity subordinate to $\mathcal{U}$* is a collection of continuous maps $\{f_i : X \to [0,1]\}_{i \in I}$ such that the following holds.

i) For each $i \in I$, we have $\operatorname{supp}(f_i) \subset U_i$.

ii) The collection $\{\operatorname{supp}(f_i)\}_{i \in I}$ is a locally finite cover of X.

iii) For each $x \in X$, we have $\sum_{i \in I} f_i(x) = 1$.

The arbitrary sum in the third condition is actually a finite sum by local finiteness.

Theorem 30.9: (Shrinking Lemma)

Let X be a paracompact T_2 space. Then, for any open cover $\mathcal{U} = \{U_i\}_{i \in I}$, there exist a locally finite open cover $\mathcal{V} = \{V_i\}_{i \in I}$ such that $V_i \subset \overline{V_i} \subset U_i$ for all $i \in I$.

Proof

Note that X is T_4 and in particular regular. Consider $\mathcal{W}$ to be the collection of open sets $W \subset X$ such that $W \subset \bar{W} \subset U_i$ for some $i \in I$. As $\mathcal{U}$ is a cover, by regularity, it follows that $\mathcal{W}$ is also a cover. Let us index it as $\mathcal{W} = \{W_j\}_{j \in J}$. We have function (by axiom of choice)

$$\theta : J \to I$$

such that $W_j \subset \overline{W_j} \subset U_{\theta(j)}$ for all $j \in J$. For each $i \in I$, denote

$$V_i = \bigcup \{W_j \mid \theta(j) = i\}.$$

Note that if $\theta^{-1}(i)$ is empty, then $V_i = \emptyset$. Consider the collection $\mathcal{V} = \{V_i\}_{i \in I}$, which is still a cover, and by construction, $V_i \subset U_i$ for all $i \in I$. Now, by local finiteness of $\mathcal{W}$, it follows that $\overline{V_i} = \bigcup_{\theta(j)=i} \overline{W_j} \subset U_i$ as well. Finally, let us check local finiteness of $\mathcal{V}$. For $x \in X$, there is an open set $N \subset X$ such that $N \cap W_j = \emptyset$ for all $j \in J \setminus F$, where $F \subset J$ is a finite set. Then, $\theta(F) \subset I$ is a finite set. Suppose, for some $i \in I \setminus \theta(F)$, we have $N \cap V_i \neq \emptyset$. Then, $N \cap W_j \neq \emptyset$

for some $j \in J$ such that $\theta(j) = i$. But then $j \in F \Rightarrow i = \theta(j) \in \theta(F)$, a contradiction. Thus, $\mathcal{W}$ is a locally finite collection. □

Theorem 30.10: (Existence of Partition of Unity)

In a paracompact T_2 space, any open cover admits a partition of unity subordinate to the cover.

Proof

Let X be a paracompact T_2 space, and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover. Applying the shrinking lemma twice, we get two open covers $\mathcal{V} = \{V_i\}_{i\in I}$ and $\mathcal{W} = \{W_i\}_{i\in I}$ such that

$$V_i \subset \overline{V_i} \subset W_i \subset \overline{W_i} \subset U_i, \qquad i \in I.$$

Note that $\overline{V_i}$ and $X\setminus W_i$ are disjoint closed sets. Then, by the Urysohn lemma, there are continuous functions $h_i : X \to [0,1]$ such that

$$h_i(\overline{V}_i) = 1, \quad h_i\left(X \setminus W_i\right) = 0.$$

Observe that

$$h_i^{-1}(0,1] \subset W_i \Rightarrow \operatorname{supp}(h_i) = \overline{h_i^{-1}(0,1]} \subset \overline{W_i} \subset U_i.$$

As $\mathcal{V} = \{V_i\}_{i\in I}$ is locally finite, it follows that $\left\{\overline{V_i}\right\}_{i\in I}$ is again a locally finite collection (Check!). Hence, $\{\operatorname{supp}(h_i)\}$ is a locally finite collection. For any $x \in X$, we have $x \in V_i$ for some $i \in I$, and then, $h_i(x) = 1$. Thus, $\{\operatorname{supp}(h_i)\}_{i\in I}$ is a locally finte cover of X. Now, local finiteness implies that $h(x) = \sum_{i\in I} h_i(x)$ is always a finite sum. Let us show that it is in fact a continuous map. Indeed, for any $x \in X$, there is a neighborhood $N \subset X$, such that $h|_N$ is a finite sum of continuous functions, which is then continuous. Since h is continuous on neighborhoods, it follows that h is continuous. Moreover, h is nowhere vanishing (Check!). In fact, we have $h : X \to [0,\infty)$ Define $f_i = \frac{h_i}{h}$, which is again continuous. Note that $f_i : X \to [0,1]$, as $h \geq 1$. Moreover, for each $x \in X$ we have

$$f(x) = \sum f_i(x) = \sum \frac{h_i(x)}{h(x)} = \frac{\sum h_i(x)}{h(x)} = \frac{h(x)}{h(x)} = 1.$$

Clearly, $\operatorname{supp}(f_i) \subset \operatorname{supp}(f_i)$. Thus, $\{f_i : X \to [0,1]\}_{i\in I}$ is partition of unity subordinate to the family $\mathcal{U}$. □