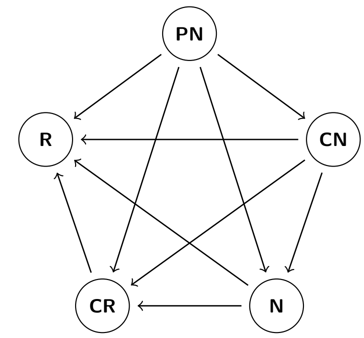
Quiz 2

13th November, 2025

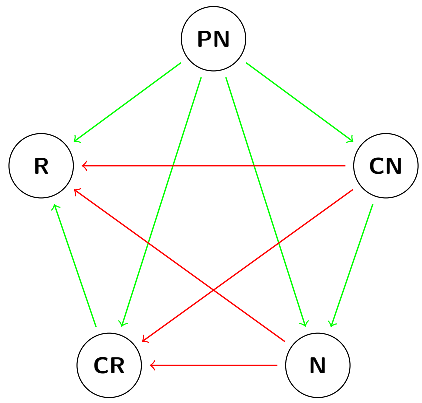
Time: 2 hrs **Total Marks:** 30

Q1. Consider the following separation properties: **p**erfectly **n**ormal (**PN**), **c**ompletely **n**ormal (**CN**), **n**ormal (**N**), **c**ompletely **r**egular (**CR**), and **r**egular (**R**). Look at the following diagram.



For each of the 10 arrows, decide whether the implication it represents is always true. Justify your answer with either a proof (if it is always true) or a counterexample (if it is not). Please clearly mention **A** $\Rightarrow$ **B** or **A** $\not\Rightarrow$ **B** for all the cases that you are attempting! $[2 \times 10 = 20]$

Solution. Here are all the true and false implications.



PN $\Rightarrow$ **CN**: Suppose X is perfectly normal. Then, for any subspace $Y \subset X$, consider some subset $A \subset Y$ (closed in the subspace topology). Then, $A = Y \cap \bar{A}$. Now, by the perfect normality of X, we have a continuous function $f : X \to [0, 1]$ such that $f^{-1}(0) = \bar{A}$. Consider the restriction $g := f|_Y : Y \to X$, which is again continuous. Clearly, $g^{-1}(0) = f^{-1}(0) \cap Y = \bar{A} \cap Y = A$. Thus, Y is again perfectly normal. In other words, perfect normality is a hereditary property. Since any perfectly normal space is normal, it follows that X is completely normal.

CN $\Rightarrow$ **N**: As a space is a subspace of itself, completely normal spaces are normal by definition.

PN $\Rightarrow$ **N**: Since **PN** $\Rightarrow$ **CN** and **CN** $\Rightarrow$ **N**, we have perfectly normal spaces are always normal.

PN $\Rightarrow$ **CR**: Suppose X is a perfectly normal space. Let $A \subset X$ be closed, and $x \in X \setminus A$. Since X is perfectly normal, it is a G_δ-space. Thus, there are open sets $U_n \subset X$ such that $A = \bigcap_{n \geq 1} U_n$. Since $x \notin A$, we must have some n_0 such that $x \notin U_{n_0}$. Consider $B = X \setminus U_{n_0}$. Then, A, B are closed sets, and $A \cap B = \emptyset$. We have a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = B$ and $f^{-1}(1) = A$. In particular, $f(x) = 0$ and $f(A) = 1$. Thus, X is completely regular.

PN $\Rightarrow$ **CR (Alt. proof):** Suppose X is perfectly normal. Let $A \subset X$ be a closed set, and $x \in X \setminus A$. Then, there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = \emptyset$ and $f^{-1}(1) = A$ (as $\emptyset$ is a closed set disjoint from A). Now, say $c = f(x)$. Clearly, $c \neq 1$, as $c \notin A = f^{-1}(A)$. Consider the map $g : [0,1] \to [0,1]$ given by

$$g(t) = \begin{cases} 0, & 0 \leq t \leq c, \\ \frac{t-c}{1-c}, & c \leq t \leq 1. \end{cases}$$

By pasting lemma, g is a continuous map. Consider $h = g \circ f : X \to [0,1]$. Then, $h(x) = g\left(f(x)\right) = g(c) = 0$, and for any $a \in A$ we have $h(a) = g\left(f(a)\right) = g(1) = 1$. Thus, h separates x and A, proving that X is completely regular.

CR $\Rightarrow$ **R**: Suppose X is completely regular. Let $A \subset X$ be closed, and $x \notin A$. Then, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(A) = 1$. Then, $U := f^{-1}\left[0, \frac{1}{4}\right)$ and $V := f^{-1}\left(\frac{3}{4}, 1\right]$ are open sets. Clearly, $x \in U, A \subset V$ and $U \cap V = \emptyset$. Thus, X is regular.

PN $\Rightarrow$ **R**: Since **PN** $\Rightarrow$ **CR** and **CR** $\Rightarrow$ **R**, we have any perfectly normal space is regular.

Counterexample: Consider the space $X = \{-1, 0, 1\}$ with the exlcuded-point topology with base $p = 0$, given as

$$\mathcal{T} = \{\emptyset, X, \{-1\}, \{1\}, \{-1, 1\}\}.$$

The closed sets in this topology are

$$\{\emptyset, X, \{0,1\}, \{0,-1\}, \{0\}\}.$$

In this space, any two nonempty closed sets always intersect. Thus, the space is normal. In fact, any (proper) open subset of this space is discrete, and hence again normal. Thus, $(X, \mathcal{T})$ is a completely normal space.

CN $\not\Rightarrow$ **R**: As noted, the space X is completely normal. Take $A = \{0\}$, which is closed, and $x = 1 \notin A$. Since the only open set containing 0 is X itself, we cannot separate x and A by open neighborhoods. Thus, X is not regular.

CN $\not\Rightarrow$ **CR**: Since **CR** $\Rightarrow$ **R**, the same space X above cannot be completely regular either.

N $\not\Rightarrow$ **R**: Again, the space X is normal. But X is not regular.

N $\not\Rightarrow$ **CR**: As the space X is normal but not completely regular, we have the claim.

Q2. Given a space X, fix a subset $\emptyset \subsetneq A \subsetneq X$. Let $a \in A$ and $b \in X \setminus A$. If there is a path joining a to b, show that the path must intersect the boundary ∂A. [5]

Solution. Let $\gamma : [0,1] \to X$ be a path, with $\gamma(0) = a, \gamma(1) = b$. We have $\partial A = \bar{A} \cap \overline{X \setminus A}$. If possible, suppose γ does not intersect ∂A. Now, $a \in A \subset \bar{A}$ and $b \in X \setminus A \subset \overline{X \setminus A}$, both of which are closed sets. Consider $P := \gamma^{-1}\left(\bar{A}\right), Q := \gamma^{-1}\left(\overline{X \setminus A}\right)$, which are closed sets of $[0,1]$. Now, $\emptyset \neq P, Q \subsetneq [0,1]$, as $0 \in P$ and $1 \in Q$. Clearly, $X = A \sqcup (X \setminus A) \subset \bar{A} \cup \overline{X \setminus A} \Rightarrow X = \bar{A} \cup \overline{X \setminus A}$, and hence,

$$[0,1] = \gamma^{-1}(X) = \gamma^{-1}\left(\bar{A} \cup \overline{X \setminus A}\right) = \gamma^{-1}\left(\bar{A}\right) \cup \gamma^{-1}\left(\overline{X \setminus A}\right) = P \cup Q.$$

As γ does not intersect $\partial A = \bar{A} \cap \overline{X \setminus A}$, we have

$$P \cap Q = \gamma^{-1}\left(\bar{A}\right) \cap \gamma^{-1}\left(\overline{X \setminus A}\right) = \gamma^{-1}\left(\partial A\right) = \emptyset.$$

Thus, P, Q are nontrivial closed sets of $[0,1]$, with $P \cap Q = \emptyset$ and $P \cup Q = [0,1]$. This contradicts the connectivity of the interval. Hence, any path joining $a \in A$ to $b \in X \setminus A$ must intersect the boundary ∂A at some point.

Q3. Let X be a second countable space. Suppose $\mathcal{B}$ is an arbitrary basis for X. Show that there exists a countable basis $\mathcal{B}'$ for X, such that $\mathcal{B}' \subset \mathcal{B}$. [5]

Solution. Fix a countable basis $\mathcal{U} = \{U_n\}_{n \geq 1}$. For each pair of numbers (m, n), if possible, choose some $B_{m,n} \in \mathcal{B}$ such that

$$U_m \subset B_{m,n} \subset U_n.$$

Denote the collection $\mathcal{B}' = \{B_{m,n}\} \subset \mathcal{B}$. Clearly this collection is countable, as it is indexed by a subset of $\mathbb{N} \times \mathbb{N}$. Let us show that this is a basis for the topology on X.

Let U be any open set, and $x \in U$ is a point. Then, there is some U_n such that $x \in U_n \subset U$. Now, $\mathcal{B}$ is a basis, and hence, there is some $B \in \mathcal{B}$ such that $x \in B \subset U_n$. Again, using the basis property of $\mathcal{U}$, there is some U_m such that $x \in U_m \subset B \subset U_n$. Now, this implies that there is $B_{m,n} \in \mathcal{B}'$, such that $U_m \subset B_{m,n} \subset U_n$. Observe that we have $x \in U_m \subset B_{m,n} \subset U_n \subset U$, i.e, $x \in B_{m,n} \subset U$. Since U and $x \in U$ was arbitrary, it follows that $\mathcal{B}'$ is a basis for X, which is also countable.