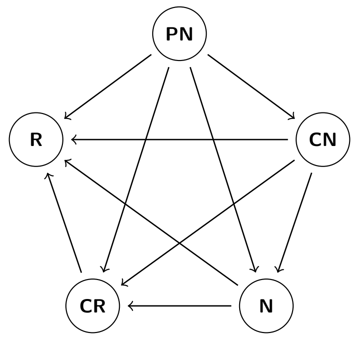
Quiz 2

13th November, 2025

Time: 2 hrs **Total Marks:** 30

Question 1 is mandatory. Attempt **one** more from the rest. You can score a maximum of **25**.

Q1. Consider the following separation properties: **p**erfectly **n**ormal (**PN**), **c**ompletely **n**ormal (**CN**), **n**ormal (**N**), **c**ompletely **r**egular (**CR**), and **r**egular (**R**). Look at the following diagram.



For each of the 10 arrows, decide whether the implication it represents is always true. Justify your answer with either a proof (if it is always true) or a counterexample (if it is not). Please clearly mention **A** $\Rightarrow$ **B** or **A** $\not\Rightarrow$ **B** for all the cases that you are attempting! $[2 \times 10 = 20]$

Hint: There are exactly **six** true implications, and **four** false ones. There is a single counterexample which demonstrates that all four of those false implications fail! You might find it easier to prove the true statements first.

Q2. Given a space X, fix a subset $\emptyset \subsetneq A \subsetneq X$. Let $a \in A$ and $b \in X \setminus A$. If there is a path joining a to b, show that the path must intersect the boundary ∂A. $[5]$

OR

Q3. Let X be a second countable space. Suppose $\mathcal{B}$ is an arbitrary basis for X. Show that there exists a countable basis $\mathcal{B}'$ for X, such that $\mathcal{B}' \subset \mathcal{B}$. $[5]$

Definitions

- X is *perfectly normal* if given any two disjoint closed sets $A, B \subset X$, there is a continuous function $f : X \to [0,1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.
- X is *completely normal* if every subspace of X is normal.
- X is *normal* if given any two disjoint closed sets $A, B \subset X$, there are open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.
- X is *completely regular* if given a closed set $A \subset X$ and a point $x \in X \setminus A$, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(A) = 1$.
- X is *regular* if given a closed set $A \subset X$ and a point $x \in X \setminus A$, there are open sets $U, V \subset X$ such that $x \in U, A \subset V$ and $U \cap V = \emptyset$.
- Given $A \subset X$, the *boundary* is defined as $\partial A = \bar{A} \cap \overline{X \setminus A}$.
- X is called *second countable* if there exists a basis for the topology, which is countable.