Quiz 1

(Supplementary)

24th September, 2025

Time: 2 hrs **Total marks:** 26

On the real line $\mathbb{R}$, let $\mathcal{T}_{\geq}$ be the collection of subsets consisting of $\emptyset$, along with the usual open sets $U \subset \mathbb{R}$ satisfying

$$\mathbb{Z}_{\geq n} := \{n, n+1, n+2, \ldots\} \subset U, \text{ for some } n \in \mathbb{Z}.$$

Attempt any question. You can get **maximum 20**.

Q1. Show that $\mathcal{T}_{\geq}$ is a topology on $\mathbb{R}$.

Solution: Given $\emptyset \in \mathcal{T}_{\geq}$. Also, $\mathbb{R} \supset \mathbb{Z}$, and hence, $\mathbb{R} \in \mathcal{T}_{\geq}$. For any collection $U_\alpha \in \mathcal{T}_{\geq}$, we have some α_0 and some $n \in \mathbb{Z}$ so that $\mathbb{Z}_{\geq n} \subset U_{\alpha_0}$. But then $\mathbb{Z}_{\geq n} \subset \bigcup U_\alpha$, which is already open in the usual topology. Thus, $\bigcup U_\alpha \in \mathcal{T}_{\geq}$. Also, for $U_1, \ldots, U_k \in \mathcal{T}_{\geq}$, we have some $n_1, \ldots, n_k \in \mathbb{Z}$ such that $\mathbb{Z}_{\geq n_i} \subset U_{n_i}$. Take $N = \max_{1 \leq i \leq k} \{n_i\}$. Then, $\mathbb{Z}_{\geq N} \subset \bigcap_{i=1}^{k} U_i$, which is already open in $\mathbb{R}$.

Q2. Compare (i.e., strictly fine, strictly coarse or incomparable) $\mathcal{T}_{\geq}$ with the following.

i) The usual topology on $\mathbb{R}$.

Solution: It is given that any open in $\mathcal{T}_{\geq}$ is open in the usual topology. Also, $(0,1) \notin \mathcal{T}_{\geq}$. Thus, $\mathcal{T}_{\geq}$ is strictly coarser than the usual topology.

ii) The lower limit topology $\mathbb{R}_l$.

Solution: Since lower limit topology is strictly finer than the usual topology, it is strictly finer than $\mathcal{T}_{\geq}$ as well.

iii) The upper limit topology $\mathbb{R}_u$.

Solution: Since upper limit topology is strictly finer than the usual topology, it is strictly finer than $\mathcal{T}_{\geq}$ as well.

iv) The topology $\mathcal{T}_{\rightarrow} = \{\emptyset, \mathbb{R}\} \bigcup \{(a, \infty) \mid a \in \mathbb{R}\}$ on $\mathbb{R}$.

Solution: For any (a, ∞), we can always get some $a < n \in \mathbb{Z}$, whence $\mathbb{Z}_{\geq n} \subset (a, \infty)$. Thus, $(a, \infty) \in \mathcal{T}_{\geq}$. On the other hand, $\mathbb{R} \setminus \{0\}$ is open in $\mathcal{T}_{\geq}$, but not open in $\mathcal{T}_{\rightarrow}$. Thus, $\mathcal{T}_{\geq}$ is strictly finer than $\mathcal{T}_{\rightarrow}$.

Q3. For $a \in \mathbb{R}$, determine (with justification) the closures of the following sets in $(\mathbb{R}, \mathcal{T}_{\geq})$. $[1 \times 5 = 5]$

i) (a, ∞).

Solution: Any open set in $\mathcal{T}_{\geq}$ is unbounded from above. Thus, any open set will intersect (a, ∞). Hence, $\overline{(a, \infty)} = \mathbb{R}$.

ii) $(-\infty, a)$.

Solution: We have $(-\infty, a]$ is closed, and clearly $\mathbb{R} \setminus (-\infty, a) = [a, \infty)$ is not even open in the usual topology. Thus, $\overline{(-\infty, a)} = (-\infty, a]$, being the smallest closed set containing it.

iii) $\{a\}$.

Solution: Since $\mathbb{R} \setminus \{a\} = (-\infty, a) \cup (a, \infty)$ is open in $\mathcal{T}_{\geq}$, we have $\overline{\{a\}} = \{a\}$.

iv) $A = \{a, a+1, a+2, \dots\}$.

Solution: If $a \in \mathbb{Z}$, then it follows that any open set will intersect A. Thus, $\bar{A} = \mathbb{R}$. On the other hand, if $a \notin \mathbb{Z}$, then it follows that $\mathbb{R} \setminus A$ is open in the usual topology, and contains $\mathbb{Z}$. Thus, $\bar{A} = A$ in that case.

v) $B = \{a, a-1, a-2, \dots\}$.

Solution: Clearly $\mathbb{R} \setminus B$ is open in the usual topology, and contains $\mathbb{Z}_{\geq n}$ for any $a < n \in \mathbb{Z}$. Thus, $\bar{B} = B$.

Q4. Determine (with justification) whether $(\mathbb{R}, \mathcal{T}_{\geq})$ is T_0, T_1, or T_2.

Solution: Since for any $a \in \mathbb{R}$, we have $\{a\}$ is closed, it follows that $(\mathbb{R}, \mathcal{T}_{\geq})$ is T_1, and hence, T_0. On the other hand, any two open sets will contain some $\mathbb{Z}_{\geq n}$, and hence, $(\mathbb{R}, \mathcal{T}_{\geq})$ is not T_2.

Q5. Prove or give counter-example to the following statements.

i) If a sequence (x_n) converges to x in $(\mathbb{R}, \mathcal{T}_{\rightarrow})$, then $x_n \to x$ in $(\mathbb{R}, \mathcal{T}_{\geq})$ as well.

Solution: Consider $x_n = n + 0.5$. Then, $\{x_n\}$ is discrete in $(\mathbb{R}, \mathcal{T}_{\geq})$, and thus, does not converge to any point of $\mathbb{R}$. On the other hand, x_n converges to any point of $\mathbb{R}$ in the topology $\mathcal{T}_{\rightarrow}$.

ii) If a sequence (x_n) converges to x in $(\mathbb{R}, \mathcal{T}_{\geq})$, then $x_n \to x$ in $(\mathbb{R}, \mathcal{T}_{\rightarrow})$ as well.

Solution: Since $\mathcal{T}_{\geq}$ is strictly finer than $\mathcal{T}_{\rightarrow}$, it follows that if $x_n \to x$ in $\mathcal{T}_{\geq}$, then $x_n \to x$ in $\mathcal{T}_{\rightarrow}$ as well.

Q6. Prove or disprove : $(\mathbb{R}, \mathcal{T}_{\geq})$ is path connected.

Solution: Since $\mathcal{T}_{\geq}$ is strictly coarser than the usual topology, for any continuous map $f : X \to \mathbb{R}_{\text{usual}}$, we have $f : X \to (\mathbb{R}, \mathcal{T}_{\geq})$ is still continuous. Since $\mathbb{R}$ is path connected with the usual topology, it follows that $(\mathbb{R}, \mathcal{T}_{\geq})$ is path connected.

Q7. Consider the equivalence relation on $\mathbb{R}$: $a \sim b$ if and only if $a - b \in \mathbb{Z}$. For any $x \in \mathbb{R}$, find the closure of the equivalence class $[x]$ in the quotient topology induced from $(\mathbb{R}, \mathcal{T}_{\geq})$.

Solution: For any $x \notin \mathbb{Z}$, we have seen that $x + \mathbb{Z}$ is closed (follows from iv) and v) of Q3). Hence, $q^{-1}([x])$ is closed. But then $\overline{\{[x]\}} = \{[x]\}$. Now, consider $[0]$. For any $x \notin \mathbb{Z}$, if we have some open set $[x] \in U \subset \mathbb{R}/_{\sim}$, then $q^{-1}(U)$, being a saturated open set, must contain $\mathbb{Z}$. Thus, $[0] \in U$. Hence, $\overline{\{[0]\}} = \mathbb{R}/_{\sim}$.

Q8. Consider the equivalence relation on $\mathbb{R}$: $a \sim b$ if and only if either

$$a, b \in \mathbb{R} \setminus \mathbb{Z}, \text{ and } a = b, \qquad \text{or}, \qquad a, b \in \mathbb{Z}.$$

For any $x \in \mathbb{R}$, find the closure of the equivalence class $[x]$ in the quotient topology induced from $(\mathbb{R}, \mathcal{T}_{\geq})$.

Solution: For any $x \notin \mathbb{Z}$, we have $q^{-1}([x]) = \{x\}$, which is closed in $\mathcal{T}_{\geq}$. Thus, $\overline{\{[x]\}} = \{[x]\}$. Next, consider the class $[0]$. For any open set $U \subset \mathbb{R}/_{\sim}$, we have $q^{-1}(U)$ intersects $\mathbb{Z}$. Thus, $[0] \in U$. Hence, $\overline{\{[0]\}} = \mathbb{R}/_{\sim}$.